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A THEORY OF PFAFFIAN ORIENTATIONS

1. PERFECT MATCHINGS AND PERMANENTS.

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Abstract

Kasteleyn stated that the generating function of the perfect matchings of a graph of genus $g$ may be written as a linear combination of $4^g$ Pfaffians. Here we prove this statement. As a consequence we present a combinatorial way to compute permanents of square matrices.

Other consequences will be presented in a forthcoming paper.
1 Introduction

We present a theory of Pfaffian orientations of graphs, introduced by Kasteleyn ([6, 5, 4]). Our approach is an extension of the treatment of toroidal rectangular lattices (see [3, 9, 6, 5, 4]). The case of general toroidal graphs was also studied by Barahona ([1]). As a consequence, we present a new technique to compute permanents of square matrices, which completes the Polya’s scheme ([8]).

\[ G = (V, E) \] will always be a graph and \( x_e \) will be a variable associated with each edge \( e \) of \( G \). We let \( x = (x_e : e \in E) \) and for \( M \subset E \) let \( x(M) \) denote the product of the variables of the edges of \( M \). An orientation of a graph \( G \) is a digraph obtained from \( G \) by fixing an orientation of each edge of \( G \).

Let \( A \Delta B \) denote the symmetric difference of the sets \( A \) and \( B \) and let \( a \equiv b \) denote \( a = b \) modulo 2.

**Definition 1.1** The generating function of the perfect matchings of \( G \) is the polynomial \( P(G, x) \) equal to the sum of \( x(P) \) over all perfect matchings \( P \) of \( G \).

**Definition 1.2** Let \( G \) be a graph and let \( D \) be an orientation of \( G \). Let \( M \) be a perfect matching of \( G \). For each perfect matching \( P \) of \( G \) let \( sgn(D, M \Delta P) = (-1)^n \) where \( n \) is the number of clockwise even alternating cycles of \( M \Delta P \), and let \( P(D, M) \) equal the sum of \( sgn(D, M \Delta P)x(P) \) over all perfect matchings \( P \) of \( G \).

**Definition 1.3** Let \( G = (V, E) \) be a graph with \( 2n \) vertices and \( D \) an orientation of \( G \). Denote by \( A(D) \) the skew-symmetric matrix with the rows and the columns indexed by \( |V| \), where \( a_{vw} = x_{(v,w)} \) in case \( (v, w) \) is an arc of \( D \), \( a_{vw} = -x_{(v,w)} \) in case \( (w, v) \) is an arc of \( D \), and \( a_{vw} = 0 \) otherwise.

The Pfaffian of the skew-symmetric matrix \( A(D) \) is defined as

\[
Pf(A(D)) = \sum_P s^s(P) a_{i_1 j_1} \cdots a_{i_n j_n}
\]

where \( P = \{\{i_1 j_1\}, \ldots, \{i_n j_n\}\} \) is a partition of the set \( \{1, \ldots, 2n\} \) into pairs, \( i_k < j_k \) for \( k = 1, \ldots, n \), and \( s^s(P) \) equals to the sign of the permutation \( (i_1 j_1 \ldots i_n j_n) \).

Each nonzero term of the expansion of the Pfaffian of \( A(D) \) equals \( x(P) \) or \( -x(P) \) where \( P \) is a perfect matching of \( G \). Let \( s(D, P) \) equal the sign of the term \( x(P) \) so that...
The following theorem was proved by Kasteleyn ([4]).

**Theorem 1.4** Let $G$ be a graph and $D$ an orientation of $G$. Let $P, M$ be two perfect matchings of $G$. Then

$$s(D, P) = s(D, M) \text{sgn}(D, M \Delta P).$$

Hence,

$$Pf(A(D)) = \sum_P s(D, M) \text{sgn}(D, M \Delta P) x(P) = s(D, M) P(D, M).$$

The following theorem is well-known (see [2]).

**Theorem 1.5** Let $G$ be a graph and let $D$ be an orientation of $G$. Then

$$Pf^2(A(D)) = \det(A(D)).$$

**Remark.** This theorem has an algorithmic application: if the edge-set of $G$ is partitioned into a bounded number of classes and the variables $x_e$ are equal in each class, then $P(D, M)$ and $Pf(A(D))$ may be determined efficiently.

Let $M = \{\{i_1 j_1\}, \ldots, \{i_n j_n\}\}$, $i_k < j_k$, be a perfect matching of $G$. Let $x'$ be defined as follows: $x'_e = x_e$, if $e \notin M$ and $x'_f = x_f z$, if $f \in M$, $z$ is a new variable. Let $A'$ be the matrix obtained from $A(D)$ by replacing each $x_e$ by $x'_e$. Using Gaussian elimination we can express efficiently $\det(A')$ as a rational function, i.e. a ratio of two polynomials. Since $\det(A')$ is a polynomial, its coefficients can be determined efficiently from the rational function. We can view $\det(A')$ as a polynomial $\det(A')(x')$ or as a polynomial $\det(A')(x, z)$. By Theorem 1.5, $Pf(A')(x, z) = \pm \sqrt{\det(A')(x, z)}$. Hence we can determine efficiently a polynomial $Q(x, z)$ such that $Pf(A')(x, z) = \pm Q(x, z)$. Note that $P(D, M) = \pm Q(x, 1)$.

There is exactly one monomial in $Q(x, z)$ containing $z^n$ and its coefficient is $+1$ or $-1$. Let $Q'(x, z)$ be the unique polynomial such that $Q'(x, z) = Q(x, z)$ or $Q'(x, z) = -Q(x, z)$ and the coefficient of $Q'(x, z)$ of the term containing $z^n$ equals to $+1$. We have $P(D, M) = +Q'(x, 1)$. Moreover,
\[ Pf(A(D)) = s(D, M) P(D, M) \]

and \( s(D, M) = s^*(M) t^*(M) \) where \( t^*(M) \) equals to the product of the signs of the elements \( a_{i,jk} \) of the matrix \( A(D) \) such that \( i_k j_k \in M \). Hence \( P(D, M) \) and \( Pf(A(D)) \) may be determined efficiently.

Kasteleyn ([4]) introduced the following notion.

**Definition 1.6** A graph \( G \) is called Pfaffian if it has a Pfaffian orientation, i.e., an orientation such that each alternating cycle with respect to an arbitrary fixed perfect matching \( M \) of \( G \) is clockwise odd.

Hence if a graph \( G \) has a Pfaffian orientation \( D \) then \( s(D, P) \), \( P \) perfect matching of \( D \), are equal and \( P(G, x)^2 = Pf^2(A(D)) = det(A(D)) \).

Kasteleyn ([4]) also observed that the planar graphs have a Pfaffian orientation.

