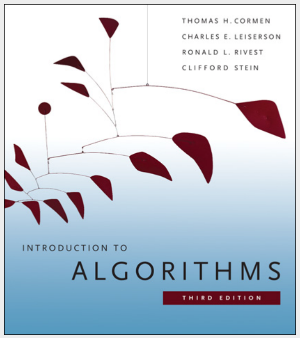
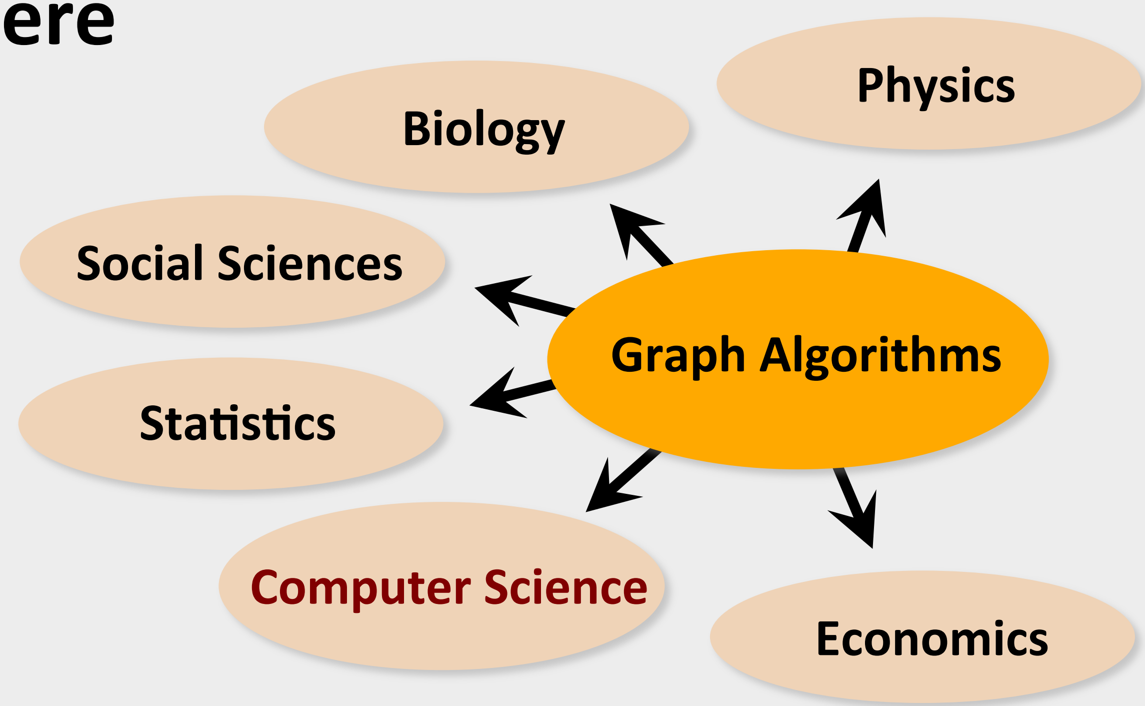
Graphs, Linear Algebra, and Continuous Optimization

Part I: Overview

Aleksander Mądry

Graphs are **everywhere**

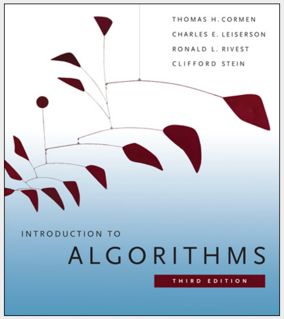




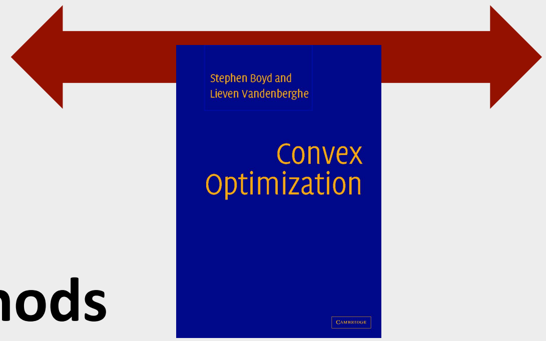
Algorithmic Graph Theory:
Shaping our understanding
of algorithms since 1950s

But: Our graph toolkit is still far from being complete

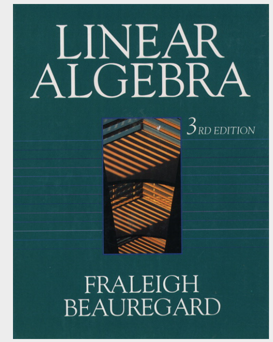
Our goal: Forging the next generation of tools for graph algorithms



Combinatorial methods
(trees, paths, partitions, matchings, routings,...)



Cont. opt. primitives
(gradient-descent, interior-point methods,...)



Linear-algebraic tools
(eigenvalues, electrical flows, linear systems,...)

Underlying theme: Merging combinatorial and continuous methods

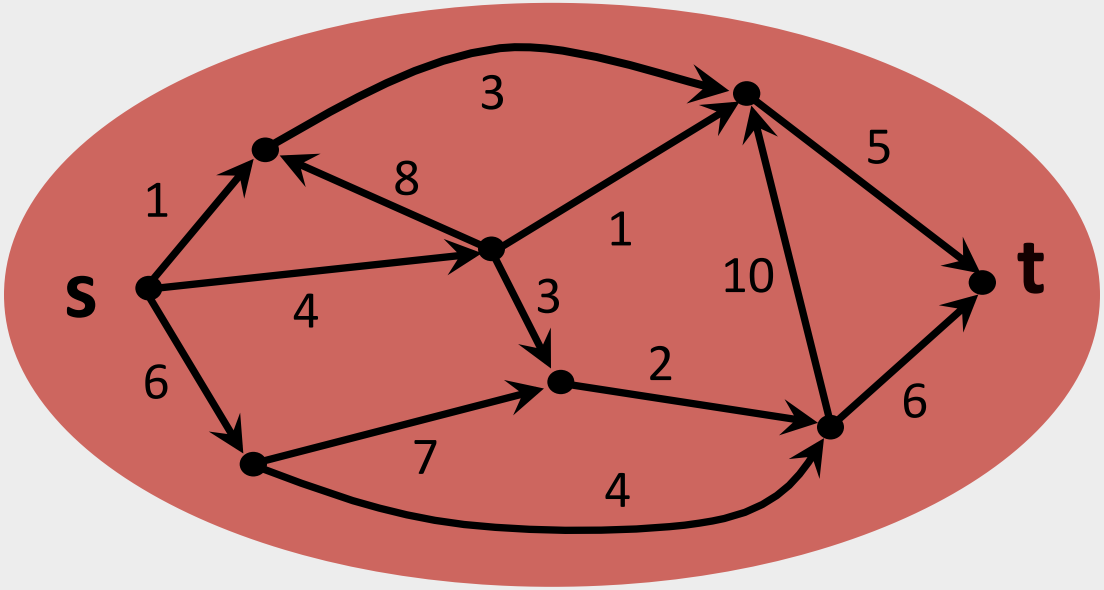
Our plan for this week:
Illustrate this theme on an example of a single problem

Problem: Maximum flow

Underlying approach:
Relate combinatorial structure of a graph to linear-algebraic properties of associated matrices

Maximum flow problem

Input: Directed graph **G**, integer **capacities u_e**, **source s** and **sink t**



Think: arcs = roads
capacities = # of lanes
s/t = origin/destination

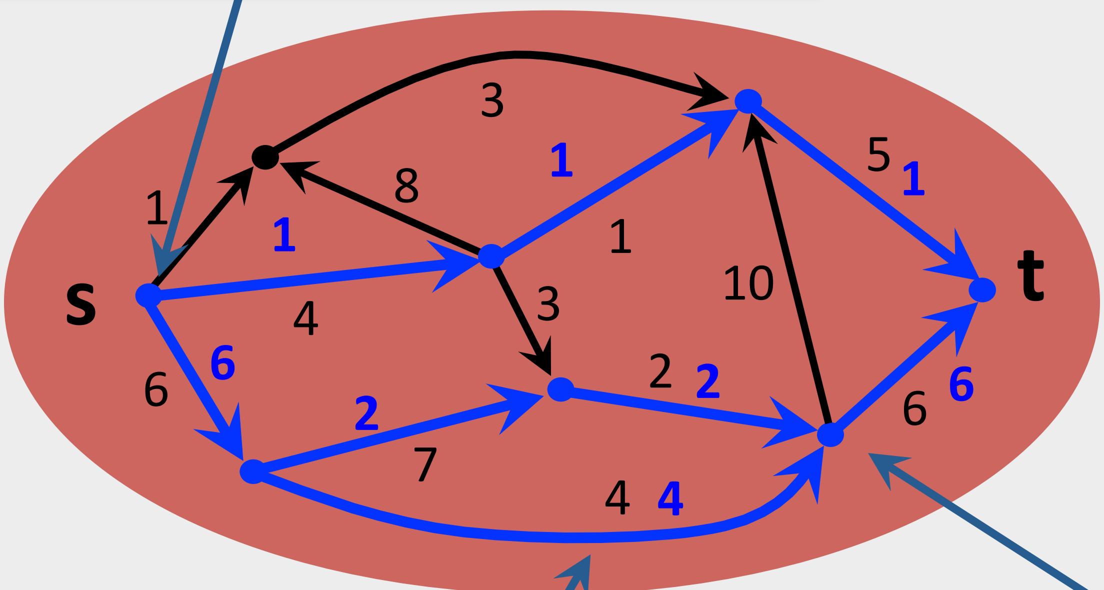
Task: Find a **feasible s-t flow** of **max value**

