

# **0-1/2 CUTS: A COMPUTATIONAL STUDY**

**G. Andreello, DEI, University of Padova**

**A. Caprara, DEIS, University of Bologna**

**M. Fischetti, DEI, University of Padova**



**Thank God! Maybe this time ... it works ...**



**Hey ... those are **MY** Aussois pictures ...**

**Hmmm, this BLOSSOM stuff is so boring ...  
but let me postpone my nap ...  
maybe new pictures will pop up!!**

## 0-1/2 CUTS

*Usual stuff...*

- A polyhedron

$$P := \{x \in \mathbb{R}^n : Ax \leq b\}, \quad \text{where } A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$$

- and the convex hull of its integer points

$$P_I := \text{conv}\{x \in \mathbb{Z}^n : Ax \leq b\}$$

- Chvátal-Gomory cuts for  $P_I$  :

$$\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$$

valid for each  $\lambda \geq 0 : \lambda^T A \in \mathbb{Z}^n$

w.l.o.g.  $\lambda \in [0,1)^m : \lambda^T b - \lfloor \lambda^T b \rfloor > 0$

- **0-1/2 cuts:** as before, but  $\lambda \in \{0, 1/2\}^m$

- **0-1/2 CUT separation:** given  $x^* \in P$ , find

$$\lambda \in \{0, 1/2\}^m : \lambda^T A \in Z^n, \lambda^T A x^* > \lfloor \lambda^T b \rfloor$$

- Can be **rephrased** as: find  $\lambda \in \{0, 1/2\}^m$  such that

- $\lambda^T A \in Z^n, \lambda^T b - \lfloor \lambda^T b \rfloor = 1/2$

- **slack**  $s^* := b - A x^* : \lambda^T s^* < 1/2$

- or equivalently: find  $2\lambda =: \mu \in \{0, 1\}^m$  such that

$$\mu^T A \text{ is } \mathbf{even}, \mu^T b \text{ is } \mathbf{odd}, \text{ and } \mu^T s^* < 1$$

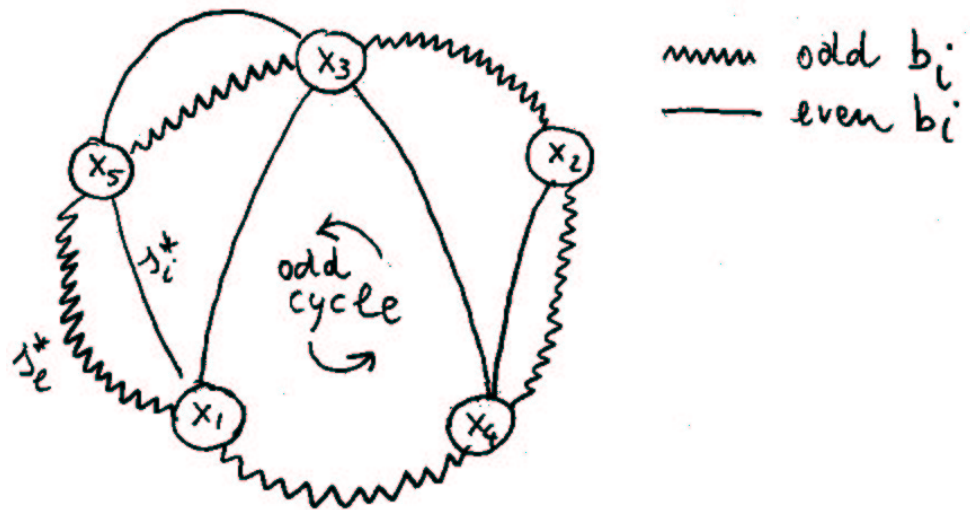
*0-1/2 cuts express parity requirements of the integer solution...*

- **0-1/2 CUT separation:** NP-hard in general, but polynomially solvable in some special cases

- In particular when **each row of matrix  $A$  contains no more than 2 odd coefficients**

- In this case, each inequality of the original system  $a_i^T x \leq b_i$  is characterized by:

- The **two** specific variables for which the coefficient is **odd**
- an **odd/even** r.h.s.  $b_i$
- **slack**  $s_i^* := b_i - a_i^T x^*$



- **0-1/2 CUT separation:** Find and **odd cycle** in the **separation graph** of weight less than 1 (polynomial)

**WHAT if some inequalities have 3 or more odd coefficients?**

**Eureka, you RELAX**



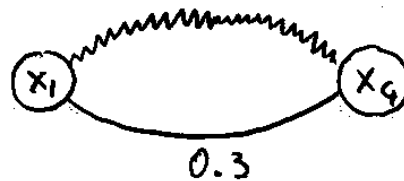
Admittedly, it does sound as good as  
*“Eureka, you shrink”* but, you know, **HE** copyrighted everything...