**Theorem 1.7** Each planar graph has a Pfaffian orientation.

**Proof.** Let \( G \) be a planar graph, and let \( M \) be its perfect matching. Consider \( G \) drawn on the plane. Orient edges of \( G \) so that each face, except possibly the outer one, is clockwise odd. Each such face ‘encircles’ no vertex of \( G \). Observe that the orientation has the property that a cycle \( C \) of \( G \) is clockwise odd if and only if \( C \) encircles an even number of vertices. Each alternating cycle with respect to \( M \) encircles an even number of vertices and hence it is clockwise odd. \( \square \)

2 The Perfect Matchings

**Definition 2.1** We define surface \( S_g \), \( g \) positive integer, as follows. It consists of a base \( B_0 \) and \( 2g \) bridges \( B^1_i \), \( i = 1, \ldots, g \) and \( j = 1, 2 \). \( B_0 \) is a convex \( 4g \)-gon with vertices \( a_1, \ldots, a_n, n = 4g \), numbered clockwise. Bridge \( B^1_i \) is a \( 4 \)-gon with vertices \( x^1_1, x^2_1, x^3_1, x^4_1 \) numbered clockwise. It is glued with \( B_0 \) so that edge \( (x^1_1, x^2_1) \) of \( B^1_i \) is identified with edge \( (a_{4(i-1)+1}, a_{4(i-1)+2}) \) of \( B_0 \) and edge \( (x^3_1, x^4_1) \) of \( B^1_i \) is identified with edge \( (a_{4(i-1)+3}, a_{4(i-1)+4}) \) of \( B_0 \).

Bridge \( B^2_i \) is a \( 4 \)-gon with vertices \( y^1_i, y^2_i, y^3_i, y^4_i \) numbered clockwise. It is glued with \( B_0 \) so that edge \( (y^1_i, y^2_i) \) of \( B^2_i \) is identified with edge \( (a_{4(i-1)+2}, a_{4(i-1)+3}) \) of \( B_0 \) and edge \( (y^3_i, y^4_i) \) of \( B^2_i \) is identified with edge \( (a_{4(i-1)+4}, a_{4(i-1)+5 \mod 4g}) \) of \( B_0 \).
The definition of surface $S_g$ corresponds to the usual definition of an orientable surface of genus $g$ in the following sense. Orientable surface $O_g$ of genus $g$ may be obtained from $S_g$ as follows: for each bridge $B$, glue together the two segments in which $B$ intersects the boundary of $B_0$, and delete $B$.

**Definition 2.2** We say that a graph $G$ is a $g$-graph if it may be drawn on $S_g$ so that all the vertices belong to the base $B_0$, and each edge uses at most one bridge. The set of the edges drawn on the base will be denoted by $E_0 = E_0(G)$ and the set of edges drawn on bridge $B^i_j$ will be denoted by $E^i_j = E^i_j(G)$.

If moreover the following conditions are satisfied then we say that $G$ is a proper $g$-graph.

1. The outer face of the subgraph embedded on $B_0$ is a cycle, and it is embedded on the boundary of $B_0$.
2. Each vertex is incident with at most one edge out of $E_0$.
3. The subgraph embedded on $B_0$ has a perfect matching.

If $G$ is a proper $g$-graph then we denote by $C_0$ the cycle which forms the outer face of $E_0$, and we denote by $M_0$ a perfect matching of the subgraph of $G$ embedded on $B_0$.

For each $g$-graph we fix its drawing on $S_g$.

**Definition 2.3** Let $G = (V, E)$ be a proper $g$-graph. The graphs $G_0 = (V, E_0)$ and $G^i_j = (V, E_0 \cup E^i_j)$ are planar. We define orientations of $G_0$ and $G^i_j$ as follows: orientation $D_0$ of $G_0$ is Pfaffian and such that each face of $G_0$ is clockwise odd (as in the proof of Theorem 1.7). Next we draw $G^i_j$ on the plane so that the drawing of $G_0$ is unchanged, and edge $(x^i_1, x^i_4)$ ($(y^i_1, y^i_4)$ respectively) of $B^i_j$ belongs to the outer face of the drawing of $G^i_j$. Now complete $D_0$ to an orientation of $G^i_j$ so that each face is clockwise odd. This defines orientation $+D^i_j$ of $E^i_j$.

$-D^i_j$ is defined by reversing the orientation of $D^i_j$.

**Remark 2.4** If $G$ is a proper $g$-graph and Pfaffian orientation $D_0$ is fixed, then $D^i_j$ is uniquely determined for each $ij$. 
Definition 2.5 Let $G$ be a proper $g$-graph, $g \geq 1$. An orientation $D$ of $G$ which equals to $D^i_j$ or $-D^i_j$ on $E^i_j$ and to $D_0$ on $E_0$ is called relevant. We define its type $r(D) \in \{+1, -1\}^{2g}$ as follows. For $i = 0, \ldots, g-1$ and $j = 1, 2$, $r(D)_{2i+j}$ equals to $+1$ or $-1$ according to the sign of $D^i_{j+1}$ in $D$.

Definition 2.6 Let $G = (V, E)$ be a proper $g$-graph and let $A$ be a subset of its edges. We define its type $t(A) \in \{+1, -1\}^{2g}$ as follows. For $i = 0, \ldots, g-1$ and $j = 1, 2$, we let $t(A)_{2i+j}$ equal to $(-1)^{s(A)_{2i+j}}$, where $s(A)_{2i+j}$ equals to the number of edges of $A$ which belong to $E^i_{j+1}$.

Let $CR(A) = \sum_{i=0}^{g-1} s(A)_{2i+1} \cdot s(A)_{2i+2}$.
Let $BR(A)$ be the subset of edges of $A$ which do not belong to $E_0$.
For each $e \in BR(A)$, let $d(e) = 2i + j$ such that $e \in E^i_{j+1}$.

Let $G$ be a proper $g$-graph. Any alternating cycle with respect to $M_0$ will be called alternating cycle. If $C$ is a cycle of even length or a cycle embedded in the plane, then we denote by $l(C)$ the number of arcs of $C$ directed clockwise, modulo 2.

We want to show that $\text{sgn}(D, M_0 \Delta P)$ depends only on $t(M_0 \Delta P)$ and $r(D)$.