(**Think:** Estimate the **max** possible rate of traffic from **s** to **t**)

Maximum flow problem

Input: Directed graph **G**, integer **capacities u_e**, **source s** and **sink t**

value = net flow out of **s**



Think: arcs = roads
capacities = # of lanes
s/t = origin/destination

Max flow value
F*=10

no overflow on arcs:
$0 \le f(e) \le u(e)$

no leaks at all **v≠s,t**

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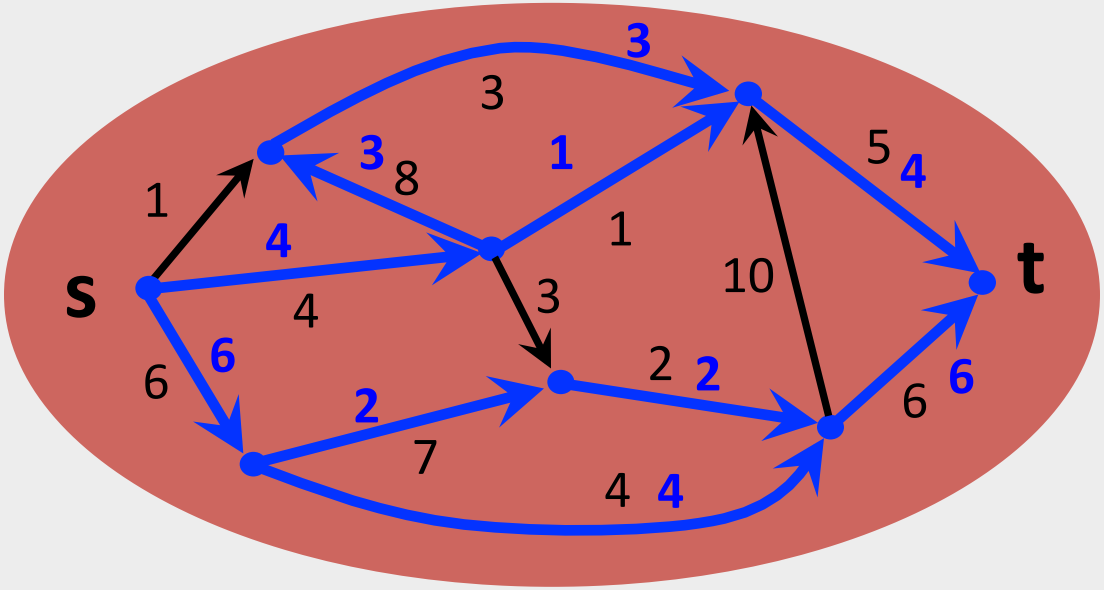

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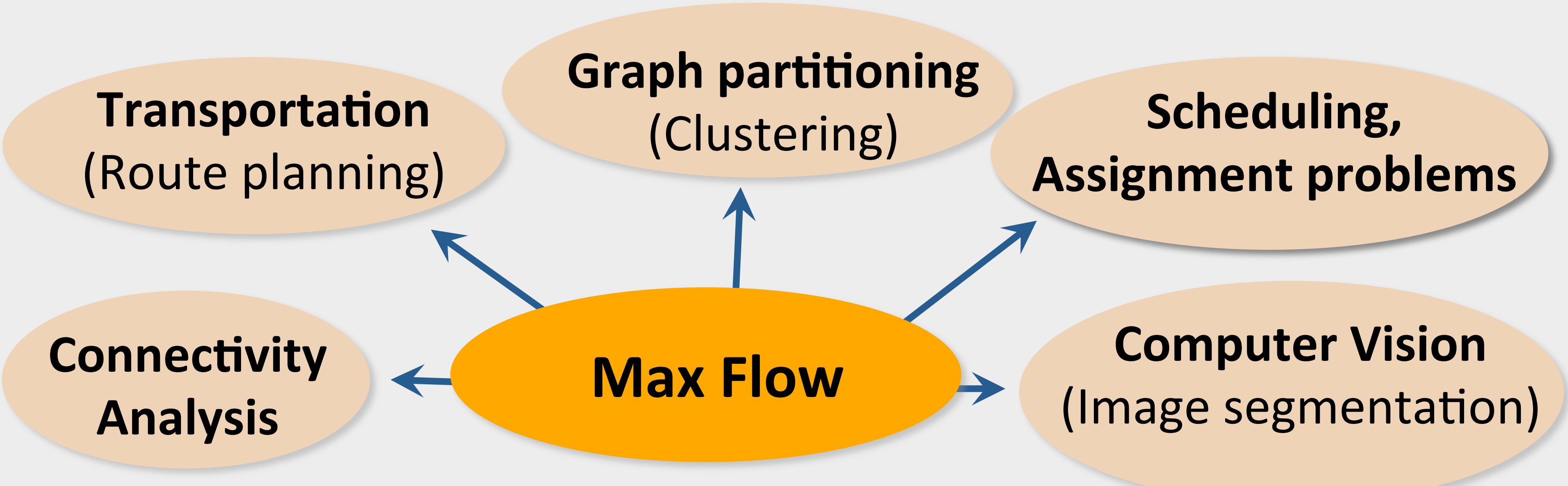
Why is this a good problem to study?

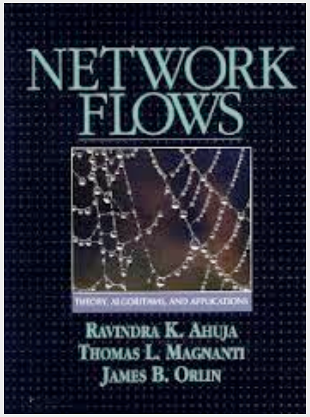
Max flow is a fundamental optimization problem

- **Extensively studied since 1930s** (classic 'textbook problem')
- **Surprisingly diverse set of applications**
- **Very influential in development of (graph) algorithms**



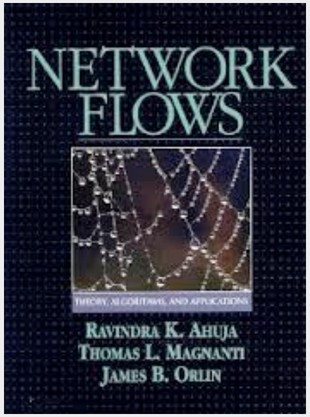
What is known about Max Flow?

A **LOT** of previous work



What is known about Max Flow?

A (very) rough history outline



[Dantzig ‘51]	$O(mn^2 U)$
[Ford Fulkerson ’56]	$O(mn U)$
[Dinitz ’70]	$O(mn^2)$
[Dinitz ‘70] [Edmonds Karp ’72]	$O(m^2 n)$
[Dinitz ‘73] [Edmonds Karp ’72]	$O(m^2 \log U)$
[Dinitz ‘73] [Gabow ’85]	$O(mn \log U)$
[Goldberg Rao ’98]	$\tilde{O}(m \min(m^{1/2}, n^{2/3}) \log U)$
[Lee Sidford ’14]	$\tilde{O}(mn^{1/2} \log U)$

Our focus: Sparse graph (**m=O(n)**) and unit-capacity (**U=1**) regime

→ It is a good benchmark for combinatorial graph algorithms

→ Already captures interesting problems, e.g., **bipartite matching**

(**n** = # of vertices, **m** = # of arcs, **U** = max capacity, **Õ()** hides polylogs)

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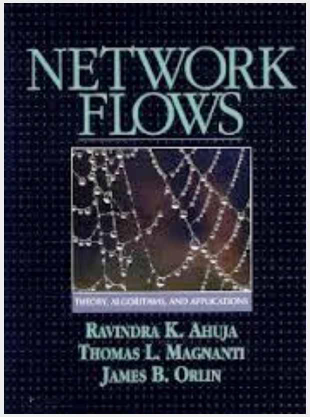

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What is known about Max Flow?