**WHAT if some inequalities have 3 or more odd coefficients? Relax it...**

$$\begin{array}{r}
 \downarrow \qquad \qquad \qquad \downarrow \\
 x_1 + 2x_2 + x_3 - x_4 \leq 1 \rightsquigarrow \text{slack } 0.1 \\
 \qquad \qquad \qquad x_3 \leq 1 \rightsquigarrow \text{slack } 0.2
 \end{array}$$

---


$$x_1 + 2x_2 + 2x_3 - x_4 \leq 2 \rightsquigarrow \text{slack } 0.3$$



~~~~~ best odd  
 ——— best even

- **Our heuristic 0-1/2 CUT separation:** do it systematically, by dynamic programming (polynomial)
- **Nice theoretical applications:** Linear Ordering, Clique Partitioning, etc.
- **General separation tool** to be invoked within a ILOG-Cplex 8.1 callback for general 0-1 ILP's

## Cut Selection Criteria

Cuts  $\gamma \implies \underline{a}^T \underline{x} \leq b$  can be characterized in terms of:

- **Euclidean distance** from the LP solution  $x^*$

$$\text{dist}(\underline{x}^*, \gamma) = \frac{|\underline{a}^T \underline{x}^* - b|}{\|\underline{a}\|}$$

- **Angle** between two cuts  $\gamma_i$  and  $\gamma_j$  (=1 parallel, =0 orthogonal)

$$\text{angle}(\gamma_i, \gamma_j) = \frac{|\underline{a}_i^T \underline{a}_j|}{\|\underline{a}_i\| \|\underline{a}_j\|}$$

A set of cuts  $\{\gamma_i : i = 1, \dots, k\}$  is supposed to be effective if

- (1)  $\forall i : \text{dist}(\underline{x}^*, \gamma_i) \geq \text{min\_dist}$
- (2)  $\forall i, j : \text{angle}(\gamma_i, \gamma_j) \leq \text{max\_angle}$



## Comparison of two algorithms X and Y over a given test bed

- The **sum of solution times**: useful, but
  - no distinction between solved and time-out instances
  - time-out instances is likely to dominate the sum.
- A **detailed table** with n. of nodes, cpu times, % gaps, etc. may be difficult to understand for **humans** (other than Paolo, of course ...)
- *à la CPLEX* (Bixby et al.): decide a time-limit and:
  1. discard the instances solved in less than 20 seconds (say) by both X and Y (too easy)
  2. compute the average **speedup  $\text{time}(X) / \text{time}(Y)$**  on the instances solved by both X and Y
  3. count the n. of instances solved by X but not by Y, and compute the average  $\text{time}(X)$
  4. count the n. of instances solved by Y but not by X, and compute the average  $\text{time}(Y)$
  5. count the n. of instances not solved by X or Y



**Average: geometric mean** is more robust than the arithmetic one when the standard deviation is high

$$\text{Arithmetic mean: } \frac{1}{n} \left( \sum_i \text{speedup}_i \right) \quad \text{Vs.} \quad \text{Geometric mean: } \left( \prod_i \text{speedup}_i \right)^{\frac{1}{n}}$$

## Our test bed

- IP formulations of about 100 **SAT** problems from DIMACS Challenge (**feasibility** problems)
- Objective function randomly generated (sum of some of the constraints of the formulation)
- Optimization stopped as soon as the first integer feasible solution is found

Main features:

- constraint matrix  $A$  has 0-1 coefficients (nice combinatorial structure expected)
- number of constraints is larger than number of variables
- **standard cuts from ILOG-CPLEX arsenal are ineffective (more combinatorial ones needed)**

## Computational results

Reference configuration for all runs:

**Default ILOG CPLEX 8.1 with a null cut callback**

(null) cut callbacks force CPLEX to disable some preprocessing

### Alternative CPLEX 8.1 configurations

|                        | Solved | Speedup | Solved by neither | Solved only by reference | Solved only by current |
|------------------------|--------|---------|-------------------|--------------------------|------------------------|
| Default w/out callback | 19     | 0.89    | 16                | 4                        | 0                      |
| Default w/out cuts     | 19     | 1.10    | 18                | 3                        | 0                      |
| Null callbk w/out cuts | 16     | 1.42    | 20                | 1                        | 2                      |

**But this is Cplex alone; what about 0-1/2 cuts embedded into Cplex?**

**Much worse than Cplex alone, I presume !!**



## 0-1/2 CUTS

- **Skip\_step:** Cut separation applied at each **skip\_step** backtrackings (Cplex-like)
- **Static cuts:** Cuts are never removed from the LP (Cplex-like)
- **Internal Pool:** used to store the 0-1/2 generated in previous calls, so as to increase the degree of freedom in the choice of the (few) cuts to add statically to the LP
- **Cut recombination:** allow for high-rank 0-1/2 cuts

### 0-1/2 CUT: Initial configurations

**Skip\_step=1, no other criteria, 5x or 0.3x n. of row in the original LP**

|                   | Solved | Speedup | Solved by neither | Solved only by Cplex 8.1 | Solved only by 0-1/2 CUT |
