Exam of Theoretical Computer Science. December 2007.

1. Show that $f : N \to N$ such that $f(x) = $ if $x \leq 1$ then 1 else $x \times f(x-2)$ is a primitive recursive function.

2. Show that the Fibonacci function from N to N, where N denotes the set of natural numbers, is a primitive recursive function.

3. Let PRF be the set of all partial-recursive-functions from N to N (p.r.f, for short). Show that the set of the total p.r.f. is not a recursive subset of PRF.

4. Give a bijection between $\bigcup_{k \in \omega} N^k$ and N.

5. Define the least Herbrand model of a definite logic program.

6. Show that $\vdash (\forall x[x = t \to A(x)]) \leftrightarrow A(t)$ if t is free for x in $A(x)$ and x does not occur in t. Show that the two conditions above are necessary.

7. Given a continuous function $f : D \to D$ where D is a cpo with bottom. Show that:
(i) $\bigsqcup_{n \in \omega} (\lambda f. f^n(\perp)) \; = \; \lambda f. (\bigsqcup_{n \in \omega} f^n(\perp))$, and
(ii) $\mathit{fix}(f) = f(\mathit{fix}(f))$ where fix is defined as follows: $\bigsqcup_{n \in \omega} (\lambda f. f^n(\perp))$.

8. (i) Find the weakest precondition of the statement
$x := 1$; $y := 0$; **while** $x \geq 1$ **do** $x := x - 1$; $y := y + 1$ **od** and the postcondition $x < y$.
(ii) Find the weakest precondition of the statement
$x := 1$; $y := 0$; **while** $x \leq 1$ **do** $x := x - 1$; $y := y + 1$ **od** and the postcondition $x < y$.

9. Find the weakest precondition of the statement $x := 0$; **while** $Q(x) \wedge x \geq 0$ **do** $x := x + 1$ and the postcondition $Q(x)$.

10. Find all formulas $P(x, y)$ such that the Hoare triple
$\{y > 1\}$ $x := 0$; **while** $y > x \wedge P(x, y)$ **do** $x := y - 1$ $\{x = 0 \wedge y > 1\}$ holds. Explain your answer.

11. Let us consider a cpo $(D, \sqsubseteq)$. U subset of D is said to be *open* iff (i) $\forall d, e \in D$. ($d \sqsubseteq e$ and $d \in U$) $\Rightarrow e \in U$, and (ii) for all chains $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in D we have $\bigsqcup_{n \in \omega} d_n \in U \;\Rightarrow\; \exists n \in \omega.\ d_n \in U$. Show that: (i) $\emptyset$ and D are open, (ii) any finite intersection of open sets is open, and (iii) the union of any set of open sets is open.

12. Discuss the following commutative diagram, where f is a continuous function:

$$\begin{tikzcd}
X \times Y \arrow[r, "f"] \arrow[d, "\mathit{curry}(f) \times \mathit{id}"'] & E \\
{[Y \to E] \times Y} \arrow[ur, "\mathit{apply}"'] &
\end{tikzcd}$$

Give the explicit definition of *apply* and *curry*(f) when: (i) $X = Y = E = N$ (where N is the set of natural numbers), (ii) f is $\lambda xy.\mathit{sum}(x, y)$, and (iii) $\mathit{sum}(0, y) = y$, $\mathit{sum}(s(x), y) = s(\mathit{sum}(x, y))$.

13. We say that a relation $\prec$ is well-founded on a set X iff there is no infinite descending sequence $\ldots \prec x_i \prec \ldots \prec x_1 \prec x_0$ of elements of X. Let $f : A \to B$ be a function and $\prec_B$ a well-founded relation on B. Show that $\prec_A$ a well-founded relation on A, where $\prec_A$ is defined as follows: $a \prec_A a'$ iff $f(a) \prec_B f(a')$.

14. Let us consider two cpo's D and E and a continuous function f from D to E. Show that if Q is an inclusive subset of E then $P = f^{-1}(Q)$ is an inclusive subset of D.
Recall that a set P is said to be *inclusive* iff for each ω-chain $d_0 \sqsubseteq d_1 \sqsubseteq \ldots$ in P we have that $(\bigsqcup_{i \in \omega} d_i) \in P$.

15. Let a be the least fixpoint of the functional $\tau \in [[N^2 \to N_\perp] \to [N^2 \to N_\perp]]$ defined as follows:

$$\begin{aligned} \tau\ \varphi\ = \lambda(m,n).\ & Cond(\lfloor m \rfloor, \lfloor n+1 \rfloor, \\ & Cond(\lfloor n \rfloor, \varphi(m-1,0), \\ & let\ l \Leftarrow \varphi(m,n-1).\ \varphi(m-1,l))) \end{aligned}$$

Show that $\forall m,n \geq 0$ we have that: $a(m,n) \neq \perp \ \wedge\ \lfloor 0 \rfloor \prec a(m,n)$, where $\lfloor x \rfloor \prec \lfloor y \rfloor$ holds iff $x < y$ with $<$ denoting the usual *less-than* relation in $N \times N$.

16. Consider the equation $f(x) = \text{if } x < 3 \text{ then } 1 \text{ else } x \times f(x-1)$ and the associated functional $\varphi = \lambda f.\lambda x.\ \text{if } x < 3 \text{ then } 1 \text{ else } x \times f(x-1)$.
(i) Compute the function $\delta_{va} : N \to N_\perp$, where N is the set of natural numbers, defined as the minimal fixpoint of φ in call-by-value semantics. (ii) Compute the function $\delta_{na} : N_\perp \to N_\perp$ defined as the minimal fixpoint of φ in call-by-name semantics.

17. Show that in the lazy-2 denotational semantics for any $F : \tau \to \tau$ and for any environment ρ, we have that $[[F(RF)]]\rho = [[RF]]\rho$, where R is $\mathbf{rec}\ y.(\lambda f.f(yf))$.

18. Write a Prolog program for evaluating the operational semantics of the Lazy-1 language.

19. Check whether or not for any environment ρ,

$$[[\mathbf{rec}\ f.(\lambda x.e)]]\,\rho = [[\lambda x.(\mathbf{let}\ f \Leftarrow (\mathbf{rec}\ f.(\lambda x.e))\ \mathbf{in}\ f(x))]]\,\rho$$

in eager, lazy-1, or lazy-2 denotational semantics.

20. Show that the bisimulation equivalence in pure CCS is an equivalence relation and not a congruence relation.

21. Assume that, given a formula A, the formula $\nu X.(A \wedge [.]X)$ holds in a state, say s, of a given process. Explain in words the meaning of $\nu X.(A \wedge [.]X)$ for the state s.

Projects.

P1. Write a Prolog program for the operational semantics of IMP. Try for the factorial program.

P2. Write a Prolog program for the operational semantics (by value and by name) of REC. Try it for the term $g(f(2))$ where f and g are defined by the following equations: $f(x) = f(x)+1$ and $g(x) = 5$.

P3. Write a Prolog program for local model checking.