Lemma 2.7 Let $G$ be a proper $g$-graph. Let $C_1, \ldots, C_k$ be vertex-disjoint cycles of $G$ and let $C$ denote their union. Then

$$CR(C) = \sum_{i=1}^{k} CR(C_i).$$

Proof. $CR(C)$ equals to the sum modulo 2 of $s(C)_{2i+1} \cdot s(C)_{2i+2}$, $i = 0, \ldots, g-1$. Now consider the drawing of cycles $C_1, \ldots, C_k$ in the plane, obtained by projecting each $E^i_j$ outside of $B_0$. The total number of crossings modulo 2 also equals to the sum modulo 2 of $s(C)_{2i+1} \cdot s(C)_{2i+2}$, $i = 0, \ldots, g-1$. Each $C_l$, $l = 1, \ldots, k$ becomes a closed curve in the plane. Each pair of curves representing $C_i$ and $C_j$ intersect each other an even number of times. Hence we can forget these crossings without influencing the sum modulo 2 of $s(C)_{2i+1} \cdot s(C)_{2i+2}$, $i = 0, \ldots, g-1$. Each of the remaining crossings is a crossing of some $C_l$, $l = 1, \ldots, k$. \hfill \Box

Next we consider the case that there is exactly one alternating cycle of $M_0 \Delta P$. 7
Theorem 2.8 Let $G$ be a proper $g$-graph and let $D$ be a relevant orientation of $G$. Let $C$ be an alternating cycle of $G$. Then

$$l(C) = |BR(C)| - 1 - CR(C) + 1/2 \sum_{e \in BR(C)} (r(D)_{d(e)} + 1).$$

Proof. We will assume without loss of generality that $G = C \cup C_0 \cup M_0$.

Claim 1. Let for each $i = 1, \ldots, g$, $C$ intersects at most one of $E_i^1, E_i^2$. Then

$$l(C) = |BR(C)| - 1 + 1/2 \sum_{e \in BR(C)} (r(D)_{d(e)} + 1).$$

Proof of Claim 1. $C$ satisfying the properties of Claim 2 may be embedded in a planar way, by projecting the non-empty bridges outside of $B_0$. Hence $l(C) = 1$ iff $|\{e \in BR(C) : r(D)_{d(e)} = -1\}| \equiv 0$. From this Claim 1 follows.

End of Claim 1.

We proceed by induction on $|BR(C)|$. The case $|BR(C)| = 0$ is considered in Claim 1.

Let theorem hold for all alternating cycles $C'$ with $|BR(C)| > |BR(C')|$. Hence for such alternating cycles, $l(C')$ depends only on $t(C')$ and $r(D)$.

We let $l(t(F), r(H))$ denote the parity of the number of edges oriented clockwise of alternating cycle $F$ with $|BR(F)| < |BR(C)|$, in relevant orientation $H$ of a proper $g$-graph.

We will make the following notational agreement: if a segment $S$ of $C_0$ is traversed clockwise then we denote it by $+S$, otherwise by $-S$.

If $P$ is a path together with a prescribed way of traversing it, we denote by $l(P)$ the parity of the number of arcs of $P$ oriented in agreement with the way of traversing $P$.

Claim 2. Let there be a bridge $B = B_j^i$ containing more than one edge of $C$. Then 2.8 holds.

Proof of Claim 2. Let $e, f$ be two edges of $C$ drawn on $B$ which ‘see’ each other on $B$, i.e. there is no other edge of $C$ drawn between them on $B$.

We remind that $C_0$ denotes the outer face of $G_0$ and $e, f$ do not belong to $M_0 \subset E_0$.

Without loss of generality let $e$ be nearer to edge $[a_{2(i-1)+j}, a_{2(i-1)+j+3}]$ of $B = B_j^i$ than $f$.

Let $R$ be the cycle of $G$ formed by $e, f$ and two subpaths $R_1, R_2$ of $C_0$ defined by the endvertices of $e, f$. By the choice of $e, f$, $R$ is a face of
the planar drawing of $G'_j = (V, E_0 \cup E'_j)$. Observe that $l(R) = 1$ in the orientation of $G'_j$ induced by the relevant orientation $D$.

Let us consider new edge $g$ (not belonging to $G$), between endpoints of $e, f$ such that one of two cycles $H_1, H_2$ formed by $g$ and $C$ and containing $g$ is alternating. Without loss of generality, let $g$ use vertex $u \in R_1 \cap e$. Hence we have that $g = u_v$ or $g = u_{v_2}$ where $u \in R_1 \cap e$ and $v_1 \in R_1 \cap f$, $i = 1, 2$.

We denote two opposite orientations of $g$ by $g_1, g_2$. Let $H_1$ use $g_1$ and $H_2$ use $g_2$.

Observe that $CR(C) = l(H_1) + l(H_2)$.

We will assume without loss of generality that $H_2$ is alternating. Hence $H_1$ contains both $e, f$.

We distinguish two cases now.

**Case 1:** $g = u_{v_1}$.

Let $G' = G \cup \{g_1, g_2\}$. We consider $g_1, g_2$ embedded on $B_0$ along $R_1$.

Observe that $CR(C) \equiv l(H_1) + CR(H_2)$: let us project all the edges of $G' - E_0$ outside of $B_0$. The edge-crossings are the same as the original edge-crossings of $G$; hence the parity of the number of edge-crossings of $G'$ equals to $CR(C)$.

In addition to these crossings, $H_1$ and $H_2$ share two vertices of $g_1, g_2$. However, in our plane drawing these two vertices are touching points between the closed curves representing $H_1, H_2$, not crossing points. Hence, by the same argument as in the proof of Lemma 2.7, $CR(C)$ equals the sum modulo 2 of the number of crossings of $H_1$ and the number of crossings of $H_2$. The number of crossings of $H_i$ modulo 2 equals $CR(H_i), i = 1, 2$. Hence summarising $CR(C) \equiv l(H_1) + CR(H_2)$.

We construct two digraphs $D_1, D_2$ as follows:

$D_1$ is obtained from $D - \{e, f, \}$ by adding new vertices $u', v'_1$ of degree 2, incident with new arcs $e', f', g'_1$. $e', f', g'_1$ are obtained from $e, f, g_1$ by replacing $u$ by $u'$ and $v_1$ by $v'_1$. We extend $B_0$ along $R_2$ and consider $e', f', g'_1$ embedded on extended $B_0$. Finally we add $g'_1$ to $M_0$. Let $H'_1$ be the cycle of $D_1$ obtained from $H_1$ by replacing $e, f, g_1$ by $e', f', g'_1$. We have $l(H_1) = l(H'_1)$.

$D_2$ is obtained from $D - \{e, f, \}$ by adding arc $g_2$. We again extend $B_0$ along $R_1$ and consider $g_2$ embedded on extended $B_0$. We let $H'_2 = H_2$.

Hence for $i = 1, 2$, $D_i$ is orientation of a proper $g$-graph and $H'_i$ is an alternating cycle of $D_i$. Moreover $|BR(H'_i)| < |BR(C)|$. We also have that $CR(H_2) \equiv CR(H'_2)$ and $CR(H_1) \equiv CR(H'_1)$.