|-------------------|--------|---------|-------------------|--------------------------|--------------------------|
| Lots of cuts (5x) | 19     | 0.35    | 14                | 5                        | 8                        |
| Few cuts (0.3 x)  | 18     | 1.07    | 17                | 1                        | 5                        |



**Relax, Bob, this is not the end of the story ...**

**Fixing max\_angle= 0.3 and min\_dist = 0.2**

### **Tuning skip\_step**

**skip\_step = \*, max\_angle = 0.3, min\_dist = 0.2**

| <b>Skip_step</b> | <b>Solved</b> | <b>Speedup</b> | <b>Solved by<br/>neither</b> | <b>Solved only<br/>by Cplex 8.1</b> | <b>Solved only<br/>by 0-1/2 CUT</b> |
|------------------|---------------|----------------|------------------------------|-------------------------------------|-------------------------------------|
| 1                | 16            | 2.76           | 4                            | 4                                   | 6                                   |
| 4                | 15            | 3.80           | 8                            | 5                                   | 9                                   |
| 5                | 17            | 3.42           | 6                            | 4                                   | 9                                   |
| 7                | 13            | 2.35           | 9                            | 6                                   | 12                                  |
| 10               | 16            | 3.03           | 8                            | 2                                   | 8                                   |
| 13               | 17            | 2.37           | 7                            | 4                                   | 9                                   |
| 16               | 15            | 3.31           | 6                            | 6                                   | 9                                   |

## Tuning max\_angle (=0 only orthogonal cuts, =1 all cuts)

skip\_step = 4, max\_angle= \*, min\_dist = 0.2

| Max_angle | Solved | Speedup | Solved by neither | Solved only by Cplex 8.1 | Solved only by 0-1/2 CUT |
|-----------|--------|---------|-------------------|--------------------------|--------------------------|
| 0.001     | 14     | 3.71    | 16                | 4                        | 9                        |
| 0.1       | 13     | 3.91    | 16                | 5                        | 9                        |
| 0.3       | 15     | 3.80    | 8                 | 5                        | 9                        |
| 0.3       | 13     | 3.14    | 13                | 5                        | 12                       |
| 0.5       | 15     | 3.05    | 15                | 2                        | 10                       |
| 0.7       | 16     | 2.34    | 15                | 2                        | 10                       |
| 0.9       | 14     | 3.16    | 18                | 4                        | 7                        |
| 1.0       | 14     | 3.67    | 18                | 4                        | 8                        |



## Tuning min\_dist

skip\_step = 4, max\_angle= 0.1, min\_dist = \*

| Min_dist | Solved | Speedup | Solved by neither | Solved only by Cplex 8.1 | Solved only by 0-1/2 CUT |
|----------|--------|---------|-------------------|--------------------------|--------------------------|
| 0.00     | 15     | 2.59    | 18                | 3                        | 8                        |
| 0.05     | 14     | 3.35    | 15                | 5                        | 11                       |
| 0.10     | 14     | 2.84    | 14                | 6                        | 12                       |
| 0.15     | 14     | 3.35    | 15                | 5                        | 11                       |
| 0.20     | 13     | 3.91    | 16                | 5                        | 9                        |
| 0.25     | 13     | 3.99    | 15                | 5                        | 11                       |

## Conclusions

On specific ILP instances with a strong combinatorial structure, the use of 0-1/2 cuts can result into a considerable speed-up

**STRONGLY RECOMMENDED FOR HARD ILP'S**

SERVIZIO SANITARIO NAZIONALE REGIONE VENETO ASL 116

Roberto Sgarbi  
Via Matteo Inglese  
Venezia

Valpurga  
Ripetitive

2010/10/10

Farmacia Civica S. NTA 172  
2010/10/10

Means: 0.3 mg of 0-1/2 cuts after each meal

**Any comment?**



**I hope they will not expect I include this \$%£@# in my next book!!**



**I hope they will not expect I include this \$%£@# in the new Cplex release!!**