THE COMPLEXITY OF $H$-COLOURING OF BOUNDED DEGREE GRAPHS

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Abstract

We investigate the complexity of the $h$-colouring problem, and, more generally, of the $H$-colouring problem, restricted to graphs of bounded degree. While the general problems are almost always \textit{NP}-complete, we present some surprising polynomial algorithms for several of these restricted colouring problems. We also give a number of \textit{NP}-completeness results, and pose some open problems. One of these may be viewed as the complement of an algorithmic version of the theorem of Brooks.
1. Introduction

Let \( k \) be a positive integer; graph \( G \) is \( k \)-bounded if all degrees in \( G \) are at most equal to \( k \). Colouring of \( k \)-bounded graphs is addressed by the well known theorem of Brooks, which asserts that, for \( k \geq 3 \), each \( k \)-bounded graph, different from \( K_{k+1} \), is \( k \)-colourable. This fact leads to a trivial polynomial algorithm for 3-colourability of \( 3 \)-bounded graphs: a graph is \( 3 \)-colourable unless it is isomorphic to a \( K_4 \) (an easy check), in which case it is not \( 3 \)-colourable. Since testing 2-colourability is polynomial (for all graphs), the \( h \)-colouring problem for \( 3 \)-bounded graphs is polynomial for any \( h \).

On the other hand, for \( 4 \)-bounded graphs, while it is still true that Brooks’ theorem guarantees \( 4 \)-colourability (except for \( K_5 \)), not all \( h \)-colourability problems are trivial. In fact, we have the following observation:

**Theorem 1.1.** Deciding whether a given \( 4 \)-bounded graph is \( 3 \)-colourable is \( NP \)-complete.

*Proof.* The class of \( 4 \)-bounded graphs contains all line graphs of \( 3 \)-regular graphs. Holyer proved that deciding whether a \( 3 \)-regular graph has a proper \( 3 \)-edge colouring is \( NP \)-complete. \( \blacksquare \)

The complexity of more general notions of colouring has been subject of much recent interest [2, 4, 6, 5]. Let \( H \) be a fixed graph; an \( H \)-colouring of \( G \) is a mapping \( c : V(G) \to V(H) \) which preserves adjacency, i.e., such that \( gg' \in E(G) \) implies \( c(g)c(g') \in E(H) \). An \( H \)-colouring of \( G \) is also called a homomorphism of \( G \) to \( H \). The \( H \)-colouring problem asks whether a given graph \( G \) is \( H \)-colourable. This is a generalization of the usual \( h \)-colouring problem, since a \( K_h \)-colouring of \( G \) corresponds to an \( h \)-colouring of \( G \). However, it turns out that this generalization introduces essentially no new polynomial cases - other than modifications of the usual 2-colouring problem: It is proved in [4] that \( H \)-colouring is \( NP \)-complete unless \( H \) is bipartite. On the other hand, the \( H \)-colouring problem for directed graphs, exhibits interesting duality properties and resulting polynomial algorithms [5].

Our interest in studying the complexity of the \( H \)-colouring problem was motivated by the following result of Häggkvist and Hell [3]:

**Theorem 1.2.** For every connected graph \( A \) there exists a graph \( U[A] \) with the following property: a \( k \)-bounded graph \( G \) is \( U[A] \)-colourable if and only if \( A \) is not \( G \)-colourable.

Since \( A \) is fixed, there is a polynomial time algorithm to test whether \( A \) is \( G \)-colourable: There are only polynomially many (in terms of the size of \( G \)) mappings from \( A \) to \( G \), and we can quickly test each of them to see if it is a \( G \)-colouring of \( A \). (The fastest known algorithm to check for the existence of fixed subgraphs is due to Nesetril and Poljak [7] and is based on matrix multiplication).

Thus, for each connected graph \( A \), there exists a polynomial time algorithm to check whether a \( k \)-bounded graph \( G \) is \( U[A] \)-colourable. Hence, many nontrivial \( H \)-colouring problems for \( k \)-bounded graphs are polynomial time solvable.

2. Polynomial cases

We now return to considering the graphs \( U[A] \). For simplicity we shall consider only cubic graphs in this section. Moreover, we shall take \( A = K_3 \). Note that saying that \( K_3 \) is not \( G \)-colourable
is equivalent to saying that $G$ is triangle free, since triangles can be mapped only to triangles under a homomorphism. Thus Theorem 1.2 says that a cubic graph $G$ is $U[K_3]$-colourable if and only if $G$ is triangle free.