We remind that $l(R) = 1$ in the orientation of $G'_j$ induced by $D$. Hence,
the number of crossings of the edges of $H$ equals the sum modulo 2 of the number of crossings of edges of $H$.

By the induction assumption for $H_1', H_2'$, $l(H_2') = l(t(H_2'), r(D_2))$ and $l(H_1') \equiv l(t(H_1'), r(D_1)) + 1$. We have that

$$l(C) \equiv l(H_1) + l(H_2) \equiv l(H_1') + l(H_2') \equiv$$

$$l(t(H_1'), r(D_1)) + l(t(H_2'), r(D_2)) + 1 \equiv$$

$$\sum_{h \in BR(C) - \{e, f\}} (r(D)_{d(h)} + 1) + 1 \equiv$$

$$\left[|BR(C)| - 2 - CR(H_1') - CR(H_2') + 1/2 \right] + 1 \equiv$$

$$\left[|BR(C)| - 1 - CR(C) + 1/2 \right] + 1 \equiv$$

**End of Case 1.**

**Case 2:** $g = uv_2$.

Let $G' = G \cup \{g_1, g_2\}$. We consider $g_1, g_2$ embedded on $B$.

Observe that $CR(C) + 1 \equiv CR(H_1) + CR(H_2)$: let us project all the edges of $G' - E_0$ outside of $B_0$. The edge-crossings are: the original crossings of $G$; the crossings of $g$, and an edge $h \neq g_{2-i}$; but the total number of these crossings is even since each such edge $h$ intersects both $g_1, g_2$.

Hence the parity of the number of edge-crossings equals $CR(C)$.

In addition to these crossings, $H_1$ and $H_2$ share two vertices of $g_1, g_2$. In our plane drawing one of them is a crossing point and the other one a touching point between the closed curves representing $H_1, H_2$.

Hence, by the same argument as in the proof of Lemma 2.7, $CR(C) + 1$ equals the sum modulo 2 of the number of crossings of edges of $H_1$ and the number of crossings of the edges of $H_2$. The number of crossings of $H_i$ modulo 2 equals to $CR(H_i)$, $i = 1, 2$. Hence summarising $CR(C) + 1 \equiv CR(H_1) + CR(H_2)$.

We construct two digraphs $D_1, D_2$ as follows:

$D_1$ is obtained from $D - \{e, f\}$ by adding new arc $g_1'$ between $v_1$ and the endvertex $u'$ of $e$ different from $u$. If $l(fg_1e) = 1$ then we let $g_1' = (v_1u')$. If $l(fg_1e) = 0$ then $l(eg_1f) = 1$ and we let $g_1' = (u', v_1)$.

We consider $g_1'$ embedded on $B$. Let $H_1'$ be obtained from $H_1$ by replacing $f, g_1, e$ by $g_1'$. We have $l(H_1) = l(H_1')$.

$D_2$ is obtained from $D - \{e, f\}$ by adding arc $g_2$. We consider $g_2$ embedded on bridge $B$. We let $H_2 = H_2'$. 

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Hence for \( i = 1, 2 \), \( D_i \) is orientation of a proper \( g \)-graph and \( H'_1 \) is a cycle of \( D_i \). Moreover \( |BR(H'_1)| < |BR(C)| \). We also have that \( CR(H_2) = CR(H'_2) \) and \( CR(H_1) = CR(H'_1) \).

We remind that \( l(R) = 1 \) in the orientation of \( G'_j \) induced by \( D \). Hence exactly one of \( g_i \) is oriented so that both cycles it makes with \( R \) are clockwise odd. Let it be \( g_2 \). Hence \( D_2 \) is relevant with \( r(D_2) = r(D) \) if and only if \( r(D)_{2(i-1)+j} = 1 \), and we have:

\[ l(H_2) = l(t(H'_2), r(D)) \] if and only if \( r(D)_{2(i-1)+j} = 1 \). Otherwise \( l(H_2) = l(t(H'_2), r(D)) + 1 \).

We will show that the same holds for \( H'_1 \): let \( R_3 \) be the segment of \( C_0 \) such that \( H = (e, -R_1, -R_3, -R_2) \) is a cycle.
We show that \( l(g'_1, -R_3, -R_2) = 1: l(g'_1, -R_3, -R_2) = l(e, g_1, f, -R_3, -R_2) = l(f, -R_3) + l(e, g_1, -R_2) \).
If \( r(D)_{2(i-1)+j} = 1 \) then \( l(f, -R_3) = 1 \) and \( l(e, g_1, -R_2) = 0 \) since \( l(e, g_2, -R_2) = 1 \).
If \( r(D)_{2(i-1)+j} = -1 \) then \( l(f, -R_3) = 0 \) and \( l(e, g_1, -R_2) = 1 \) since \( l(e, g_2, -R_2) = 0 \).

Summarising \( l(g'_1, -R_3, -R_2) = 1. \)
Hence \( D_1 \) is relevant with \( r(D_1) = r(D) \) if and only if \( r(D)_{2(i-1)+j} = 1 \), and we again have:

\[ l(H_1) = l(H'_1) = l(t(H'_1), r(D)) \] if and only if \( r(D)_{2(i-1)+j} = 1 \). Otherwise \( l(H_1) = l(H'_1) = l(t(H'_1), r(D)) + 1 \).

Summarising and using the induction assumption of 2.8 for \( H'_1, H'_2 \) we get:

\[
l(C) = l(H_1) + l(H'_2) \\
l(t(H'_1), r(D)) + l(t(H'_2), r(D)) \]

\[
[[BR(C)] - 2 - CR(C)] + 1 + \\
1/2 \sum_{h \in BR(C) \setminus \{e, f\}} (r(D)_{d(h)} + 1) \\
1/2(r(D)_{d(g_2)} + 1) \\
1/2(r(D)_{d(g'_1)} + 1) \]

\[
[[BR(C)] - 1 - CR(C)] + 1/2 \sum_{h \in B(C)} (r(D)_{d(h)} + 1)].
\]

End of Case 2.
End of Claim 2.
Claim 3. Let there be $i$ such that $C$ contains exactly one edge from both $E_i^1$ and $E_i^2$. Then theorem 2.8 holds.

Proof of Claim 3. Let $e, f$ be the two edges on bridges $E_i^1$ and $E_i^2$, respectively, and let $C_1$ and $C_2$ be two paths such that $C = (C_1, e, C_2, f)$.

The end-vertices of $e, f$ belong to $C_0$. Let us assume that along the boundary of $B_0$ from $a_{4(i-1)+1}$ to $a_{4i+1}$, the endvertices of $e, f$ appear in the order $(v_1, u_1, v_2, u_2)$ where $e = u_1 u_2$ and $f = v_1 v_2$.

