Applications of Submodular Function Minimization

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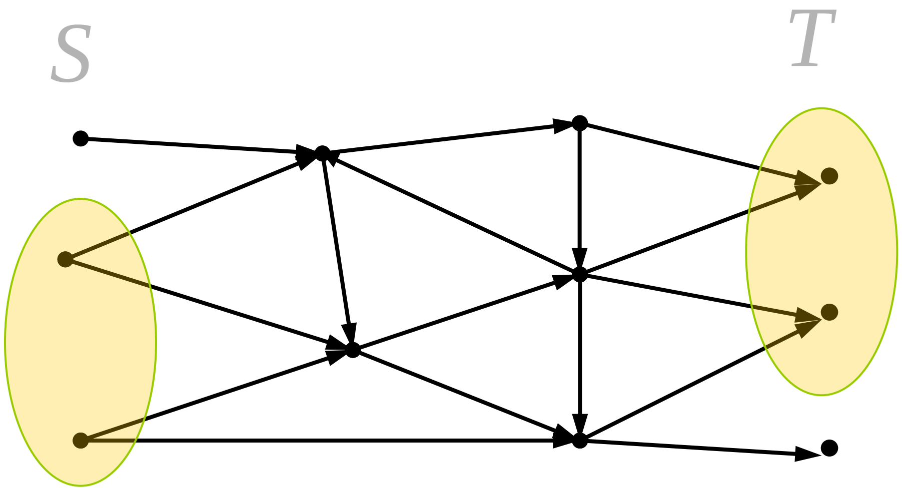
Applications

- Evacuation Problem
- Multi-terminal Source Coding
- Multi-class Queueing System
- Valued Constraint Satisfaction Problem
- Potts Model (Anglès d'Auriac, Iglói, Preissmann, Sebő [2002])
- Testing Branch-width (Oum & Seymour [2007])

Evacuation Problem（Dynamic Flow）



Hoppe, Tardos (2000)

$c(a)$: Capacity

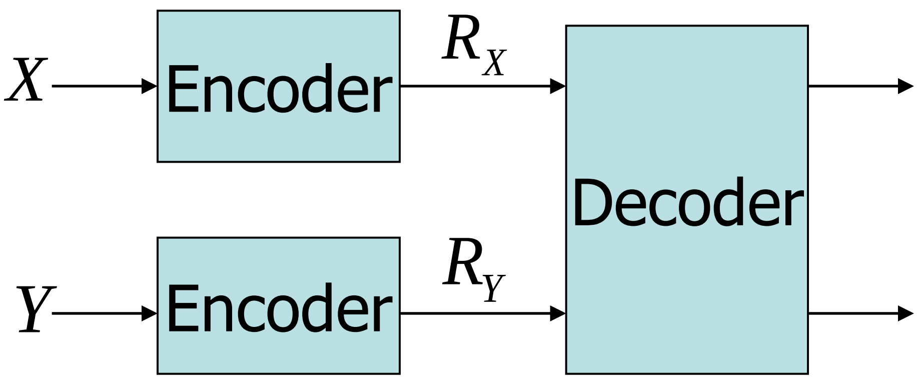
$\tau(a)$: Transit Time

$b(v)$: Supply/Demand

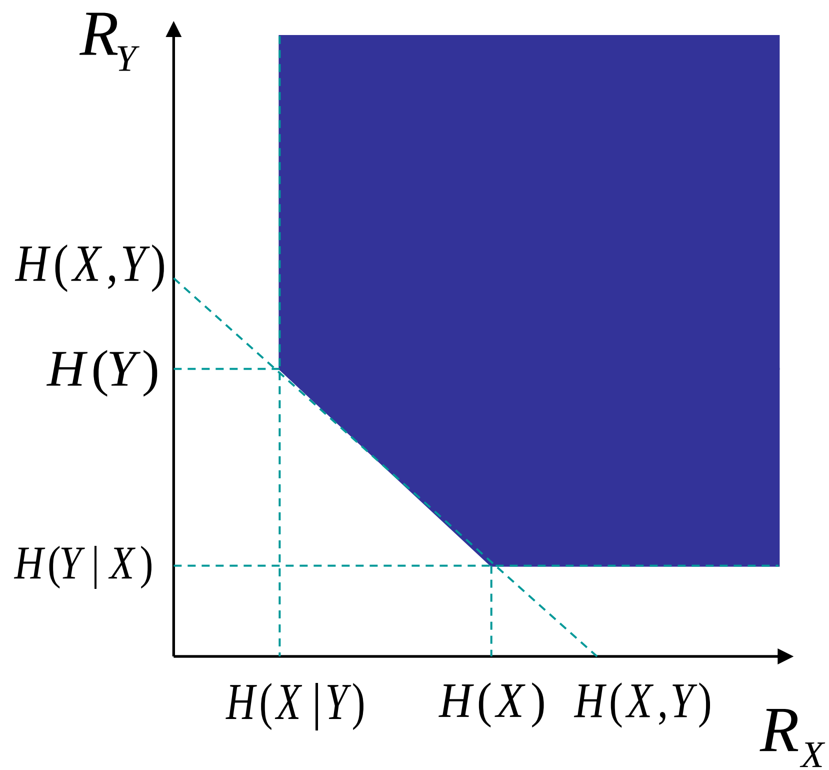
$o(X)$: Maximum Amount of Flow from $X \cap S$ to $T \setminus X$.

