**Exam of Theoretical Computer Science. 28–02–2003.**

1. Define the set of partial recursive functions from $N$ to $N$, where $N$ denotes the set of natural numbers.

2. Let PRF be the set of all partial-recursive-functions from $N$ to $N$ (p.r.f, for short). Show that the set of the total p.r.f. is not a recursive subset of PRF.

3. Show that $f : N \to N$ such that $f(x) = \text{if } x \leq 1 \text{ then } 1 \text{ else } x \times f(x-2)$ is a primitive recursive function.

4. Give a bijection between $N^k$ and $N$ for any $k \geq 0$.

5. Define the least Herbrand model of a definite logic program.

6. Show that $\vdash (\forall x[x = t \to A(x)]) \leftrightarrow A(t)$ if $t$ is free for $x$ in $A(x)$ and $x$ does not occur in $t$. Show that the two conditions above are necessary.

7. Given a continuous function $f : D \to D$ where $D$ is a cpo with bottom. Show that:
(i) $\bigsqcup_{n\in\omega}(\lambda f.f^n(\bot)) \;=\; \lambda f.(\bigsqcup_{n\in\omega} f^n(\bot))$, and
(ii) $\mathit{fix}(f) = f(\mathit{fix}(f))$ where $\mathit{fix}$ is defined as follows: $\bigsqcup_{n\in\omega}(\lambda f.f^n(\bot))$.

8. What is the weakest precondition of the statement $\quad x := 0;\ \textbf{while}\ Q \wedge x \geq 0\ \textbf{do}\ x := x+1 \quad$ and the postcondition $Q$?

9. Find a formula $P$ such that the Hoare's triple
$\{y>1\}\ x := 0;\ \textbf{while}\ y > x \wedge P\ \textbf{do}\ x := y-1\ \{x=0 \wedge y>1\}$ holds. Explain your answer.

10. Let us consider a cpo $(D, \sqsubseteq)$. $U$ subset of $D$ is said to be *open* iff (i) $\forall d, e \in D.\ (d \sqsubseteq e$ and $d \in U) \Rightarrow e \in U$, and (ii) for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in $D$ we have $\bigsqcup_{n\in\omega} d_n \in U \;\Rightarrow\; \exists n \in \omega.\ d_n \in U$. Show that: (i) $\emptyset$ and $D$ are open, (ii) any finite intersection of open sets is open, and (iii) the union of any set of open sets is open.

11. Discuss the following commutative diagram, where $f$ is a continuous function:

$$\begin{tikzcd}
X \times Y \arrow[r, "f"] \arrow[d, "curry(f) \times id"'] & E \\
{[Y \to E] \times Y} \arrow[ur, "apply"'] &
\end{tikzcd}$$

Give the explicit definition of *apply* and *curry*$(f)$ when: (i) $X = Y = E = N$ (where $N$ is the set of natural numbers), (ii) $f$ is $\lambda xy.sum(x,y)$, and (iii) $sum(0,y) = y$, $sum(s(x),y) = s(sum(x,y))$.

12. We say that a relation $\prec$ is well-founded on a set $X$ iff there is no infinite descending sequence $\ldots \prec x_i \prec \ldots \prec x_1 \prec x_0$ of elements of $X$. Let $f : A \to B$ be a function and $\prec_B$ a well-founded relation on $B$. Show that $\prec_A$ a well-founded relation on $A$, where $\prec_A$ is defined as follows: $a \prec_A a'$ iff $f(a) \prec_B f(a')$.

13. Consider the equation $f(x) = \text{if } x < 3 \text{ then } 1 \text{ else } x \times f(x-1)$ and the associated functional $\varphi = \lambda f.\lambda x.\ \text{if } x < 3 \text{ then } 1 \text{ else } x \times f(x-1)$.
(i) Compute the function $\delta_{va} : N \to N_\bot$, where $N$ is the set of natural numbers, defined as the minimal fixpoint of $\varphi$ in call-by-value semantics. (ii) Compute the function $\delta_{na} : N_\bot \to N_\bot$ defined as the minimal fixpoint of $\varphi$ in call-by-name semantics.

14. Show that in the lazy-2 denotational semantics for any $F : \tau \to \tau$ and for any environment $\rho$, we have that $[[F(RF)]]\rho = [[RF]]\rho$, where $R$ is $\textbf{rec}\ y.(\lambda f.f(yf))$.

15. Write a Prolog program for evaluating the operational semantics of the Lazy-1 language.

16. Check whether or not for any environment $\rho$,
$[[\mathbf{rec}\ f.(\lambda x.e)]]\rho = [[\lambda x.(\mathbf{let}\ f \Leftarrow (\mathbf{rec}\ f.(\lambda x.e))\ \mathbf{in}\ f(x))]]$
in eager, lazy-1, or lazy-2 denotational semantics.

17. Show that the bisimulation equivalence in pure CCS is an equivalence relation and not a congruence relation.

18. Give the operational semantics of the guarded commands of Dijkstra.

19. What does it mean that given a formula $A$, $\nu X.(A \wedge [.]X)$ holds in a given state of a process?