Let $R_1, R_2$ be two disjoint subpaths of the segment of $C_0$ between $a_{4(i-1)+1}$ and $a_{4i+1}$, which cover the endvertices of $e, f$. $R_1, R_2$ contain no other vertex of $G$ incident with an edge out of $E_0$, by the choice of $i$. $R_1, R_2, e, f$ form a cycle $R$, and without loss of generality, let $R = (R_1, e, R_2, f)$ where $R_1$ is traversed clockwise, i.e. in agreement with the indices of the vertices along $C_0$, and $R_2$ is traversed anticlockwise.

Let us consider new edge $g$ (not belonging to $G$), between endpoints of $e, f$ such that one of two cycles $I_1, I_2$ formed by $g$ and $C$ and containing $g$ is alternating. Without loss of generality let $g$ use vertex $u_1 \in R_1 \cap e$. Hence we have that $g = u_1 v_1$ or $g = u_1 v_2$ where $u_1 \in R_1 \cap e$ and $v_1 \in R_i \cap f$, $i = 1, 2$.

We denote two opposite orientations of $g$ by $g_1, g_2$. Let $I_1$ use $g_1$ and $I_2$ use $g_2$.

Observe that $l(C) = l(I_1) + l(I_2)$.

We will assume without loss of generality that $I_2$ is alternating. Hence $I_1$ contains both $e, f$.

Let $R_3$ denote the segment of $C_0$ between $u_1$ and $v_2$.

Again we distinguish two cases.

Case 1: $g = u_1 v_1$.

In this case $g$ forms a cycle with $R_1$.

Let $G' = G \cup \{g_1, g_2\}$. We consider $g_1, g_2$ embedded on $B_0$ along $R_1$.

Observe that $CR(C) = CR(I_1) + CR(I_2)$: let us project all the edges of $G' - E_0$ outside of $B_0$. The edge-crossings are the same as the original edge-crossings of $G$. Hence the parity of the number of edge-crossings equals $CR(C)$.

In addition to these crossings, $I_1$ and $I_2$ share two vertices of $g_1, g_2$. However, in our plane drawing these two vertices are touching points between the closed curves representing $I_1, I_2$, not crossing points. Hence, by the same argument as in the proof of Lemma 2.7, $CR(C)$ equals the sum modulo 2 of the number of crossings of $I_1$ and the number of crossings of $I_2$. The number of crossings of $I_i$ equals to $CR(I_i), i = 1, 2$. Hence summarising
We construct two digraphs $D_1, D_2$ as follows:

$D_1$ is obtained from $D - \{e, f\}$ by adding new vertices $u'_1, v'_1$ of degree 2, incident with new arcs $e', f', g'_1$. $e', f', g'_1$ are obtained from $e, f, g_1$ by replacing $u_1$ by $u'_1$ and $v_1$ by $v'_1$. We extend $B_0$ along $R_2$ and consider $e', f', g'_1$ embedded on extended $B_0$. Finally we add $g'_1$ to $M_0$. Let $I'_1$ be the cycle of $D_1$ obtained from $I_1$ by replacing $e, f, g_1$ by $e', f', g'_1$. We have $l(I_1) = l(I'_1)$.

$D_2$ is obtained from $D - \{e, f\}$ by adding arc $g_2$. We again extend $B_0$ along $R_1$ and consider $g_2$ embedded on extended $B_0$. We let $I'_2 = I_2$.

Hence for $i = 1, 2$, $D_i$ is orientation of a proper $g$-graph and $I'_i$ is an alternating cycle of $D_i$. Moreover $|BR(I'_i)| < |BR(C)|$. We also have that $CR(I_2) = CR(I'_2)$ and $CR(I_1) - 1 = CR(I'_1)$.

Let us assume without loss of generality that $g_2$ is directed so that the cycle $l(-R_1, g_2) = 1$. Hence $D_2$ is a relevant orientation.

Observe that $D_1$ is a relevant orientation if and only if $r(D)_{d(e)} = r(D)_{d(f)}$:

first we prove if $r(D)_{d(e)} = r(D)_{d(f)} = 1$ then $D_1$ is a relevant orientation.

In this case we need to show that $l(-R_2, f', g'_1, e') = 1$.

We have $l(g'_1, f', -R_3) = l(g_2, f, -R_3) = 0$ since $r(D_{d(f)}) = 1$ and thus $l(-R_3, g_2, f) = 1$. Moreover $l(g'_1, f') = l(f', g'_1)$ since it does not matter which way two consecutive arcs are traversed. Hence $l(f', g'_1) = l(-R_3)$.

Moreover $l(-R_2, -R_3, e') = l(-R_2, -R_3, e) = 1$ since $r(D)_{d(e)} = 1$. Replacing $f'g'_1$ for $-R_3$ gives what we needed.

Analogously, if $r(D)_{d(e)} = r(D)_{d(f)} = -1$ then $l(-R_2, f', g'_1, e') = 1$ and $D_1$ is a relevant orientation.

On the other hand, if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then $D_1$ is obtained from a relevant orientation by reversing one arc; hence $D_1$ is not relevant.

Summarizing,

\[
l(I_1) + l(I_2) \geq l(I'_1) + l(I'_2) \geq l(t(I'_1), r(D_1)) + l(t(I'_2), r(D_2)) + \frac{1}{2}(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1).
\]

Using the induction assumption of 2.8 for $I'_1, I'_2$ we get:

\[
l(C) \geq l(I_1) + l(I_2) \geq l(t(I'_1), r(D_1)) + l(t(I'_2), r(D_2)) + \frac{1}{2}(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1).
\]
\[
\frac{1}{2}(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1) \equiv \\
\left[|BR(C)| - 4 - CR(C) + 1 + 1/2 \sum_{h \in BR(C) - \{e,f\}} (r(D)_{d(h)} + 1) \right] \equiv \\
\frac{1}{2}(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1) \equiv \\
\left[|BR(C)| - 1 - CR(C) + 1/2 \sum_{h \in BR(C)} (r(D)_{d(h)} + 1) \right].
\]

End of Case 1.

Case 2: \(g = uv_2\).

In this case \(g\) forms a cycle with \(R_3\).

Let \(G' = G \cup \{g_1, g_2\}\). We consider \(g_1, g_2\) embedded on \(B_0\) along \(R_3\).

Observe that \(CR(C) \equiv CR(I_1) + CR(I_2)\): Let us project all the edges of \(G' - E_0\) outside of \(B_0\). The edge-crossings are the same as the original crossings of \(G\).

Hence the parity of the number of edge-crossings equals \(CR(C)\).