Feasible $\Longleftrightarrow$ $b(X) \leq o(X), \forall X \subseteq S \cup T$

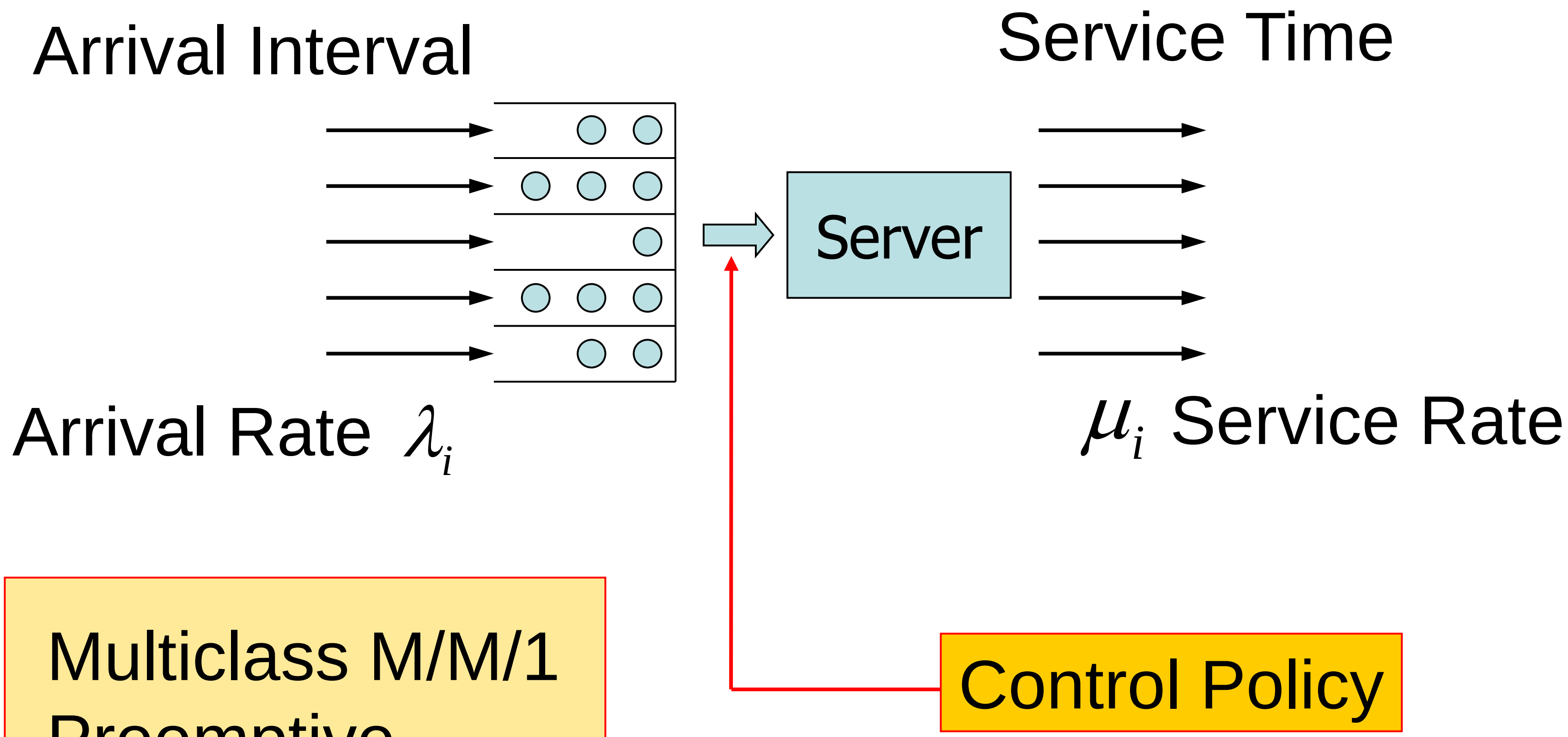
Multiterminal Source Coding



Slepian, Wolf (1973)



