Minimizing Symmetric Submodular Functions

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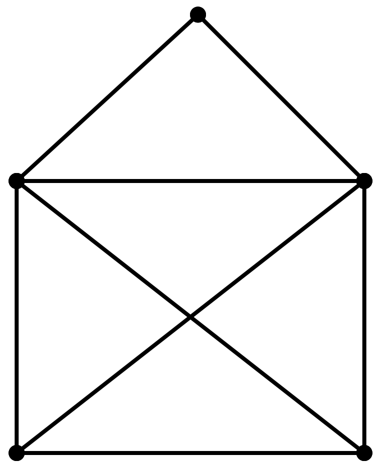
Minimax Acyclic Orientation

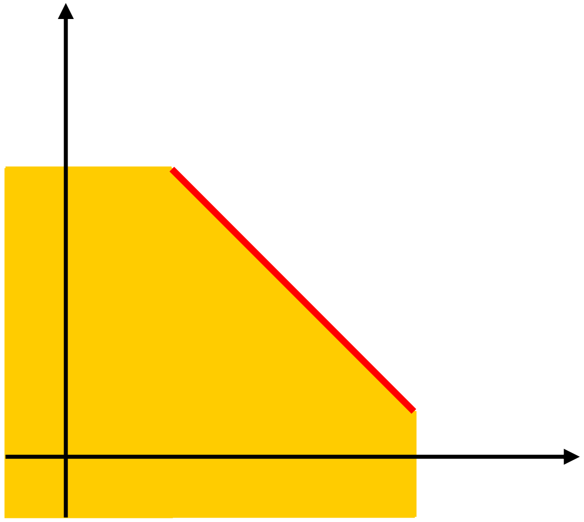
$G=(V,E)$: Graph

Find an acyclic orientation that minimizes the maximum in-degree



Minimax Extreme Base

Find an extreme point y of $B(f)$ that minimizes $\max\{y(v) \mid v \in V\}$.



Minimax Extreme Base

$S := \phi;$

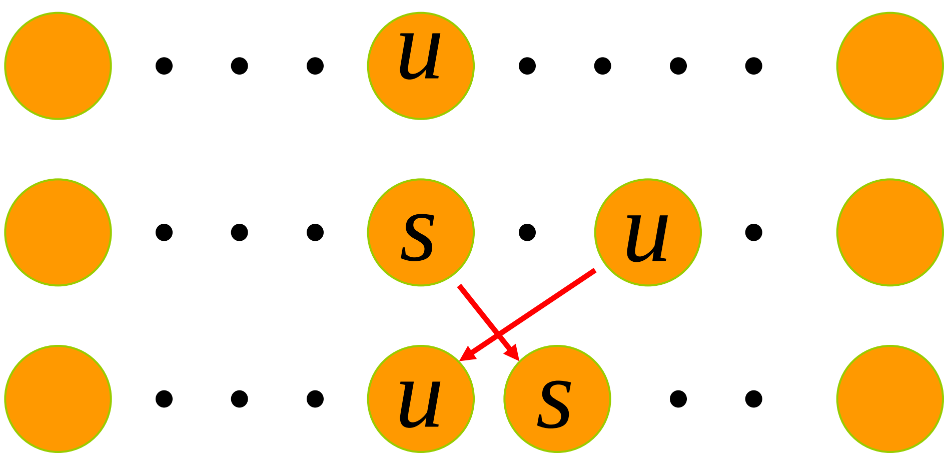
repeat

$\quad u := \arg\min\{ f(S \cup \{u\}) \mid v \in V \setminus S \};$

$\quad y(u) := f(S \cup \{u\}) - f(S);$

$\quad S := S \cup \{u\};$

untill $S = V.$



Output y

Optimal y^*

y'

$$y'(u) \le y^*(s)$$

Symmetric Submodular Functions

$$f : 2^V \to \mathbf{R}$$

Symmetric $\quad f(X) = f(V \setminus X), \quad \forall X \subseteq V.$

Crossing Submodular

$$X \cap Y \neq \phi,\ X \cup Y \neq V \Rightarrow$$
$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$$

Symmetric Submodular Function Minimization

$$\min\{ f(X) \mid \phi \neq X \subset V, X \neq V\}?$$

Maximum Adjacency Ordering

- Minimum Cut Algorithm by MA-ordering

  Nagamochi & Ibaraki (1992)
- Simpler Proofs

  Frank (1994), Stoer & Wagner (1997)
- Symmetric Submodular Functions

  Queyranne (1998)
- Alternative Proofs

  Fujishige (1998), Rizzi (2000)

Minimum Degree Ordering

Nagamochi (2007)

Finding the family of all extreme sets for symmetric crossing submodular functions in $O(n^3\gamma)$ time.