Emerging barrier: $O(n^{3/2})$

[Even Tarjan '75, Karzanov '73]: Achieved this bound for **U=1** long time ago

Last 40 years: Matching this bound in increasingly more general settings, but **no improvement**

This indicates a fundamental limitation of our techniques

Our goal: Show a new approach finally breaking this barrier

(**n** = # of vertices, **m** = # of arcs, **U** = max capacity, **Õ()** hides polylogs)

Breaking the $\Omega(n^{3/2})$ barrier

Undirected graphs and approx. answers ($\Omega(n^{3/2})$ barrier still holds here)

[M '10]: **Crude approx. of** max flow **value** in **close to linear** time

[CKMST '11]: **(1-ε)-approx.** to max flow in $\tilde{O}(n^{4/3}\varepsilon^{-3})$ time

[LSR '13, S '13, KLOS '14, P '14]: **(1-ε)-approx.** in $\tilde{O}(n\varepsilon^{-2})$ time

But: What about the **directed** and **exact** setting?

[M '13]: Exact $\tilde{O}(n^{10/7})=\tilde{O}(n^{1.43})$-time alg.

This week

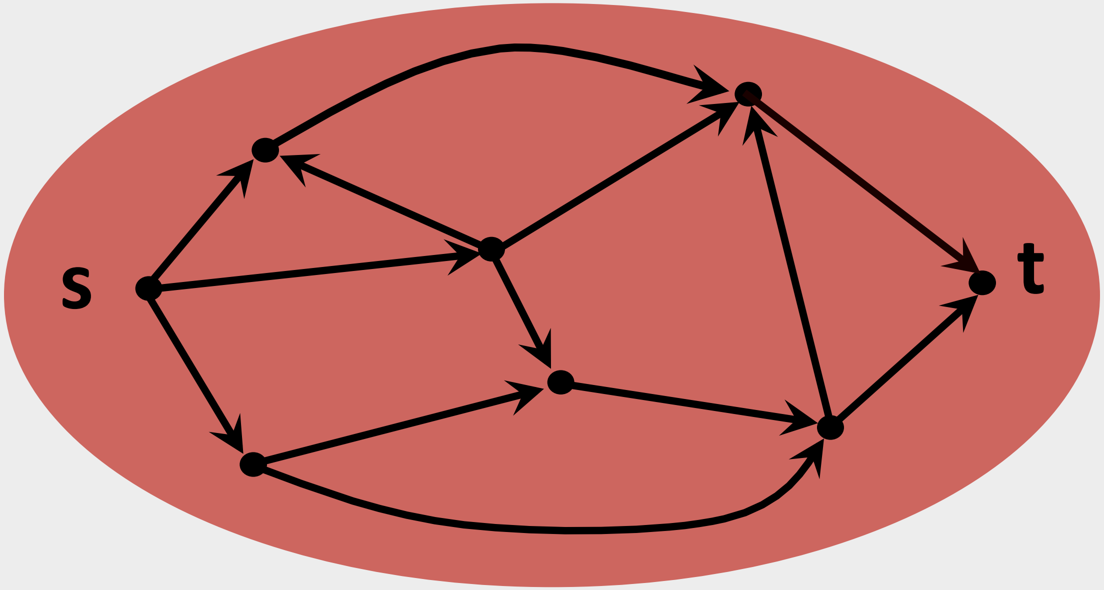
(**n** = # of vertices, $\tilde{O}()$ hides polylog factors)

Previous approach

Augmenting paths framework

[Ford Fulkerson '56]

Basic idea: Repeatedly find **s-t paths** in the **residual graph**



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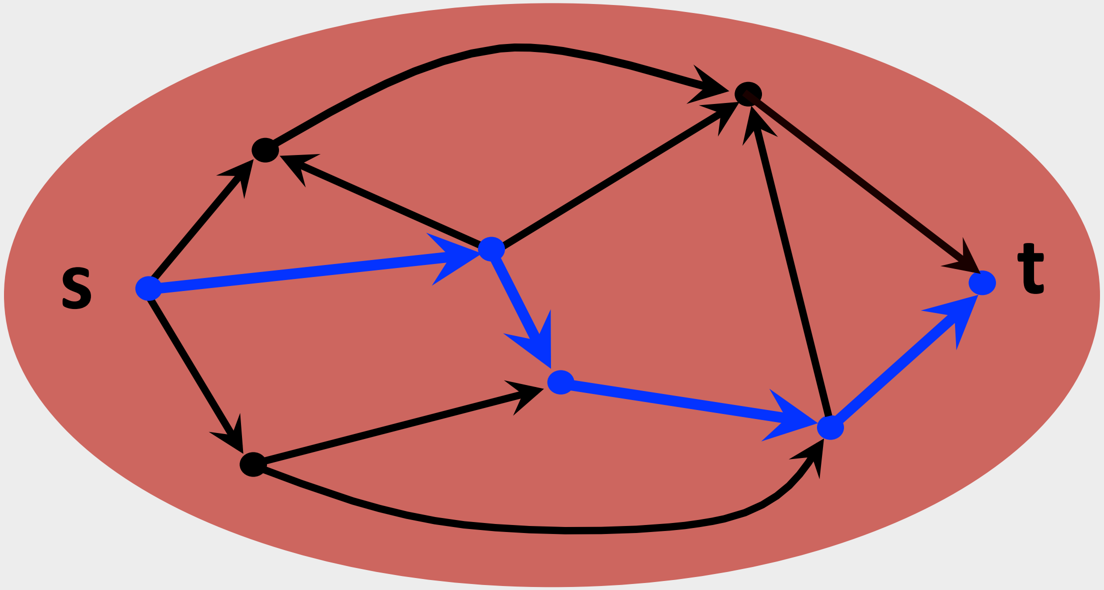

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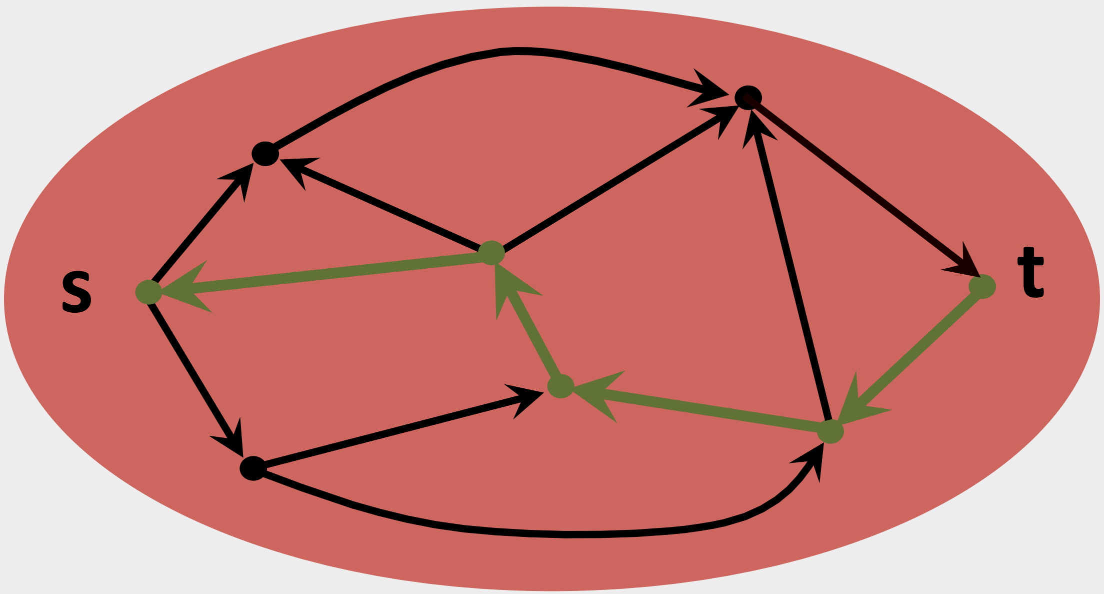