We now describe a general construction that may be used to define $U[K_3]$. Let $X$ be any set. We define a graph $U_X$ as follows:

- The vertices of $U_X$ are ordered pairs $(x, T)$ where $T$ is a 3-element subset of $X$ and $x \notin T$;
- Two vertices $(x, T)$ and $(x', T')$ are adjacent if and only if $T \cap T' = \emptyset$ and $x \in T', x' \in T$.

(We note this construction is related to the symmetric line graphs of oriented hypergraphs as studied by Ausiello et al [1].)

We now claim that when $X$ has at least 22 elements, then $U_X$ can be chosen to be $U[K_3]$. In other words, every triangle free cubic graph $G$ admits a $U_X$-colouring. Indeed, suppose that $G$ is triangle free and cubic. Then it is easy to see that the graph $G^2$ constructed from $G$ by making adjacent all vertices at distance less than or equal to 3 has maximum degree 21. Hence, by Brooks’ theorem, it admits an $|X|$-colouring, say $c$. We assign to each vertex $v \in V$ of colour $i$, the image $(i, \{j, k, l\})$ where $j, k, l$ are the three (distinct) colours of the neighbours of $v$ in $G$. We shall show that this mapping preserves adjacency. Let $xy$ be an edge of $G$ and let $i, j$ be the colours of $x, y$, respectively, in $c$. Their images are $(i, \{j, k, l\})$ and $(j, \{i, p, r\})$ and they are adjacent in $U_X$; in fact, there are 4 distinct neighbors of $x$ and $y$, since $G$ is triangle free, and they all receive different colours in $c$ since their distances are at most 3. Thus, $\{j, k, l\} \cap \{i, p, r\} = \emptyset$. Observe that we have not only found that an $U_X$-colouring exists, but we have actually constructed it (in polynomial time).

We shall show now that the graphs $U_X$ tend to have high chromatic numbers.

**Theorem 2.1.** 1. For every $k$ there exists $X$ such that the chromatic number of $U_X$ is at least $k$.

2. For every $X$ with at least 15 elements the chromatic number of $U_X$ is at least 4.

**Proof.** Suppose $X = \{1, \ldots, n\}$.

**Part 1.** Consider the following graph $S_X$: the vertices of $S_X$ are all 3-element subsets of $X$, and two such subsets, say $\{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$, and $\{y_1, y_2, y_3\}$ with $y_1 < y_2 < y_3$, are adjacent if $x_2 = y_1$ and $x_3 = y_2$. This is a variant of a general construction of type graphs defined in [8]. Note that $S_X$ is a directed graph but we will also call $S_X$ its underlying undirected graph. It follows from the Ramsey theorem for partition of triples that the chromatic number of $S_X$ may be arbitrarily large if $n$ is large.

We now claim that $S_X$ is isomorphic to a subgraph of some $U_X$ where $X'$ contains $X$. Let $f$ be a bijection from the set of all 3-element subsets of $X$ to a set $Y$ disjoint from $X$ and let $X' = X \cup Y$. Now, for $\{x_1, x_2, x_3\} \in V(S_X)$ with $x_1 < x_2 < x_3$, we let $g(x_1, x_2, x_3) = (x_2, \{x_1, x_3, f(x_1, x_2, x_3)\}) \in V(U_{X'})$. It is easy to see that $g$ is an injective homomorphism from $V(S_X)$ to $V(U_{X'})$, i.e. $S_X$ is isomorphic to a subgraph of $U_{X'}$, and hence, the chromatic number of $U_{X'}$ is at least as large as the chromatic number of $S_X$. (In fact, it is easy to see $g$ is an isomorphism onto an induced subgraph of $U_{X'}$).

**Part 2.** Suppose that $U_X$ has a proper 3-colouring $c : V(U_X) \to \{1, 2, 3\}$. Consider a 4-element subset $M$ of $X$, say $M = \{x_1, x_2, x_3, x_4\}$, and the four vertices of $U_X$ corresponding
to $M$, namely $(x_1, \{x_2, x_3, x_4\})$, $(x_2, \{x_1, x_3, x_4\})$, $(x_3, \{x_1, x_2, x_4\})$ and $(x_4, \{x_1, x_2, x_3\})$. The 3-coloring $c$ assigns the same colour to some two of these vertices. We let $c(M)$ be such a colour.

