NATURAL DEDUCTION

$$\frac{}{\Gamma, A \vdash A}$$ (Assumption Axiom)

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$ (Introduction of Assumption)

$$\frac{\Gamma, A \vdash B \quad \Gamma, \neg A \vdash B}{\Gamma \vdash B}$$ (Elimination of Assumption)

$$\frac{}{\Gamma \vdash \mathit{true}}$$ (*true* Axiom)

$$\frac{}{\Gamma \vdash \neg \mathit{false}}$$ (*false* Axiom)

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}$$ ($\vee$ Introduction)

$$\frac{\Gamma \vdash A \vee B \quad \Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma \vdash C}$$ ($\vee$ Elimination)

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$ ($\wedge$ Introduction)

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}$$ ($\wedge$ Elimination)

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$$ ($\to$ Introduction)
(also called Discharge Rule)

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A \to B}{\Gamma \vdash B}$$ ($\to$ Elimination)
(this rule corresponds to Modus Ponens)

$$\frac{\Gamma \vdash A \to B \quad \Gamma \vdash B \to A}{\Gamma \vdash A \leftrightarrow B}$$ ($\leftrightarrow$ Introduction)

$$\frac{\Gamma \vdash A \leftrightarrow B}{\Gamma \vdash A \to B} \qquad \frac{\Gamma \vdash A \leftrightarrow B}{\Gamma \vdash B \to A}$$ ($\leftrightarrow$ Elimination)

$$\frac{\Gamma \vdash A}{\Gamma \vdash \neg\neg A} \qquad (\neg\neg \text{ Introduction})$$

$$\frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A} \qquad (\neg\neg \text{ Elimination})$$

$$\frac{\Gamma, A \vdash B \quad \Gamma, A \vdash \neg B}{\Gamma \vdash \neg A} \qquad (\neg \text{ Introduction})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash \neg A}{\Gamma \vdash B} \qquad (\neg \text{ Elimination})$$

A term t *is free* (or *can be substituted*) *for the variable* x in a formula $A(x)$ iff no *free* occurrence of x in $A(x)$ is in the scope of a quantifier which binds a variable of t.

We have that: (i) both $r(x)$ and $r(y)$ are free for x in $\exists x\, q(x)$, and
(ii) $r(y)$ is *not* free for x in $\exists y\, leq(x, y)$.

Recall that $A(t)$ denotes the term obtained by replacing *all free* occurrences of x in $A(x)$ by the term t.

$\dfrac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x\, A(x)}$	if x is not free in Γ	($\forall$ Introduction)
$\dfrac{\Gamma \vdash \forall x\, A(x)}{\Gamma \vdash A(t)}$	if t is free for x in $A(x)$	($\forall$ Elimination)
$\dfrac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x\, A(x)}$	if t is free for x in $A(x)$	($\exists$ Introduction)
$\dfrac{\Gamma \vdash \exists x\, A(x) \quad \Gamma, A(b) \vdash C}{\Gamma \vdash C}$	if b is a constant occurring neither in Γ nor in $\exists x\, A(x)$ nor in C	($\exists$ Elimination)

THEOREM. **[Deduction Theorem]** Given a set Γ of formulas and two formulas φ and ψ, we have that $\Gamma \vdash \varphi \to \psi$ iff $\Gamma, \varphi \vdash \psi$.

PROOF. The *if* direction is shown by the $\to$ Introduction rule. The *only-if* direction is shown as follows. By the Introduction of Assumption rule, from $\Gamma \vdash \varphi \to \psi$ we get:
(1) $\Gamma, \varphi \vdash \varphi \to \psi$
By the Assumption axiom from (1) we get: (2) $\Gamma, \varphi \vdash \varphi$

From (2) and (1) by the $\rightarrow$ Elimination rule we get $\Gamma, \varphi \vdash \psi$. $\square$

Theorem. For every set Γ of *closed* formulas (and, in particular, for the empty set of formulas) and for any (closed or open) formula φ, there exists a derivation of φ from Γ in the Classical presentation, that is, $\Gamma \vdash \varphi$ holds iff $\Gamma \vdash \varphi$ is as sequent which holds in the Natural Deduction presentation.

Rules for equality

$$\frac{}{\Gamma \vdash t{=}t} \qquad\qquad (= \text{Axiom})$$

$$\frac{\Gamma \vdash t_1{=}t_2}{\Gamma \vdash A(t_1){\leftrightarrow}A(t_2)} \quad \text{if } t_1 \text{ and } t_2 \text{ are free for } x \text{ in } A(x) \quad (= \text{Rule})$$

An equivalence for existential variables in Horn clauses

$$\begin{aligned} \forall x\,(A(x) \rightarrow B) \;\leftrightarrow\; & \forall x\,(\neg A(x) \vee B) \;\leftrightarrow\; (\forall x\, \neg A(x)) \vee B \;\leftrightarrow \\ \leftrightarrow\; & (\neg \exists x\, A(x)) \vee B \;\leftrightarrow\; (\exists x\, A(x) \rightarrow B). \end{aligned}$$

Two interesting equivalences involving equality

Equivalence (A): $\forall x\,(x{=}t \rightarrow A(x)) \;\leftrightarrow\; A(t)$

where t is free for x in $A(x)$ and x does not occur in t.