In addition to these crossings, \(I_1\) and \(I_2\) share two vertices of \(g_1, g_2\). In the plane drawing both of them are touching points between the closed curves representing \(I_1, I_2\).

Hence, by the same argument as in the proof of Lemma 2.7, \(CR(C)\) equals the sum modulo 2 of the number of crossings of \(I_1\) and the number of crossings of \(I_2\). The number of crossings of \(I_i\) equals to \(CR(I_i), i = 1, 2\). Hence summarising \(CR(C) \equiv CR(I_1) + CR(I_2)\).

We construct two digraphs \(D_1, D_2\) as follows:

- \(D_1\) is obtained from \(D - \{e, f\}\) by adding new vertices \(u'_1, v'_2\) of degree 2, incident with new arcs \(e', f', g_1'\). \(e', f', g_1'\) are obtained from \(e, f, g_1\) by replacing \(u_1\) by \(u'_1\) and \(v_2\) by \(v'_2\). We extend \(B_0\) along \(+R_1 + R_3 + R_2\) and consider \(e', f', g_1'\) embedded on extended \(B_0\). Finally we add \(g'_2\) to \(M_0\). Let \(I_1'\) be the cycle of \(D_1\) obtained from \(I_1\) by replacing \(e, f, g_1\) by \(e', f', g_1'\). We have \(l(I_1) = l(I_1')\).

- \(D_2\) is obtained from \(D - \{e, f, g_2\}\) by adding arc \(g_2\). We again extend \(B_0\) along \(R_3\) and consider \(g_2\) embedded on extended \(B_0\). We let \(I_2' = I_2\).

Hence for \(i = 1, 2, D_i\) is orientation of a proper \(g\)-graph and \(I_1'\) is an alternating cycle of \(D_i\). Moreover \(|BR(I'_1)| < |BR(C)|\). We also have that \(CR(I_2) = CR(I_2')\) and \(CR(I_1) - 1 \equiv CR(I_1')\).

Let us assume without loss of generality that \(g_2\) is directed so that \(l(-R_3, g_2) = 1\). Hence \(D_2\) is a relevant orientation.

Observe that \(D_1\) is a relevant orientation if and only if \(r(D)_{d(e)} = r(D)_{d(f)}\):
It again suffices to consider the case that \( r(D_{d(e)}) = r(D_{d(f)}) = +1 \).
In this case we need to show that \( l(-R_2, -R_3, -R_1, f', g'_1, e') = 1 \):
We have \( l(-R_1, f', g'_1) = 0 \) since \( r(D_{d(f)}) = 1 \) and thus \( l(-R_1, f, g_2) = 1 \).
Moreover \( l(-R_2, -R_3, e') = l(-R_2, -R_3, e) = 1 \) since \( r(D_{d(e)}) = +1 \).
Summarising \( l(-R_2, -R_3, -R_1, f', g'_1, e') = 1 \).

Hence

\[
l(I_1) + l(I_2) = l(I'_1) + l(I'_2) = l(t(I'_1), r(D_1)) + l(t(I'_2), r(D_2)) + 1/2(r(D_{d(e)}) - 1 + r(D_{d(f)}) - 1).
\]

Using the induction assumption of 2.8 for \( I_1, I_2 \) we get:

\[
l(C) = l(I_1) + l(I_2) = l(t(I'_1), r(D_1)) + l(t(I'_2), r(D_2)) + 1/2(r(D_{d(e)}) - 1 + r(D_{d(f)}) - 1) =
\]

\[
|BR(C)| - 4 - CR(C) + 1 + 1/2 \sum_{h \in BR(C) - \{e, f\}} (r(D_{d(h)}) + 1) + 1/2(r(D_{d(e)}) - 1 + r(D_{d(f)}) - 1) =
\]

\[
|BR(C)| - 1 - CR(C) + 1/2 \sum_{h \in BR(C)} (r(D_{d(h)}) + 1).
\]

**End of Case 2.**

**End of Claim 3.**

Theorem follows from Claim 1, 2, 3. \( \square \)

Next we show that the statement of Theorem 2.8 holds for the set of the alternating cycles of \( M_0 \Delta P \) as well.

**Theorem 2.9** Let \( G \) be a proper \( g \)-graph and let \( D \) be a relevant orientation of \( G \). Let \( P \) be a perfect matching of \( G \). Then

\[
\text{sgn}(D, M_0 \Delta P) = (-1)^q,
\]

where

\[
q = |BR(M_0 \Delta P)| - CR(M_0 \Delta P) + 1/2 \sum_{e \in BR(M \Delta P)} (r(D_{d(e)}) + 1).
\]

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Proof. Let $C_1, ..., C_k$ be the alternating cycles of $M_0 \Delta P$.
We have that $\text{sgn}(D, M_0 \Delta P) = (-1)^q$, where $q = l(C_1) + ... + l(C_k) - k$.
Using Theorem 2.8 for $C_1, ..., C_k$ it remains to show:

$$CR(M_0 \Delta P) = \sum_{j=1}^{k} CR(C_j).$$

This is true by Lemma 2.7.

Corollary 2.10 Let $G$ be a proper 1-graph and $D$ a relevant orientation of $G$. Let $C$ be an alternating cycle with respect to $M_0$. Then

1. if $r(D) = (1, 1)$ then $\text{sgn}(D, C) = 1$ iff $t(C) = (1, 1)$ or $t(C) = (1, -1)$ or $t(C) = (-1, 1)$.

2. If $r(D) = (1, -1)$ then $\text{sgn}(D, C) = 1$ iff $t(C) = (1, 1)$ or $t(C) = (-1, -1)$ or $t(C) = (-1, 1)$.

3. If $r(D) = (-1, 1)$ then $\text{sgn}(D, C) = 1$ iff $t(C) = (1, 1)$ or $t(C) = (-1, -1)$ or $t(C) = (1, -1)$.

4. If $r(D) = (-1, -1)$ then $\text{sgn}(D, C) = 1$ iff $t(C) = (1, 1)$.

Definition 2.11 Let $G$ be a proper $g$-graph and let $D$ be a relevant orientation of $G$. Let $r(D) = (r_1, ..., r_{2g})$. We let $c(r(D))$ equal to the product of $c_i$, $i = 0, ..., g - 1$, where $c_i = c(r_{2i+1}, r_{2i+2})$ and $c(1, 1) = c(1, -1) = c(-1, 1) = 1/2$ and $c(-1, -1) = -1/2$.

Observe that $c(r(D)) = (-1)^n 2^{-g}$, where $n = |\{i; r_{2i+1} = r_{2i+2} = -1\}|$.