Next consider the set $Z$ consisting of the ordered pairs $(A, M)$ where $M$ is a 4-element subset of $X$ and $A$ is a 2-element subset of $M$, say $a_1, a_2$, such that $c(a_1, M-a_1) = c(a_2, M-a_2) = c(M)$. Each $M$ admits at least one $A$ such that $(A, M) \in Z$ by the definition of $c(M)$. Thus $Z$ has at least $\binom{n}{4}$ elements.

We now claim that each $A$ can occur in at most $n-3$ elements $(A, M)$ of $Z$. In fact, consider two elements $(A, M)$ and $(A, M')$ of $Z$ and assume that $A = \{a_1, a_2\}$, $M = \{a_1, a_2, u, v\}$ and $M' = \{a_1, a_2, x, y\}$. Then $c(a_1, \{a_2, u, v\}) = c(M) = c(a_2, \{a_1, x, y\})$ and hence cannot be adjacent in $U_X$. Therefore $\{u, v\} \cap \{x, y\} \neq \emptyset$. Thus the set of 2-element subsets $B$ of $X-A$ such that $(A, A \cup B) \in Z$ has the property that any two subsets $B$ intersect, and hence has at most $n-3$ elements. Hence, $Z$ contains at most $\binom{n}{2}(n-3)$ elements.

We conclude that $\binom{n}{4} \leq |Z| \leq \binom{n}{2}(n-3)$ and then $n \leq 14$. □

According to the above theorem, the only known triangle free nonbipartite graphs $H$, for which we have a polynomial time solvable $H$-colouring problem for cubic graphs, namely $H = U_X$ and $|X| = 22$, have chromatic number greater than 3. This motivates the conjecture in the last section of the paper.

3. NP-complete cases

In this section, we show that for several families of triangle free nonbipartite graphs $H$, the $H$-colouring problem for 3-bounded graphs is NP-complete.

Let us start by considering the $C_{2k+1}$-colouring problem. We prove that this problem is NP-complete for the class of 3-bounded graphs.

We reduce a graph $G$ into a graph $G^s$ of degree at most three as follows: we replace each vertex $x$ of degree $t$ in $G$ with $t$ odd cycles $C^j = \{v_1^j, \ldots, v_{2k+1}^j\}$ (numbered clockwise), $j = 1, \ldots, t$, in $G^s$ such that each edge $e_j \in \delta(x)$ is incident to $v_1^j$, and moreover $v_s^j v_3^j = v_s^{j+1} v_3^{j+1}$ for $j = 1, \ldots, t$. Then:

**Theorem 3.1.** $G$ is $C_{2k+1}$-colourable if and only if $G^s$ is $C_{2k+1}$-colourable.

**Proof.** It suffices to observe that any valid colouring of $G^s$ assigns the same colour to the vertices $v_1^j$, $j = 1, \ldots, t$. Hence, any valid colouring of $G^s$ leads to a valid colouring of $G$ and vice versa. □

Since the $C_{2k+1}$-colouring problem is NP-complete for graphs with no degree constraints [4], the NP-completeness of our problem follows.

More generally, we can prove NP-complete every $H$-colouring problem where the graph $H$ has odd girth $2k+1$, every vertex belongs to a $C_{2k+1}$ and no two copies of $C_{2k+1}$ share more than one edge. For simplicity we describe this result in the case $k = 2$.

**Theorem 3.2.** Let $H$ be a triangle free graph in which each vertex belongs to a pentagon, and in which no two pentagons share more than one edge. Then the $H$-colouring problem for 3-bounded graphs is NP-complete.
Proof. Given a graph $G = G_0$ with $n$ vertices and no degree bound (i.e., $n$-bounded), we construct in polynomial time a sequence $G_0, G_1, \ldots, G_l$ of graphs such that $G_l$ is 3-bounded and $G \rightarrow H$ if and only if $G_l \rightarrow H$.

Suppose $G_i$ has already been constructed, and suppose it is $k$-bounded. Then $G_{i+1}$ is constructed by replacing each vertex $v$ of $G_i$ with a gadget $\Gamma(v)$ as follows: the edges at each $v$ are partitioned into $t = \lceil \sqrt{k} \rceil$ groups of size approximately $\sqrt{k}$ and each group is made incident with a separate vertex of $\Gamma(v)$ as suggested by the figure.