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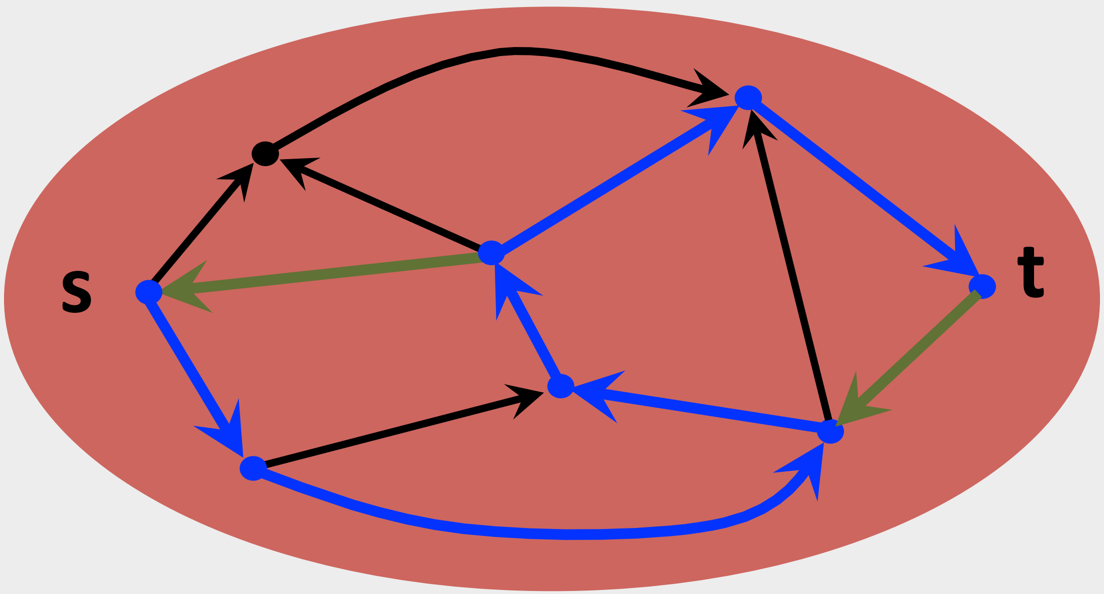

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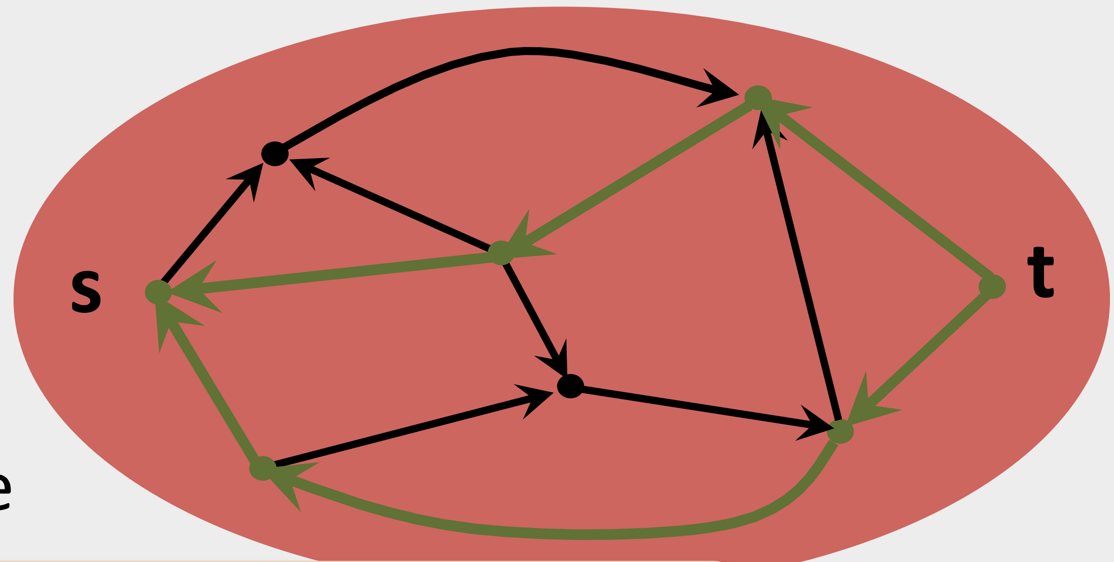
Basic idea: Repeatedly find **s-t paths** in the **residual graph**

Advantage: Simple, purely combinatorial and greedy (flow is built path-by-path)



Problem:
Very difficult to analyze

Naïve impl

(≤ **n** augme path)

Unclear how to get
a further speed-up via this route

Sophisticated implementation and arguments:

$O(n^{3/2})$ time **[Karzanov ‘73] [Even Tarjan ‘75]**

Beyond augmenting paths

New approach:
Bring linear-algebraic techniques into play

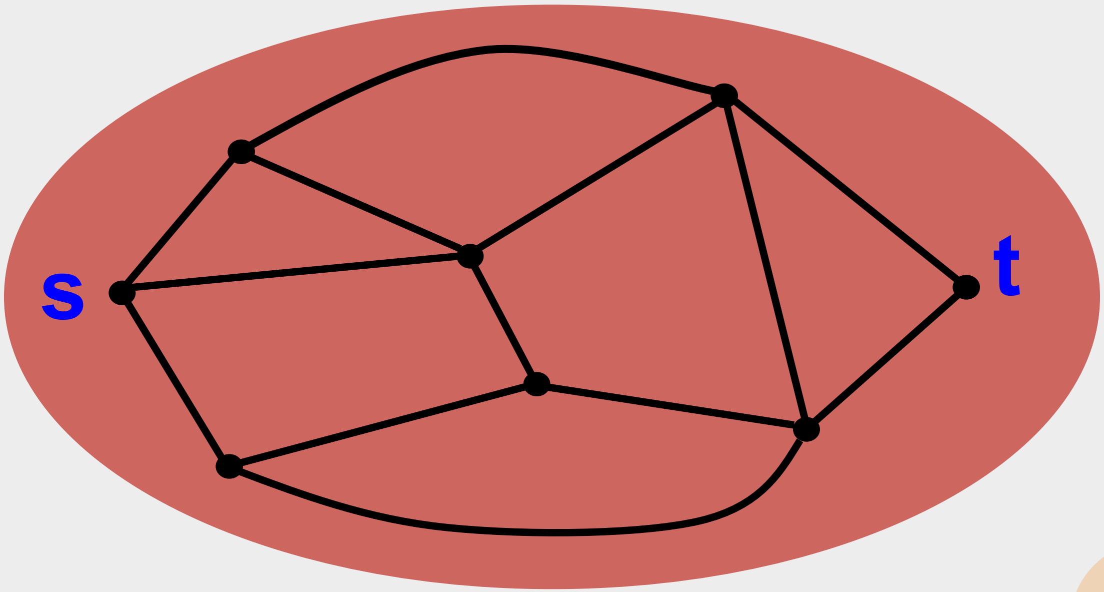
Idea: Probe the **global flow structure** of the graph by **solving linear systems**

How to relate **flow structure** to **linear algebra**?
(And why should it even help?)

Key object: Electrical flows

Electrical flows (Take I)

Input: **Undirected** graph **G**, **resistances** r_e, **source s** and **sink t**



Recipe for elec. flow:

1) Treat edges as **resistors**

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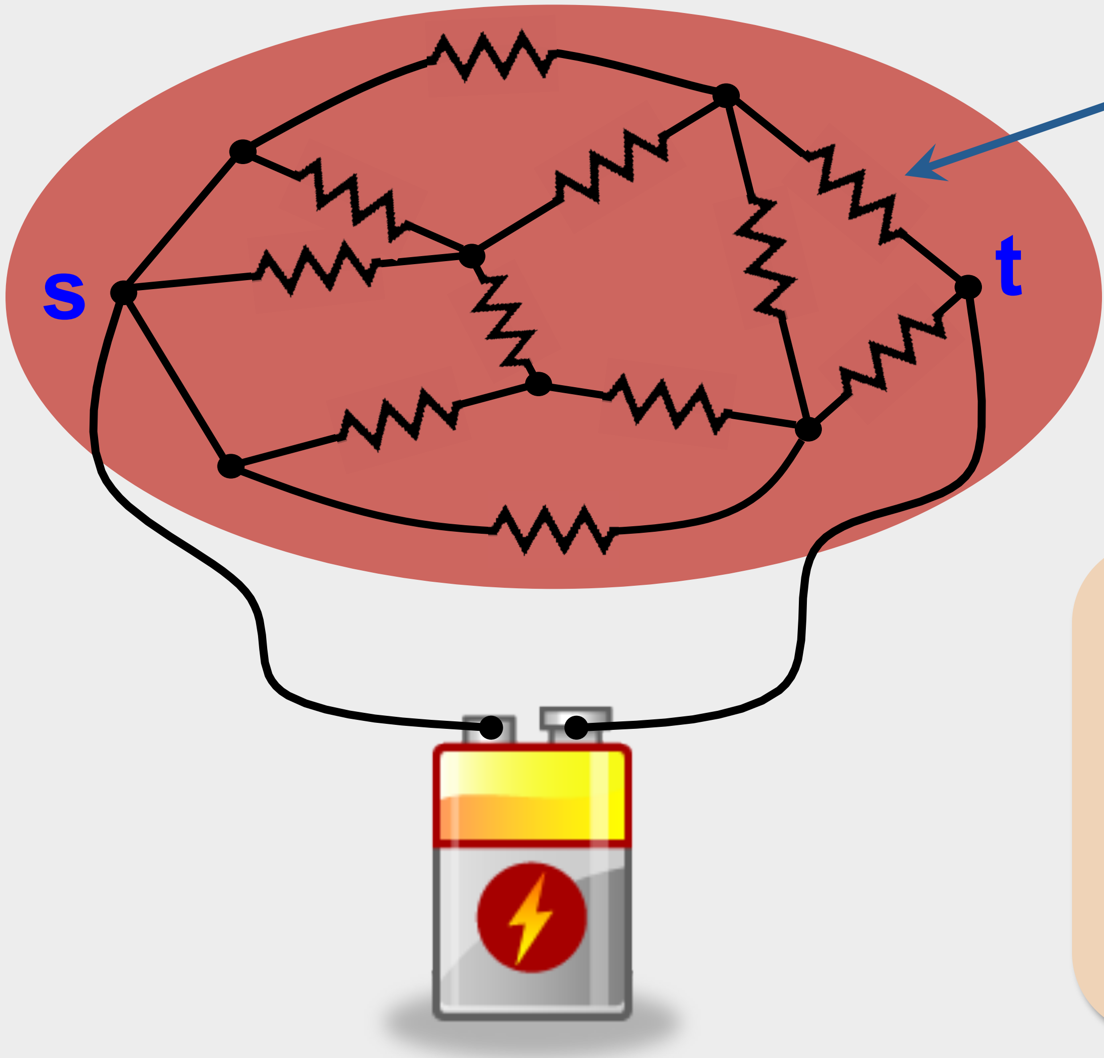

resistance r_e

Recipe for elec. flow:

1) Treat edges as **resistors**

2) Connect a **battery** to **s** and **t**

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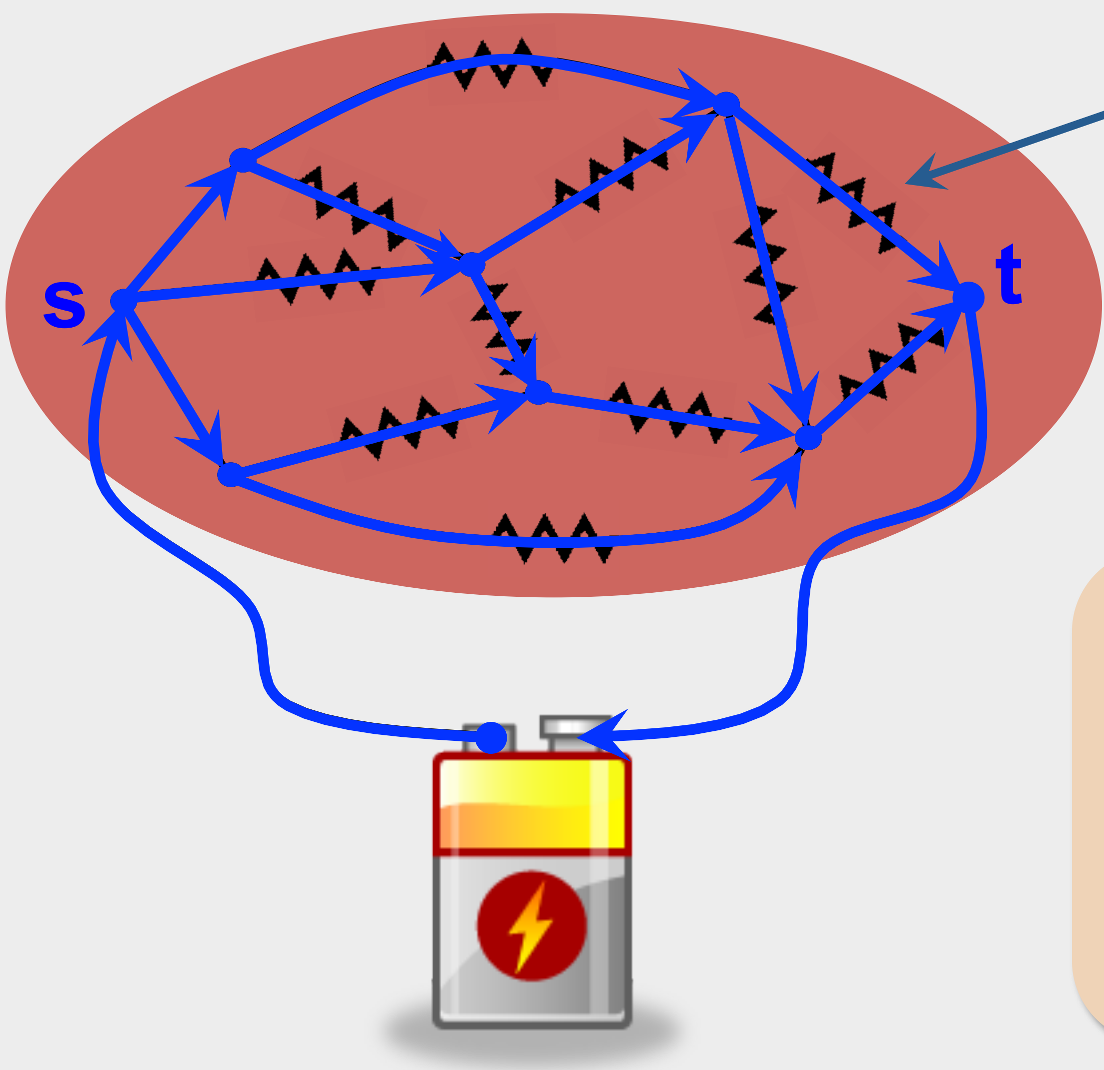


resistance r_e

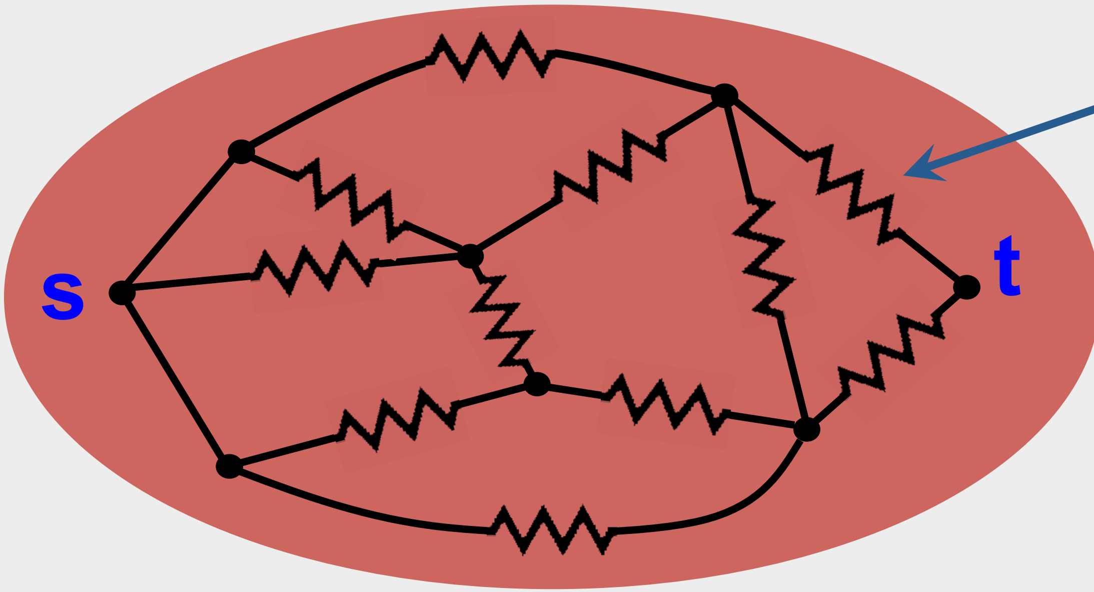
Recipe for elec. flow:

1) Treat edges as **resistors**

2) Connect a **battery** to **s** and **t**

Electrical flows (Take II)

Input: **Undirected** graph **G**, **resistances** r_e, **source s** and **sink t**