Corollary 2.12 Let $G$ be a proper 1-graph. Let $D_1, D_2, D_3, D_4$ be the relevant orientations of $G$. Then

$$\mathcal{P}(G, x) = \sum_{i=1}^{4} c(r(D_i)) \mathcal{P}(D_i, M_0).$$

Corollary 2.12 holds for all proper $g$-graphs. In order to deduce it we start with another corollary of Theorem 2.9.
Corollary 2.13 Let $G$ be a proper $g$-graph, $D$ a relevant orientation of $G$ and let $P$ be a perfect matching of $G$. Then $\text{sgn}(D, M_0\Delta P)$ is a function of $r(D)$ and $t(M_0\Delta P)$ only. Let us denote this function by $\sigma(r(D), t(M_0\Delta P))$.

Lemma 2.14 Let $r = (r_1, \ldots, r_{2g})$ and $t = (t_1, \ldots, t_{2g})$ where $r_i, t_i \in \{1, -1\}$. Let $r^j = (r_{2j+1}, r_{2j+2})$ and $t^j = (t_{2j+1}, t_{2j+2})$, $j = 0, \ldots, g - 1$. Then

$$\sigma(r, t) = \prod_{j=0}^{g-1} \sigma(r^j, t^j).$$

**Proof.** By Corollary 2.13, $\sigma(r, t) = \text{sgn}(D, C)$ where $r(D) = r$, $t(C) = t$ and $D$ is a relevant orientation of a proper $g$-graph $G$ such that $G = C_0 \cup C$. $C$ is a set of vertex-disjoint cycles $C_1, \ldots, C_k$ with the following properties:

1. each $C_i$ is alternating with respect to a perfect matching $M_0$ of $G_0$,
2. for each $i, j |E^j_i| \leq 1$,
3. for each $i$ there is at most one $j$ such that $|C_j \cap (E^i_1 \cup E^i_2)| \geq 1$,
4. for each $C_j$ there is exactly one $i$ such that $C_j$ intersects $E^i_1 \cup E^i_2$.

Hence,

$$\sigma(r, t) = \text{sgn}(D, C) = \prod_{i=1}^{k} \text{sgn}(D, C_i) = \prod_{i=1}^{k} \text{sgn}(D^i, C_i)$$

where $D^i$ is the restriction of $D$ to $C_0 \cup C_i$. Finally observe that, by Corollary 2.10, $\sigma(z_1, z_2) = 1$ if $z_2 = (1, 1)$. Hence, using Corollary 2.13, we have that $\prod_{i=1}^{k} \text{sgn}(D^i, C_i) = \prod_{j=0}^{g-1} \sigma(r^j, t^j)$, which we needed to prove. \qed

Theorem 2.15 Let $G$ be a proper $g$-graph. Then

$$\mathcal{P}(G, x) = \mathcal{L}_g(G, x) = \sum_{i=1}^{4^g} c(r(D_i))\mathcal{P}(D_i, M_0)$$

where $D_i$, $i = 1, \ldots, 4^g$, are the relevant orientations of $G$.

**Proof.** Let $P$ be a perfect matching of $G$. In each $\mathcal{P}(D_i, M_0)$, $x(P)$ has coefficient equal to $\text{sgn}(D_i, M_0\Delta P)$. By Corollary 2.13, $\text{sgn}(D_i, M_0\Delta P) = \sigma(r(D_i), t(M_0\Delta P))$. 

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Let
\[ K_g(t(M_0 \Delta P)) = \sum_{i=1}^{4^g} c(r(D_i))\sigma(r(D_i), t(M_0 \Delta P)) \].

Theorem follows from the following claim:

Claim. \( K_g(t(M_0 \Delta P)) = 1 \) for each \( t(M_0 \Delta P) \).

Proof of Claim. We proceed by induction on \( g \).

The basis of the induction when \( g = 1 \) is Corollary 2.12.

To prove the induction step we introduce the following notation. If \( z \) is a vector in \( \{-1, 1\}^{2g} \) then we let \( z = (z(0), \ldots, z(g-1)) \) where \( z(i) = (z_{2i+1}, z_{2i+2}) \).

We call two relevant orientations \( D, D' \) of \( G \) equivalent if \( (r(D)(1), \ldots, r(D)(g-1)) = (r(D')(1), \ldots, r(D')(g-1)) \). Clearly, the equivalence classes consist of 4 elements; let \( R_1, \ldots, R_{4^{g-1}} \) be the equivalence classes of the relevant orientations of \( G \) and let \( R_j = \{D^1_j, D^2_j, D^3_j, D^4_j\}, j = 1, \ldots, 4^{g-1} \).

Finally let \( r(D^j_k)(k) = r^j_k(k), k = 0, \ldots, g-1 \) and \( t = t(M_0 \Delta P) \). We have that
\[ K_g(t) = \sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r(D^j_i))\sigma(r(D^j_i), t). \]

Now, by Lemma 2.14, this equals to
\[ \sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r^j_i(0))c(r^j_i(1), \ldots, r^j_i(g-1)) \prod_{k=0}^{g-1} \sigma(r^j_i(k), t(k)). \]

By the definition of the equivalence classes, \( r^1_i(k) = r^2_i(k) = r^3_i(k) = r^4_i(k) \) for \( k \geq 1 \) and \( j = 1, \ldots, 4^{g-1} \). Hence, we let \( r^j_i(k) = r^j_k(k) \) and write the above summation as:
\[ \sum_{j=1}^{4^{g-1}} c(r^j(1), \ldots, r^j(g-1)) \prod_{k=1}^{g-1} \sigma(r^j(k), t(k)) \sum_{i=1}^{4} c(r^j_i(0))\sigma(r^j_i(0), t(0)) \].

The internal sum equals to 1 for each \( j = 1, \ldots, 4^{g-1} \) by the basis step of the induction, and hence, using Lemma 2.14 for the first sum, we can write the above summation as
\[\sum_{j=1}^{4^g-1} c(r^j(1), \ldots, r^j(g-1)) \sigma((r^j(1), \ldots, r^j(g-1), (t(1), \ldots, t(g-1))) = K_{g-1}(t(1), \ldots, t(g-1)) = 1,\]

by the induction hypothesis for \( g - 1 \).

As a consequence of Theorem 1.4 and Theorem 2.15, we get

Corollary 2.16 Let \( G \) be a proper \( g \)-graph. Then \( s(D_i, M_0) = s(D_j, M_0) \) for each \( i, j \in \{1, \ldots, 4^g\} \) and

\[P(G, x) = L'_g(G, x) = s(D_1, M_0) \sum_{i=1}^{4^g} c(r(D_i)) Pf(A(D_i))\]

where \( D_i, i = 1, \ldots, 4^g, \) are the relevant orientations of \( G \).