Multiclass Queueing Systems



Multiclass M/M/1
Preemptive

Performance Region

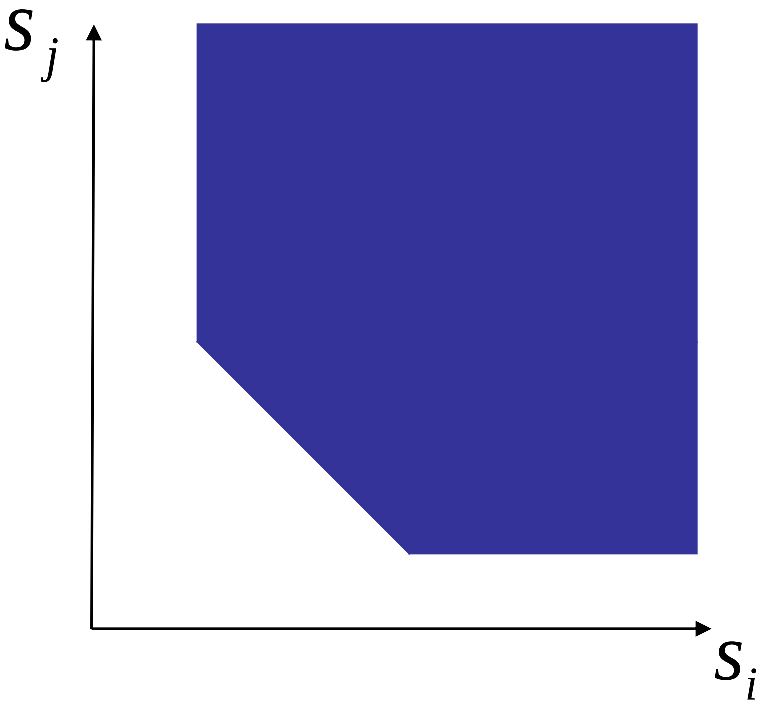
s_j : Expected Staying Time of a Job in *j*

$\rho_i := \lambda_i / \mu_i, \quad \sum_{i \in V} \rho_i < 1$

S : Achievable

$$\sum_{i \in X} \rho_i S_i \geq \frac{\sum_{i \in X} \rho_i / \mu_i}{1 - \sum_{i \in X} \rho_i}, \forall X \subseteq V$$

Coffman, Mitrani (1980)



A Class of Submodular Functions

$x, y, z \in \mathrm{R}_{+}^{V}$ Itoko & I. (2005)

$h:$ Nonnegative, Nondecreasing, Convex

$$\boxed{f(X) = z(X) - y(X)h(x(X))} \quad (X \subseteq V)$$

Submodular

$$\sum_{i \in X} \rho_i S_i \geq \frac{\sum_{i \in X} \rho_i / \mu_i}{1 - \sum_{i \in X} \rho_i}, \forall X \subseteq V$$

$$z_i := \rho_i S_i \qquad y_i := \frac{\rho_i}{\mu_i}$$

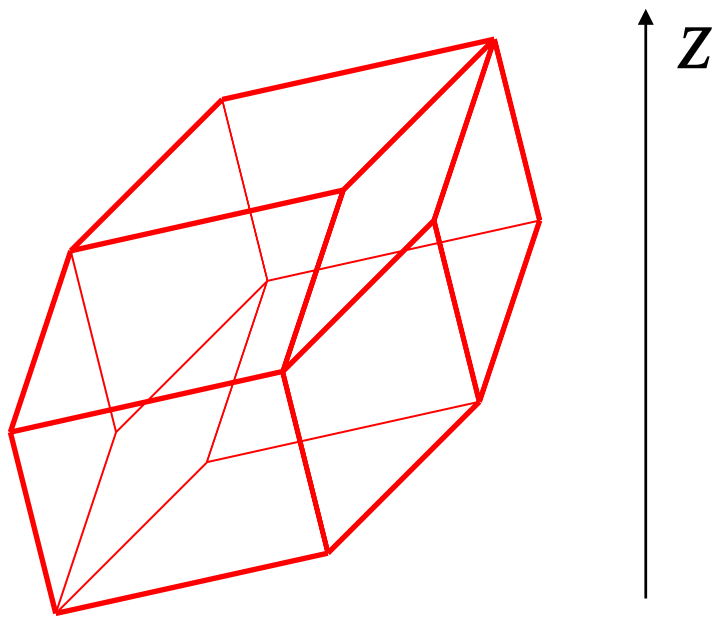
$$x_i := \rho_i \qquad h(x) := \frac{1}{1-x}$$