Proof. At line 10: $t{=}t \rightarrow A(t)$ should be the result of substituting all free occurrences of x in $x{=}t \rightarrow A(x)$ by t.

1.	$A(t),\ x{=}t \vdash x{=}t$	Assumption Axiom
2.	$A(t),\ x{=}t \vdash A(x) \leftrightarrow A(t)$	$=$ Rule (t is free for x in $A(x)$ and x is free for x in $A(x)$)
3.	$A(t),\ x{=}t \vdash A(t) \rightarrow A(x)$	$\leftrightarrow$ Elimination
4.	$A(t),\ x{=}t \vdash A(t)$	Assumption Axiom
5.	$A(t),\ x{=}t \vdash A(x)$	$\rightarrow$ Elimination $(3, 4)$
6.	$A(t) \vdash x{=}t \rightarrow A(x)$	$\rightarrow$ Introduction
7.	$A(t) \vdash \forall x\,(x{=}t \rightarrow A(x))$	$\forall$ Introduction (x is not free in $A(t)$)
$\boxed{8.}$	$\vdash A(t) \rightarrow \forall x\,(x{=}t \rightarrow A(x))$	$\rightarrow$ Introduction
9.	$\forall x(x{=}t \rightarrow A(x)) \vdash \forall x(x{=}t \rightarrow A(x))$	Assumption Axiom
10.	$\forall x\,(x{=}t \rightarrow A(x)) \vdash t{=}t \rightarrow A(t)$	$\forall$ Elimination (t is free for x in $(x{=}t \rightarrow A(x))$ and x does not occur in t)
11.	$\forall x\,(x{=}t \rightarrow A(x)) \vdash t{=}t$	$=$ Axiom
12.	$\forall x\,(x{=}t \rightarrow A(x)) \vdash A(t)$	$\rightarrow$ Elimination $(10, 11)$
$\boxed{13.}$	$\vdash (\forall x\,(x{=}t \rightarrow A(x))) \rightarrow A(t)$	$\rightarrow$ Introduction
14.	$\vdash (\forall x\,(x{=}t \rightarrow A(x))) \leftrightarrow A(t)$	$\leftrightarrow$ Introduction $(8, 13)$

Equivalence (B): $\exists x\,(x{=}t \wedge A(x)) \;\leftrightarrow\; A(t)$
where t is free for x in $A(x)$ and x does not occur in t.

Proof. The symbol b stands for a constant, not occurring elsewhere. Note that at line 12, since x does not occur in t, the formula $b{=}t \wedge A(b)$ is the result of substituting b for x in $x{=}t \wedge A(x)$. At line 3: $t{=}t \rightarrow A(t)$ should be the result of substituting all free occurrences of x in $x{=}t \rightarrow A(x)$ by t.

1.	$A(t) \vdash A(t)$	Assumption Axiom
2.	$A(t) \vdash t{=}t$	$=$ Axiom
3.	$A(t) \vdash t{=}t \wedge A(t)$	$\wedge$ Introduction $(2,1)$
4.	$A(t) \vdash \exists x\,(x{=}t \wedge A(x))$	$\exists$ Introduction (x does not occur in t and t is free for x in $(x{=}t \rightarrow A(x))$)
$\boxed{5.}$	$\vdash A(t) \rightarrow \exists x\,(x{=}t \wedge A(x))$	$\rightarrow$ Introduction
6.	$\exists x\,(x{=}t \wedge A(x)) \vdash \exists x\,(x{=}t \wedge A(x))$	Assumption Axiom
7.	$\exists x\,(x{=}t \wedge A(x)),\ b{=}t \wedge A(b) \vdash b{=}t \wedge A(b)$	Assumption Axiom
8.	$\exists x\,(x{=}t \wedge A(x)), b{=}t \wedge A(b) \vdash b{=}t$	$\wedge$ Elimination (7)
9.	$\exists x\,(x{=}t \wedge A(x)), b{=}t \wedge A(b) \vdash A(b)$	$\wedge$ Elimination (7)
10.	$\exists x\,(x{=}t \wedge A(x)), b{=}t \wedge A(b) \vdash A(b) \leftrightarrow A(t)$	$=$ Rule (8) (b and t are free for x in $A(x)$)
11.	$\exists x\,(x{=}t \wedge A(x)), b{=}t \wedge A(b) \vdash A(b) \rightarrow A(t)$	$\leftrightarrow$ Elimination
12.	$\exists x\,(x{=}t \wedge A(x)), b{=}t \wedge A(b) \vdash A(t)$	$\rightarrow$ Elimination $(9,11)$
13.	$\exists x\,(x{=}t \wedge A(x)) \vdash A(t)$	$\exists$ Elimination $(6,12)$ (x does not occur in t)
$\boxed{14.}$	$\vdash \exists x\,(x{=}t \wedge A(x)) \rightarrow A(t)$	$\rightarrow$ Introduction
15.	$\vdash \exists x\,(x{=}t \wedge A(x)) \leftrightarrow A(t)$	$\leftrightarrow$ Introduction $(5,14)$