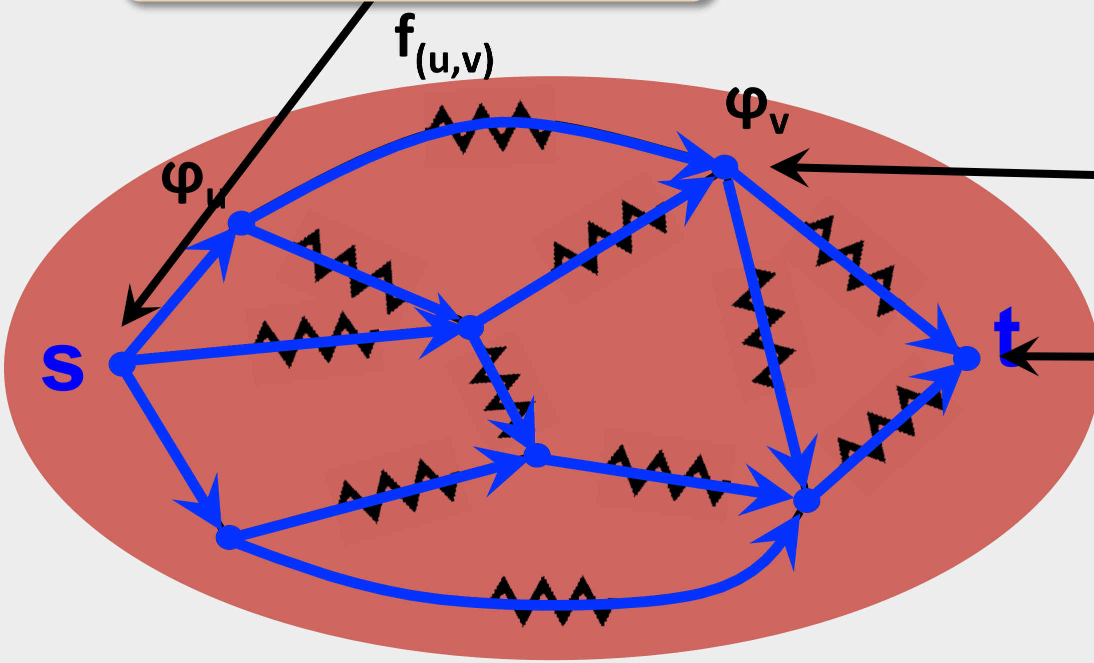
resistance r_e

(Another) recipe for electrical flow (of value F):

Ele… ake II)

Input: Undirected graph **G**, **resistances** r_e, **source s** and **sink t**

deficit of **F** at **s**



no leaks at all **v≠s,t**

excess of **F** at **t**

(Another) recipe for electrical flow (of value F):

Find **vertex potentials** φ_v such that setting, for all **(u,v)**

$$f_{(u,v)} \leftarrow (\varphi_v - \varphi_u)/r_{(u,v)} \qquad \textbf{(Ohm's law)}$$

gives a **valid s-t flow** of **value F**

Electrical flows (Take III)

Input: **Undirected** graph **G**, **resistances** r_e, **source s** and **sink t**

Principle of least energy

Electrical flow of value F:
The unique minimizer of the **energy**

$$\mathbf{E}(f) = \Sigma_e\, r_e\, f(e)^2$$

among all **s-t** flows **f** of value **F**

Electrical flows = ℓ_2-minimization

How to compute an electrical flow?

Input: Graph **G=(V,E)**, **resistances r_e**, **source s** and **sink t, value F=1**

Solve a linear system!

Wlog as elect. flow are invariant under scaling

How to compute an electrical flow?

Input: Graph **G=(V,E)**, **resistances r_e**, **source s** and **sink t, value F=1**

Solve a linear system!

Observe: It suffices to compute **vertex potentials φ_v**

Ohm's law: If **φ** is an (**|V|-dim**) vector of **vertex potentials** then

$$f = R^{-1}B^T \varphi$$

is the corresponding flow

Here:

→ **f** is an **|E|**-dim vector with **$|f_e|$** giving the amount of flow on **e** and **sign(f_e)** encoding its direction (wrt edge orientation)

→ **R** is an **|E|×|E| diagonal** matrix with $R_{ee} = r_e$

→ **B** is an **|V|×|E|** matrix with **e-th** column, for **e=(v,u)**, having **-1** (resp. **+1**) at its **v-th** (resp. **u-th**) coordinate and **0** everywhere else

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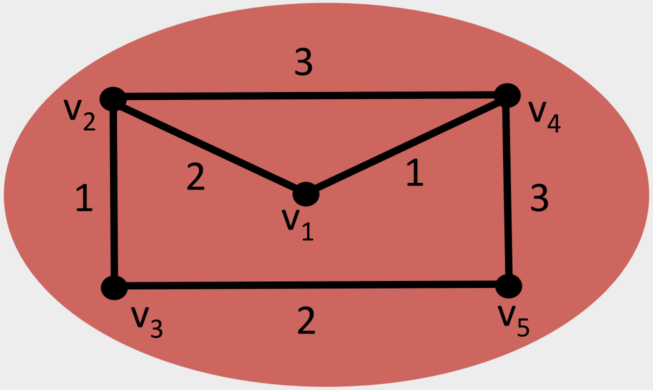
Example:



$$B = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

|V|=5, |E|=6, all edges oriented **(v_i,v_j)** with **i<j**

$$R = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

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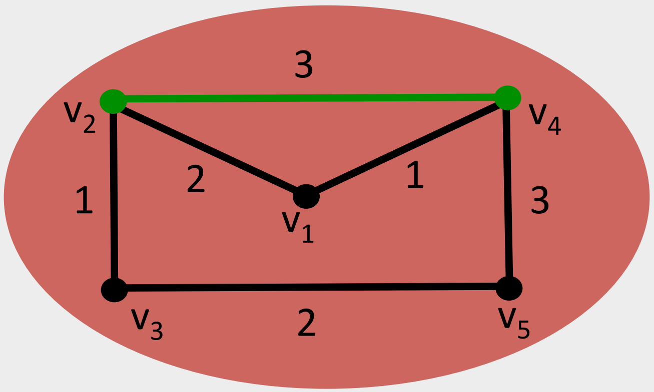

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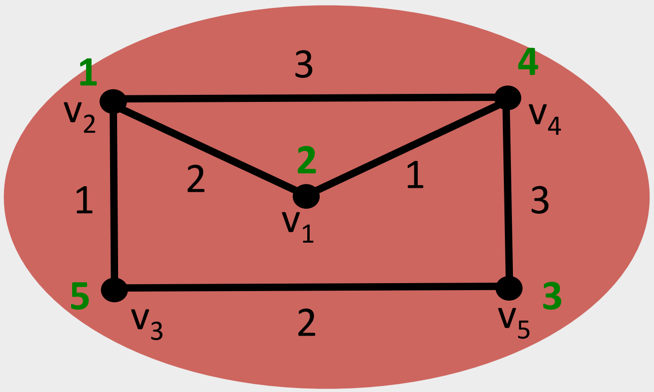

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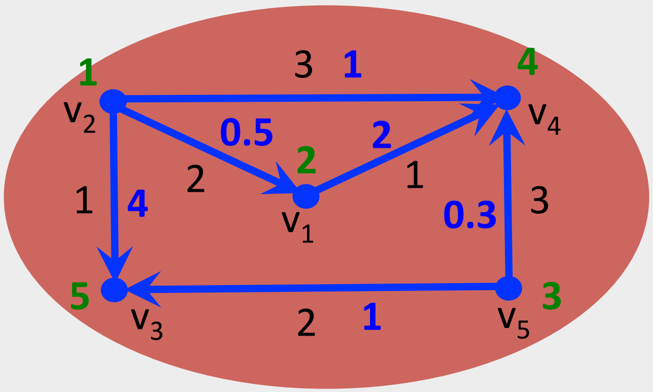

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Recall: **φ** induces an electrical flow **f** iff

f is a valid **s-t** flow
(i.e., satisfies flow conservation constraints)

Equivalently: **φ** induces an electrical flow **f** iff

$$\mathbf{B}\,\mathbf{f} = \boldsymbol{\chi}_{s,t}$$

where $\boldsymbol{\chi}_{s,t}$ has a **1** at **t**, **-1** at **s** and **0**s everywhere else

Note: $(\mathbf{Bf})_v$ is the excess/deficit of **f** at **v**

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Putting it together: **φ** induces an electrical flow iff

$$B\,R^{-1}B^{T}\varphi = \chi_{s,t}$$

How to compute an electrical flow?