Zonotope in 3D

$$w(X) = (x(X), y(X), z(X))$$

$$Z = \mathrm{conv}\{w(X) \mid X \subseteq V\}$$

Zonotope

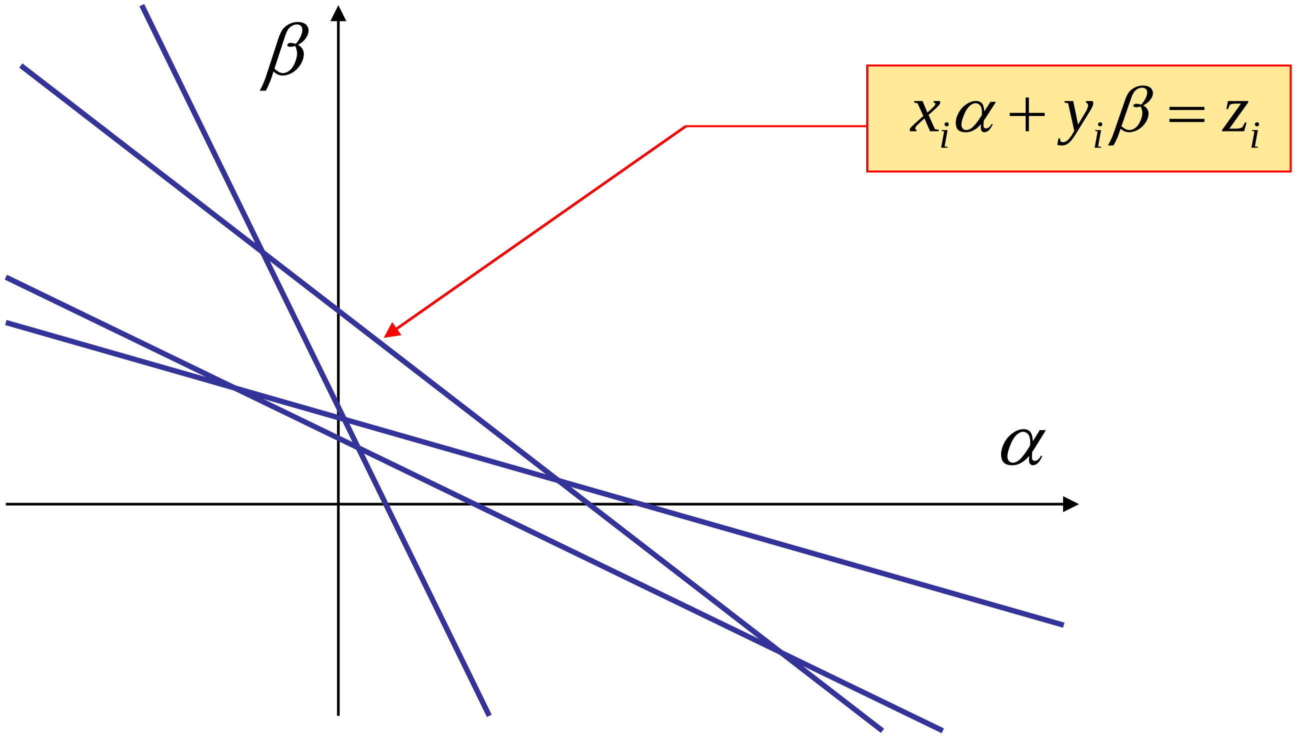


$$\tilde{f}(x, y, z) = z - yh(x)$$

$$\min\{f(X) \mid X \subseteq V\}$$
$$= \min\{\tilde{f}(x, y, z) \mid (x, y, z) : \text{Lower Extreme Point of } Z\}$$

Remark: $\tilde{f}(x, y, z)$ is NOT concave!

Line Arrangement



Enumerating All the Cells

Topological Sweeping Method
Edelsbrunner, Guibas (1989)

$\rightarrow O(n^2)$

Summary

- Submodular Functions Arise Everywhere.
- Discrete Analogue of Convexity.
- General SFM Algorithms Available.
- Exploit Special Structures of Problems.

Pointers

- S. Fujishige: *Submodular Functions and Optimization*, Elsevier, 2005.

- S. T. McCormick: Submodular Function Minimization, *Discrete Optimization*, K. Aardal, G. Nemhauser, R. Weismantel, eds., Handbooks in Operations Research, Elsevier, 2005.