$\Downarrow$

Symmetric Submodular Function Minimization

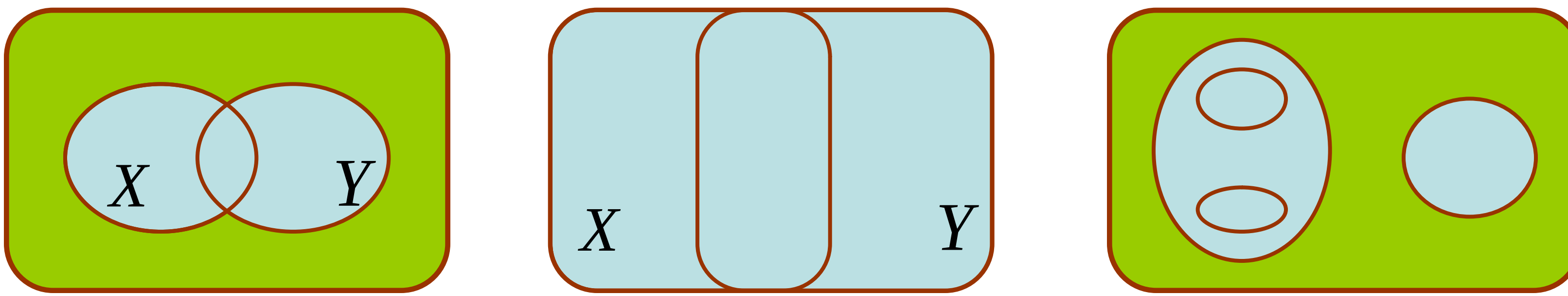
Extreme Sets

f : Symmetric Crossing Submodular Function

X : Extreme Set

$$f(Z) > f(X), \quad \forall Z \subset X : \phi \neq Z \neq X.$$

The family of all extreme sets forms a laminar.

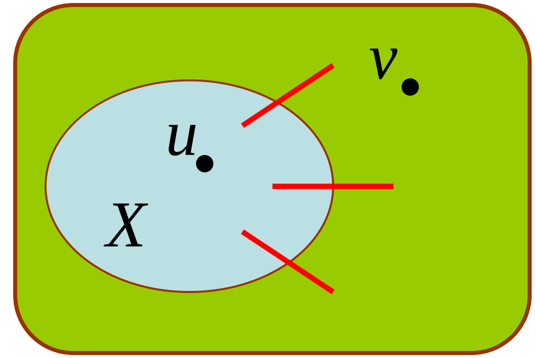


$$f(X) + f(Y) \geq f(X \setminus Y) + f(Y \setminus X)$$

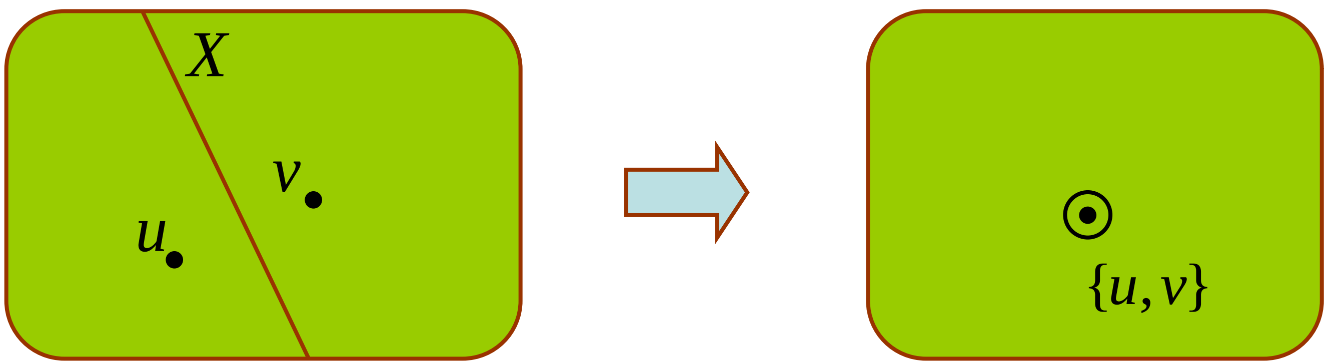
Flat Pair for Graphs

$G=(V,E)$ Undirected Graph

Flat Pair $\{u,v\}\subseteq V \quad (u\neq v)$



$$d(X)\geq \min\{d(x)\mid x\in X\},\quad \forall X\subseteq V \text{ s.t. } |X\cap\{u,v\}|=1.$$



No Extreme Sets
Separate u and v.