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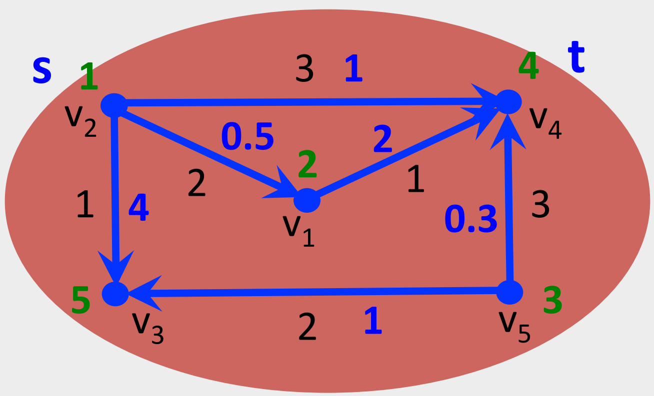

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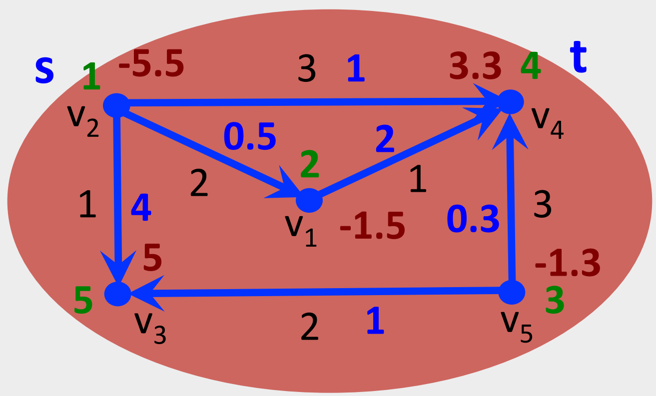

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✗

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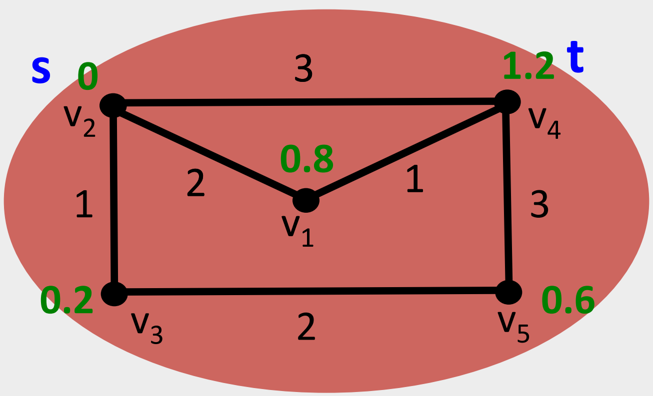
$$B\,R^{-1}B^{T}\,\varphi = \chi_{s,t}$$

Example:



$$B = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$|V|=5$, $|E|=6$, all edges oriented **(v_i,v_j)** with **$i<j$**

$$\varphi = \begin{bmatrix} 0.8 \\ 0 \\ 0.2 \\ 1.2 \\ 0.6 \end{bmatrix}$$

$$\chi_{s,t} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

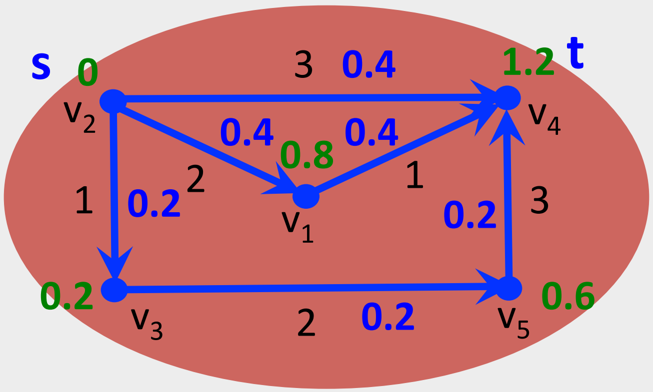
$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

How to compute an electrical flow?

Putting it together: φ induces an electrical flow iff

$$B\, R^{-1} B^{T} \varphi = \chi_{s,t}$$

Example:



$$B = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

|V|=5, |E|=6, all edges oriented (v_i, v_j) with **i<j**

$$\varphi = \begin{bmatrix} 0.8 \\ 0 \\ 0.2 \\ 1.2 \\ 0.6 \end{bmatrix} \xrightarrow{R^{-1}B^{T}} f = \begin{bmatrix} -0.4 \\ 0.4 \\ 0.4 \\ 0.2 \\ 0.2 \\ -0.2 \end{bmatrix}$$

$$\chi_{s,t} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

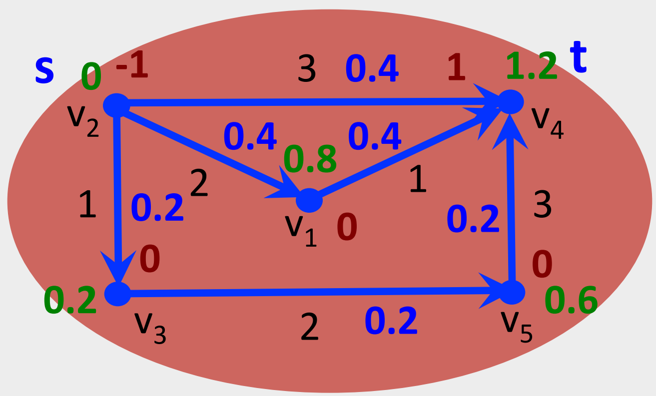
$$R = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

How to compute an electrical flow?

Putting it together: φ induces an electrical flow iff

$$B\,R^{-1}B^T\varphi = \chi_{s,t}$$

Example:



$$B = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$|V|=5$, $|E|=6$, all edges oriented **(v_i,v_j)** with **$i<j$**

$$\varphi = \begin{bmatrix} 0.8 \\ 0 \\ 0.2 \\ 1.2 \\ 0.6 \end{bmatrix} \xrightarrow{R^{-1}B^T} f = \begin{bmatrix} -0.4 \\ 0.4 \\ 0.4 \\ 0.2 \\ 0.2 \\ -0.2 \end{bmatrix} \xrightarrow{B} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \chi_{s,t}$$

✔

$$R = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

How to compute an electrical flow?

Bottom line:

Electrical flow computation

$$BR^{-1}B^{T} \varphi = \chi_{s,t}$$

Solving a **linear system**

Bad news: Solving a linear system can take $\mathbf{O(n^{\omega})=O(n^{2.373})}$

(Prohibitive!)

Key observation:

$\mathbf{BR^{-1}B^{T}}$ is the **Laplacian** matrix $\mathbf{L}$

of the underlying graph

How to compute an electrical flow?

Bot

Laplacian = key object of spectral graph theory (will get back to this)

Electrical flow computation

$L \varphi = \chi_{s,t}$

Solving a **Laplacian system**

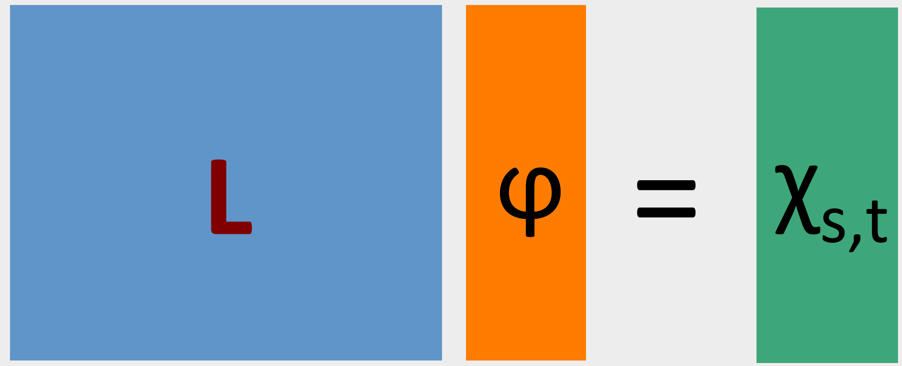
Bad news: Solving a linear system can take $\mathbf{O(n^{\omega})=O(n^{2.373})}$ (Prohibitive!)

Key observation:
$\mathbf{BR^{-1}B^{T}}$ is the **Laplacian** matrix **L** of the underlying graph

How to compute an electrical flow?