Observe that:

i) $G_{i+1}$ is $2 + \lceil \sqrt{k} \rceil$-bounded;

ii) $|V(G_{i+1})| = |V(G_i)|(3 + 2\lceil \sqrt{k} \rceil)$;

iii) $G_{i+1} \rightarrow H$ if and only if $G_i \rightarrow H$.

The first two items follow immediately from the construction of $G_{i+1}$. We will now prove (iii). Suppose first that $G_i$ is $H$-colourable and let $c(v)$ be the colour of a vertex $v$ of $G_i$. Since each vertex of $H$ belongs to a pentagon, we have that $c(v)$ belongs to a pentagon, say $\{c(v) = a, b, c, d, e\}$. Then we may colour $\Gamma(v)$ as follows: all vertices $a_j, j = 1, \ldots, \lceil \sqrt{k} \rceil$, receive the colour $c(v) = a$, all vertices $b_j$ receive colour $b$ and so on. It is easy to see that this induces an $H$-colouring of $G_{i+1}$.

Let us suppose conversely that $G_{i+1}$ is $H$-colourable. We claim that all vertices $a_j$, for any $j$, have the same colour in any $H$-colouring. In fact, since $H$ is triangle free, each pentagon of $G_{i+1}$ maps into a pentagon of $H$ and if two $a_j$'s were different then their corresponding pentagons in $\Gamma(v)$ would be mapped into two different pentagons of $H$ sharing two consecutive edges, contradicting the assumption. Hence, any $H$-colouring of $G_{i+1}$ may be easily transformed into an $H$-colouring of $G_i$.

What remains to show is that the construction converges to a 3-bounded graph in a number of steps which is polynomial in $n$. Consider the recurrence relation:

$q_0 = n$

$q_{i+1} = 2 + \lceil \sqrt{q_i} \rceil$

While $q_i$ is not too small ($q_i > 50$, say) this recurrence is decreasing like $n^{1/3}$, i.e., quite fast. In fact, after $O(\log \log n)$ steps we have $q_i = O(1)$. When $q_i$ becomes smaller than 6, the influence
of the addition of 2 and of the ceiling affects the situation, and in fact $2 + |\sqrt{3}| = 5$. However, for 6-bounded (and hence also 5-bounded) graphs the construction yields a 4-bounded graph $G_{l-1}$.

Finally, we construct $G_l$ from $G_{l-1}$ as follows:

It is easy to verify that $G_l$ is 3-bounded and that $G_{l-1} \rightarrow H$ if and only if $G_l \rightarrow H$.

Thus, the above construction provides a reduction from a graph $G = G_0$ to a graph $G_l$ which is 3-bounded and such that $G \rightarrow H$ if and only if $G_l \rightarrow H$. It can be accomplished in polynomial time (in $n$) since there are $O(\log n)$ intermediate graphs and the total number of vertices of $G_l$ is only $O(n^{1+1/\log n + \ldots}) = O(n^2)$. Since the $H$-colouring problem is $NP$-complete for general graphs unless $H$ is bipartite [4], the theorem follows.

Note that, in the above theorem, $H$ may have arbitrary large chromatic number. As a consequence of the above construction we can prove that, for any integers $k, g$, there exists a graph $H_k$, with $\chi(H_k) = k$, such that $H_k$-colourability is $NP$-complete for cubic graphs of girth $g$.

4. Conclusions

We have investigated the computational complexity of the $H$-colouring problem for the class of 3-bounded graphs. From our results, some questions arise naturally. In particular, as a consequence of Theorem 2.1, we know that the only triangle free nonbipartite graphs $H$ for which we have a polynomial time solvable $H$-colouring problem for cubic graphs, namely the graphs $H = U_X$ with $|X| \geq 22$, have chromatic number greater than 3. Since graphs $H$ with chromatic number 3, and which contain triangles, have polynomial time solvable $H$-colouring problems for cubic graphs (by virtue of Brooks' theorem), we can ask whether all problems of this type, which are not solvable by Brooks' theorem techniques, are in fact $NP$-complete. As shown in Section 3, several classes of triangle free graphs $H$ for which the corresponding $H$-colouring problem for 3-bounded graphs is $NP$-complete exist. Perhaps the simplest unsolved case not covered by Theorem 3.2 is the Petersen graph; we conjecture that this is also an $NP$-complete case. The results of this paper suggest the following conjecture:
Conjecture. Let $H$ be a triangle free graph with chromatic number 3. Then the $H$-colouring problem for 3-bounded graphs is NP-complete.

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References