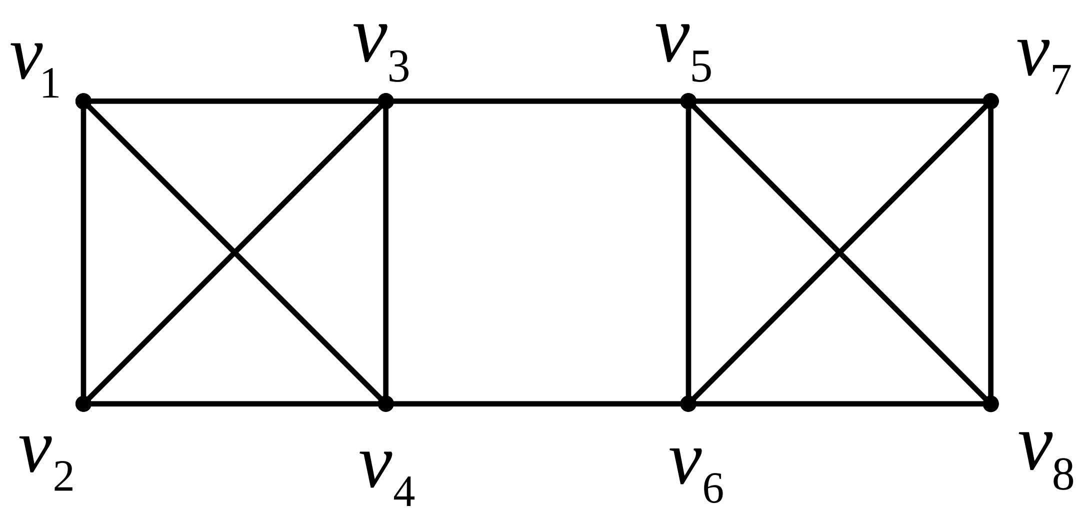
Shrink $\{u,v\}$ into
a single vertex.

Minimum Degree Ordering

MD-ordering $v_1, v_2, \ldots, v_{n-1}, v_n \in V \quad (n = |V|)$

$$V_i = \{v_1, v_2, \ldots, v_i\}$$

Each v_j has minimum degree in $G[V \setminus V_{j-1}]$.



Minimum Degree Ordering

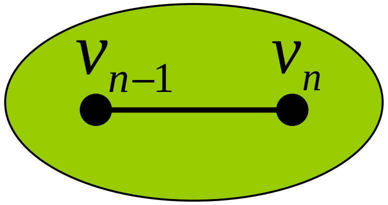
The last two vertices v_{n-1}, v_n of an MD-ordering form a flat pair.

Proof by Induction:

$\{v_{n-1}, v_n\}$: Flat Pair in $G[V \setminus V_i]$

$i = n-2, \ldots, 1, 0.$

$i = n-2$



Suppose $\{v_{n-1}, v_n\}$: flat pair in $G[V \setminus V_j]$

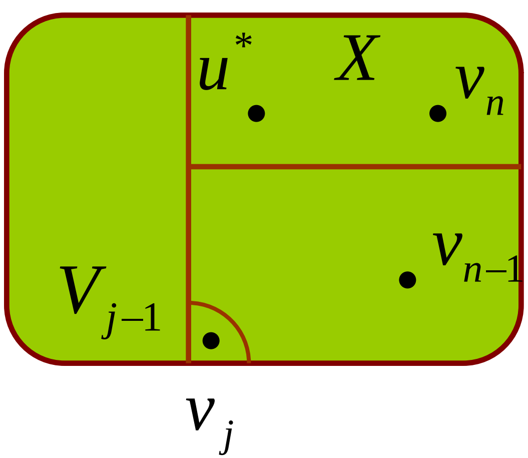
Show $\{v_{n-1}, v_n\}$: flat pair in $G[V \setminus V_{j-1}]$

d_i : cut function in $G[V \setminus V_i]$

$$\boxed{X \subseteq V \setminus V_{j-1}, \quad |X \cap \{v_{n-1}, v_n\}| = 1} \Rightarrow \boxed{d_{j-1}(X) \geq \min\{d_{j-1}(u) \mid u \in X\}.}$$

$$\boxed{v_j \notin X}$$

$$d_j(X) \geq \min\{d_j(u) \mid u \in X\} = d_j(u^*)$$



$$\begin{aligned} \underline{d_{j-1}(X)} &= d_j(X) + d_{j-1}(v_j, X) \\ &\geq d_j(u^*) + d_{j-1}(v_j, u^*) \\ &= \underline{d_{j-1}(u^*)} \end{aligned}$$