Bottom line:

Electrical flow
computation



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$\mathbf{BR^{-1}B^{T}}$ is the **Laplacian** matrix **L**
of the underlying graph

(will get back to this)

Result: Electrical flow is a **nearly-linear time** primitive

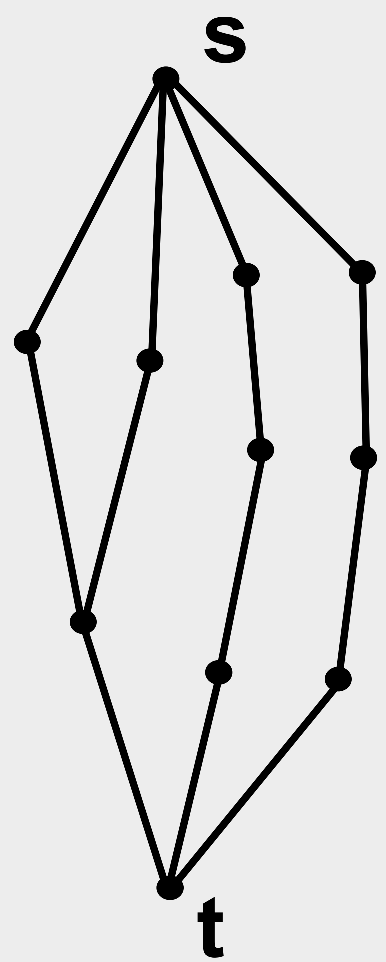
From electrical flows to undirected max flow

[CKMST '11]

Approx. undirected max flow via electrical flows

Assume: F* known (via binary search)

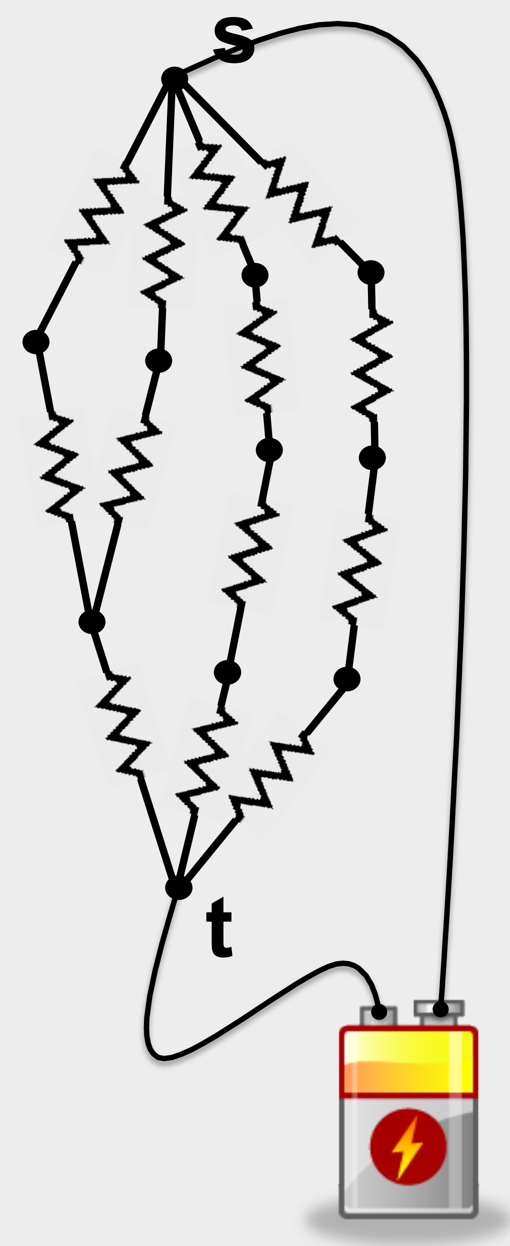
→ Treat edges as resistors of resistance **1**



Approx. undirected max flow via electrical flows

Assume: $\mathbf{F}^*$ known (via binary search)

→ Treat edges as resistors of resistance **1**
→ Compute electrical flow of value $\mathbf{F}^*$



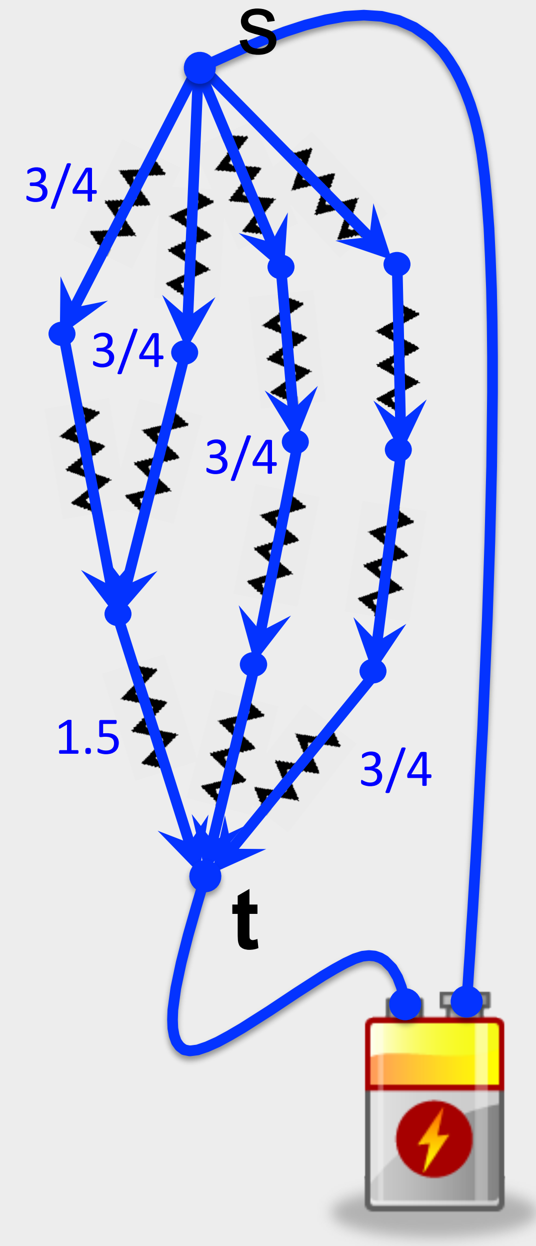
Approx. undirected max flow via electrical flows

Assume: F^* known (via binary search)

→ Treat edges as resistors of resistance **1**

→ Compute electrical flow of value **F^***

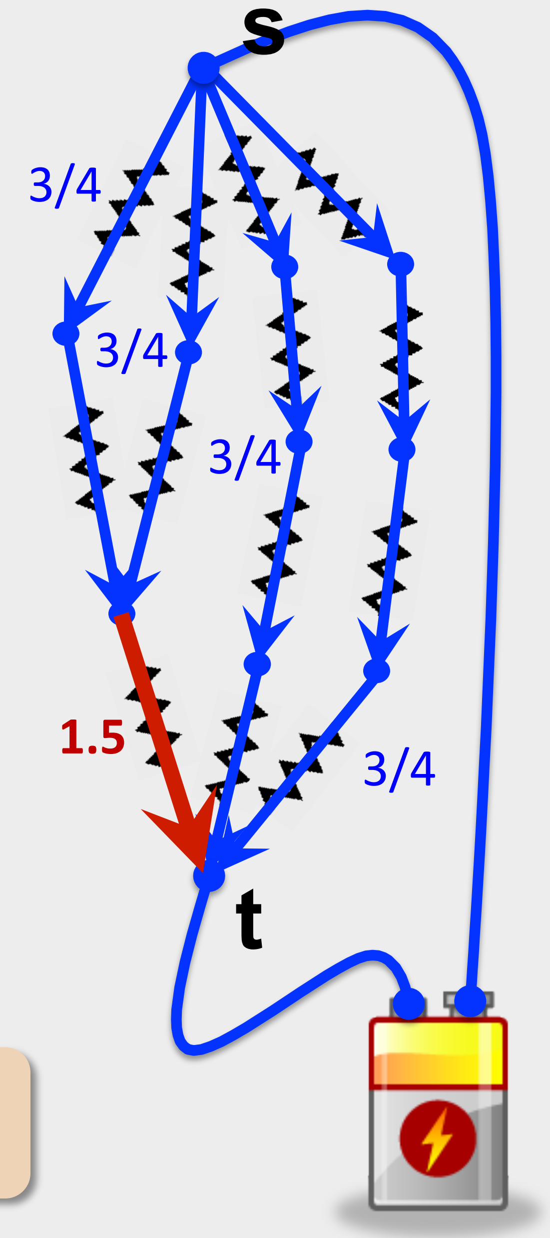
(This flow has **no leaks**, but **can overflow** some edges)



Approx. undirected max flow via electrical flows

Assume: F* known (via binary search)

→ Treat edges as resistors of resistance **1**
→ Compute electrical flow of value **F***
(This flow has **no leaks**, but **can overflow** some edges)
→ To fix that: **Increase resistances** on the overflowing edges
Repeat (**hope:** it doesn't happen too often)



Surprisingly: This approach can be made work!

Tomorrow: Will discuss how to fill in the blanks

Thank you