Corollary 2.17 Let \( G \) be a graph embeddable on an orientable surface of genus \( g \). Then \( P(G, x) \) may be expressed as a linear combination of \( 4^g \) Pfaffians of matrices \( A(D) \), where each \( D \) is an orientation of \( G \).

Proof. Orientable surface \( O_g \) of genus \( g \) may be obtained from \( S_g \) as follows: for each bridge \( B \), glue together the two segments in which \( B \) intersects the boundary of \( B_0 \), and delete \( B \).

If a graph \( G \) is embeddable on an orientable surface \( O_g \) of genus \( g \), then without loss of generality no vertex belongs to the boundary of \( B_0 \).

Then, splitting the boundaries back to the bridges, this gives a drawing of the graph on \( S_g \) such that each vertex lies on \( B_0 \) but some edges may use several bridges.

We construct a graph \( G' \) so that we replace each edge \( e \) which uses \( k \) bridges, \( k \geq 1 \), by a path \( P_e \) of edges \( (e_1, \ldots, e_{2k+1}) \) and vertices \( v_1, \ldots, v_{2k} \). We let \( x'_{e_1} = x_e \) and \( x'_{e_i} = 1 \) for each \( i > 1 \). Next we add edges so that the outer face of the planar part is a cycle. We let \( x'_e = 0 \) for each such edge \( e \).

Note that \( G' \) is a proper \( g \)-graph and \( P(G', x') = P(G, x) \).

By Theorem 2.15, \( P(G', x') \) may be written as a linear combination of \( 4^g \) Pfaffians of \( A(D') \), where each \( D' \) is a relevant orientation of \( G' \).

Claim. For each relevant orientation \( D' \) of \( G' \) there is an orientation \( D \) of \( G \) so that \( Pf(A(D')) = Pf(A(D)) \) or \( Pf(A(D')) = -Pf(A(D)) \).

Proof of Claim. We construct \( D \) from \( D' \) in two steps:
1. delete the edges $e$ of $G' - G$ with $x'_e = 0$,

2. for each edge $e$ of $G$ which was changed into a path $P_e$ of odd length in the construction of $G'$, orient $e$ in the direction in which an odd number of edges of $P_e$ is directed in $D'$: this is uniquely determined since $P_e$ has odd length.

If $P$ is a perfect matching of $G$ then there is a unique perfect matching $P'$ of $G'$ such that $x(P) = x'(P')$.

Observe that $\text{sgn}(D, P \Delta Q) = \text{sgn}(D', P' \Delta Q')$ for each pair of perfect matchings $P, Q$ of $G$. The claim now follows from Theorem 1.4.

End of Claim.

This finishes the proof of the Corollary.

\[\square\]

3 Exact Matching, Pfaffian Orientation, and Permanents.

The results of the previous sections have interesting algorithmic consequences.

**Theorem 3.1** Let $S$ be an orientable surface and $k$ a fixed positive integer. Let $\mathcal{G}$ be the class of graphs which may be embedded on $S$ and such that the edges are partitioned into at most $k$ classes and variables $x_e$ are equal in each class. Then $\mathcal{P}(G, x)$ may be determined efficiently for $G \in \mathcal{G}$. As a consequence, it is possible to verify efficiently whether $G \in \mathcal{G}$ is Pfaffian.

**Proof.** It follows from Theorems 2.15 and 2.17 that $\mathcal{P}(G, x)$ may be determined efficiently.

Concerning the problem of recognizing whether $G$ is Pfaffian, we proceed as follows: it was proved by Vazirani and Yannakakis (see the proof of theorem 3.1 of [12]) that there is an orientation $D$ of graph $G$ so that $G$ is Pfaffian if and only if $D$ is its Pfaffian orientation, and moreover $D$ may be constructed efficiently. Hence $Pf(A(D))$ equals to the number of the perfect matchings of $G$ if and only if $G$ is Pfaffian, and it means that we can decide efficiently whether a graph is Pfaffian once we can compute efficiently its number of perfect matchings.

\[\square\]
In particular the following well-known problem may be solved efficiently for the graphs embeddable on an arbitrary fixed orientable surface:

**Exact Matching Problem.** Given a graph $G$ with some edges coloured red, and a number $h$. Find out whether $G$ has a perfect matching with exactly $h$ red edges.

Next we consider permanents of square matrices.

In 1913, Polya ([8]) suggested computing the permanent of a matrix $A$ by changing the signs of some entries of $A$ so that the determinant of the resulting matrix equals the permanent of $A$. Let us call a 0,1-matrix $A$ *convertible* if such a change is possible.

Szegö ([10]) pointed out in the same year that not all matrices are convertible.

This may be explained nowadays using a complexity argument. There is an efficient algorithm to compute the determinant, while Valiant proved that the problem of computing the permanent of a 0,1 matrix is #$P$-complete (see [11]).

The computational problem of recognition of convertible matrices has been proved recently to admit a polynomial algorithm by McCuaig, Robertson, Seymour and Thomas (see [7]).

The recognition of convertible matrices is equivalent to the problem of recognition of bipartite Pfaffian graphs, and to the ‘Even Cycle Problem’: given a directed graph, decide whether it contains a directed cycle of even length.

Let $A$ be a square matrix. Denote by $G(A)$ the bipartite graph whose two bipartition classes are indexed by the rows and the columns of $A$, and for each edge $ij$, $a_{ij} = x_{ij}$. Then $\text{per}(A) = \mathcal{P}(G(A), x)$.

This means that Corollary 2.17 provides a new combinatorial way to compute permanents of square matrices: $\text{per}(A)$ may be written as a linear combination of $4^g$ terms of form $\mathcal{P}f(A(D))$, where $D$ is an orientation of $G(A)$ and $g$ is the genus of $G(A)$.

Since $G(A)$ is a bipartite graph, the non-zero entries of $A(D)$ belong to two blocks $A_1, A_2$, where $A_1$ is obtained from $A$ by changing the sign of some entries and $A_2 = -A_1$. Moreover $|\mathcal{P}f(A(D))| = |\text{det}(A_1)| = |\text{det}(A_2)|$ by Theorem 1.5.

This means that the method of Polya may be completed as follows:

**Corollary 3.2** Let $A$ be a square matrix. Then $\text{per}(A)$ may be expressed as a linear combination of terms of form $\text{det}(A^i)$, $i = 1, ..., 4^g$, where each $A^i$ is obtained from $A$ by changing the sign of some entries.
Concluding Remarks. In a continuation of this paper which is in preparation we consider the following related problems:

1. The generating function of $T$-joins and edge-cuts,
2. The evaluation of Tutte polynomial and the weight enumerator of a binary code,
3. A theory of crystal structures and application of the proposed method to solve the Ising problem for three-dimensional crystal structures.

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References