Suppose $\{v_{n-1}, v_n\}$: flat pair in $G[V \setminus V_j]$

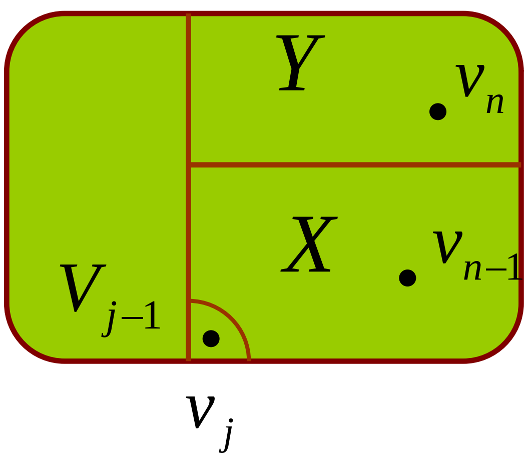
Show $\{v_{n-1}, v_n\}$: flat pair in $G[V \setminus V_{j-1}]$

d_i : cut function in $G[V \setminus V_i]$

$X \subseteq V \setminus V_{j-1},$
$|X \cap \{v_{n-1}, v_n\}| = 1$ $\Rightarrow$ $d_{j-1}(X) \geq \min\{d_{j-1}(u) \mid u \in X\}.$

$v_j \in X$

$Y := (V \setminus V_{j-1}) \setminus X$



$$\underline{d_{j-1}(X)} = d_{j-1}(Y)$$
$$\geq \min\{d_{j-1}(u) \mid u \in Y\}$$
$$\geq \underline{d_{j-1}(v_j)}$$

Time Complexity

- Finding an MD-ordering in $O(m)$ time.
  For weighted graphs: $O(m + n\log n)$ time.

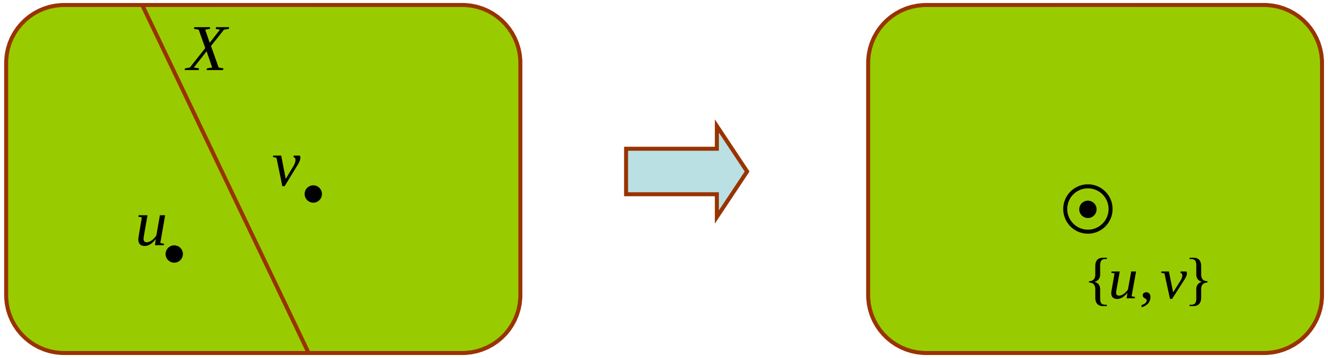
- Finding all the extreme sets in $O(nm)$ time.
  For weighted graphs: $O(nm + n^2 \log n)$ time.

Flat Pair for Symmetric Submodular Functions

Flat Pair $\{u,v\} \subseteq V \quad (u \neq v)$

$$f(X) \geq \min\{f(x) \mid x \in X\},$$

$$\forall X \subseteq V \text{ s.t.} |X \cap \{u,v\}| = 1.$$

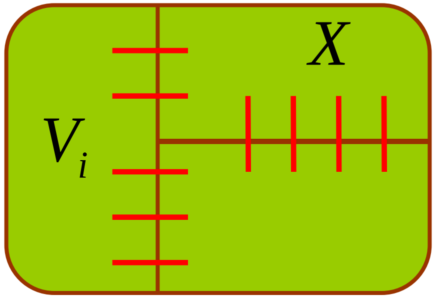


No Extreme Sets Separate u and v.

