Exam of Theoretical Computer Science. November–December 2003.

1. Show that $f : N \to N$ such that $f(x) = \text{if } x \leq 1 \text{ then } 1 \text{ else } x \times f(x-2)$ is a primitive recursive function.

2. Show that the Fibonacci function from N to N, where N denotes the set of natural numbers, is a primitive recursive function.

3. Let PRF be the set of all partial-recursive-functions from N to N (p.r.f, for short). Show that the set of the total p.r.f. is not a recursive subset of PRF.

4. Give a bijection between $\bigcup_{k=0}^{\infty} N^k$ and N.

5. Define the least Herbrand model of a definite logic program.

6. Show that $\vdash (\forall x[x = t \to A(x)]) \leftrightarrow A(t)$ if t is free for x in $A(x)$ and x does not occur in t. Show that the two conditions above are necessary.

7. Given a continuous function $f : D \to D$ where D is a cpo with bottom. Show that:
   (i) $\bigsqcup_{n\in\omega}(\lambda f.f^n(\bot)) \;=\; \lambda f.(\bigsqcup_{n\in\omega} f^n(\bot))$, and
   (ii) $\mathit{fix}(f) = f(\mathit{fix}(f))$ where fix is defined as follows: $\bigsqcup_{n\in\omega}(\lambda f.f^n(\bot))$.

8. (i) Find the weakest precondition of the statement
   $x := 1;\ y := 0;\ \textbf{while}\ x \geq 1\ \textbf{do}\ x := x - 1;\ \ y := y + 1\ \textbf{od}$ and the postcondition $x < y$.
   (ii) Find the weakest precondition of the statement
   $x := 1;\ y := 0;\ \textbf{while}\ x \leq 1\ \textbf{do}\ x := x - 1;\ \ y := y + 1\ \textbf{od}$ and the postcondition $x < y$.

9. Find the weakest precondition of the statement $x := 0;\ \textbf{while}\ Q \wedge x \geq 0\ \textbf{do}\ x := x + 1$ and the postcondition Q.

10. Find a formula P such that the Hoare's triple
$\{y > 1\}\ x := 0;\ \textbf{while}\ y > x \wedge P\ \textbf{do}\ x := y - 1\ \{x = 0 \wedge y > 1\}$ holds. Explain your answer.

11. Let us consider a cpo $(D, \sqsubseteq)$. U subset of D is said to be *open* iff (i) $\forall d, e \in D.\ (d \sqsubseteq e$ and $d \in U) \Rightarrow e \in U$, and (ii) for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D we have $\bigsqcup_{n\in\omega} d_n \in U \;\Rightarrow\; \exists n \in \omega.\ d_n \in U$. Show that: (i) $\emptyset$ and D are open, (ii) any finite intersection of open sets is open, and (iii) the union of any set of open sets is open.

12. Discuss the following commutative diagram, where f is a continuous function:

$$\begin{tikzcd}
X \times Y \arrow[r, "f"] \arrow[d, "curry(f) \times id"'] & E \\
{[Y \to E] \times Y} \arrow[ur, "apply"'] &
\end{tikzcd}$$

Give the explicit definition of *apply* and $curry(f)$ when: (i) $X = Y = E = N$ (where N is the set of natural numbers), (ii) f is $\lambda xy.sum(x,y)$, and (iii) $sum(0,y) = y$, $sum(s(x),y) = s(sum(x,y))$.

13. We say that a relation $\prec$ is well-founded on a set X iff there is no infinite descending sequence $\ldots \prec x_i \prec \ldots \prec x_1 \prec x_0$ of elements of X. Let $f : A \to B$ be a function and $\prec_B$ a well-founded relation on B. Show that $\prec_A$ a well-founded relation on A, where $\prec_A$ is defined as follows: $a \prec_A a'$ iff $f(a) \prec_B f(a')$.

14. Let us consider two cpo's D and E and a continuous function f from D to E. Show that if Q is an inclusive subset of E then $P = f^{-1}(Q)$ is an inclusive subset of D.

Recall that a set P is said to be *inclusive* iff for each ω-chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in P we have that $(\bigsqcup_{i\in\omega} d_i) \in P$.

15. Let a be the least fixpoint of the functional $\tau \in [[N^2 \to N_\perp] \to [N^2 \to N_\perp]]$ defined as follows:

$$\begin{aligned}\tau\ \varphi\ = \lambda(m,n).\ & Cond(\lfloor m\rfloor, \lfloor n+1\rfloor,\\ & Cond(\lfloor n\rfloor, \varphi(m-1,n),\\ & let\ l \Leftarrow \varphi(m,n-1).\ \varphi(m-1,l)))\end{aligned}$$

Show that $\forall m,n \geq 0$ we have that: $a(m,n) \neq \perp\ \wedge\ \lfloor 0\rfloor \prec a(m,n)$, where $\lfloor x\rfloor \prec \lfloor y\rfloor$ holds iff $x < y$ with $<$ denoting the usual *less-than* relation in $N \times N$.

16. Consider the equation $f(x) =$ if $x < 3$ then 1 else $x \times f(x-1)$ and the associated functional $\varphi = \lambda f.\lambda x.$ if $x < 3$ then 1 else $x \times f(x-1)$.
(i) Compute the function $\delta_{va} : N \to N_\perp$, where N is the set of natural numbers, defined as the minimal fixpoint of φ in call-by-value semantics. (ii) Compute the function $\delta_{na} : N_\perp \to N_\perp$ defined as the minimal fixpoint of φ in call-by-name semantics.

17. Show that in the lazy-2 denotational semantics for any $F : \tau \to \tau$ and for any environment ρ, we have that $[[F(RF)]]\rho = [[RF]]\rho$, where R is $\mathbf{rec}\ y.(\lambda f.f(yf))$.

18. Write a Prolog program for evaluating the operational semantics of the Lazy-1 language.

19. Check whether or not for any environment ρ,

$$[[\mathbf{rec}\ f.(\lambda x.e)]]\rho = [[\lambda x.(\mathbf{let}\ f \Leftarrow (\mathbf{rec}\ f.(\lambda x.e))\ \mathbf{in}\ f(x))]]$$

in eager, lazy-1, or lazy-2 denotational semantics.

20. Show that the bisimulation equivalence in pure CCS is an equivalence relation and not a congruence relation.

21. Assume that, given a formula A, the formula $\nu X.(A \wedge [.]X)$ holds in a state, say s, of a given process. Explain in words the meaning of $\nu X.(A \wedge [.]X)$ for the state s.

Projects.

P1. Write a Prolog program for the operational semantics of IMP. Try for the factorial program.

P2. Write a Prolog program for the operational semantics (by value and by name) of REC. Try it for the term $g(f(2))$ where f and g are defined by the following equations: $f(x) = f(x)+1$ and $g(x) = 5$.

P3. Write a Prolog program for local model checking.