Shrink $\{u,v\}$ into a single vertex.

MD-Ordering for Symmetric Submodular Functions

Cut Function in $G[V \setminus V_i]$



$$d_i(X) = \frac{d(X) + d(V_i \cup X) - d(V_i)}{2}$$

$$f_i(X) := f(X) + f(V_i \cup X) \qquad (X \subseteq V \setminus V_i)$$

Symmetric, Crossing Submodular

MD-ordering $v_1, v_2, \ldots, v_{n-1}, v_n \in V$

Each v_j has minimum value of $f_{j-1}(v)$ among $v \in V \setminus V_{j-1}$.

MD-Ordering for Symmetric Submodular Functions

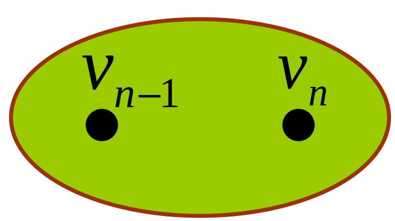
The last two vertices v_{n-1}, v_n of an MD-ordering form a flat pair.

Proof by Induction:

$\{v_{n-1}, v_n\}$: Flat Pair for f_i on $V \setminus V_i$

$i = n-2, \ldots, 1, 0.$

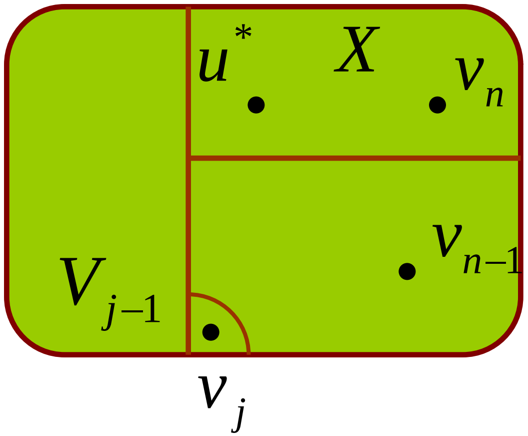
$i = n-2$



$\{v_{n-1}, v_n\}$: flat pair for f_j ➡ flat pair in f_{j-1}

$$X \subseteq V \setminus V_{j-1}$$

$v_j \notin X$



$$f_j(X) \geq \min\{f_j(u) \mid u \in X\} = f_j(u^*)$$

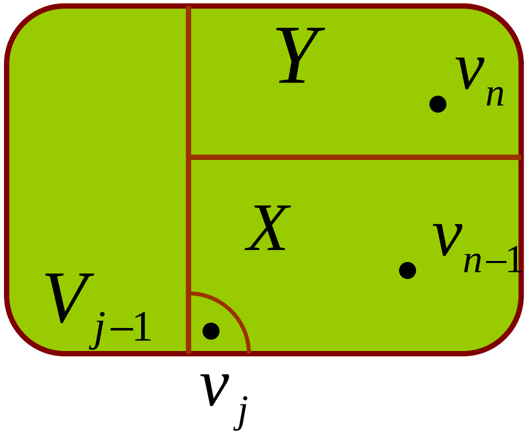
$$\begin{aligned} f_{j-1}(X) - f_{j-1}(u^*) &= f(X) + f(V_{j-1} \cup X) - f(V_{j-1} \cup \{u^*\}) - f(u^*) \\ &\geq f(X) + f(V_j \cup X) - f(V_j \cup \{u^*\}) - f(u^*) \\ &= f_j(X) - f_j(u^*) \geq 0 \end{aligned}$$

$\{v_{n-1}, v_n\}$: flat pair for f_j ➡ flat pair in f_{j-1}

$X \subseteq V \setminus V_{j-1}$

$v_j \in X$

$Y := (V \setminus V_{j-1}) \setminus X$



$$\begin{aligned} f_{j-1}(X) &= f_{j-1}(Y) \\ &\geq \min\{ f_{j-1}(u) \mid u \in Y \} \\ &\geq f_{j-1}(v_j) \end{aligned}$$

Time Complexity

- Finding an MD-ordering in $O(n^2\gamma)$ time.
- Finding all the extreme sets in $O(n^3\gamma)$ time.
- Minimizing symmetric crossing submodular functions in $O(n^3\gamma)$ time.

Conclusion

Minimum Degree Ordering

→ Minimax In-degree Orientation,

Minimax Extreme Base

→ Extreme Sets

Symmetric Submodular Function

Minimization