On the Stabbing Number of
Matchings, Trees, and Triangulations

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The Problem(s)

Set $P$ of $n$ points in the plane, $n$ even

Stabbing number of a line:
# intersections with matching $M$

Stabbing number of $M$:
maximum over all lines

Stabbing number of $P$:
minimum over all $M$

Perfect matchings
Spanning trees
Triangulations

<table>
<thead>
<tr>
<th>two directions</th>
<th>minimize average</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$ directions</td>
<td>minimize maximum</td>
</tr>
<tr>
<td>all directions</td>
<td></td>
</tr>
</tbody>
</table>


Why Stabbing?

Ray shooting queries, e.g., Hershberger and Suri 1993

Hidden surface removal
Collision detection

Interesting link: \(m\)-guillotine subdivisions (Mitchell 1999)
**Previous Work**

**Worst case bounds** on the stabbing number

Chazelle 1988, Agarwal 1992, Welzl 1993: $\Theta(\sqrt{n})$ for trees

de Berg, van Kreveld 1994
de decompose a simple rectilinear polygon into rectangles, $\Theta(\log n)$

Shewchuk 2002: Delaunay tetrahedralizations, $\Theta(n^2)$ example

No algorithmic results!
(Counter) Intuition for Matchings

- Recursive approaches
  
- Independent sets

- Minimum weight matching
Theorem Orthogonal stabbing number for matchings is strongly \( \mathcal{NP} \)-hard.

Proof Reduction from 3SAT.
Complexity II: Clauses

True and false setting of a literal

\[ (x_a \lor x_b \lor \bar{x}_c) \]
\[ (x_a \lor \bar{x}_b \lor \bar{x}_c) \]

Mechanism: Three false literals make a stabbing number of 6!
∃ matching of stabbing number 5  \iff\  3SAT expression is satisfiable
Corollaries

**Theorem** There is no $\alpha$-approximation algorithm for minimizing the orthogonal stabbing number for matchings with $\alpha < 6/5$.

**Theorem** Minimizing the stabbing number for matchings in *all* directions is weakly $\mathcal{NP}$-hard.

There is a similar construction for spanning trees.
**Complexity IV: Key Idea for Triangulations**

**Lemma** The horizontal stabber of this row stabs any triangulation of $P$ at least $a + b - 2$ times.

**Red edges** are barriers!

Variables:

\[ x_i \quad \bar{x}_i \]
Complexity V: Overall Layout for Triangulations
Linear Programs: Matchings and Trees

\[ \text{minimize } k \]
\[ \sum_{ij \in \delta(i)} x_{ij} = 1 \quad i \in V \]
\[ \sum_{ij \in \delta(S)} x_{ij} \geq 1 \quad S \subseteq V, \ |S| \text{ odd} \]
\[ \sum_{ij : ij \cap \ell \neq \emptyset} x_{ij} \leq k \quad \text{stabbing line } \ell \]
\[ x \geq 0 \]

\( \Theta(n) \) orthogonal stabbing constraints
\( \Theta(n^2) \) stabbing constraints for “all” directions

LP can be solved in polynomial time
IP is the first exact approach to the minimization problem
Integrality Gap

\[ \Theta(n) \text{ points} \]
\[ \Theta(\sqrt{n}) \text{ rows} \]
\[ \Theta(\sqrt{n}) \text{ columns} \]

fractional \( \Theta(1) \)
integral \( \Theta(\sqrt{n}) \)
Optimal Fractional Matching

30 points
$k = 2.498$
optimal $k = 3$
heuristic $k = 4$
Larger Instances: 400 points
Remarks, Current Research

First approach to minimizing some stabbing number

First answers to computational complexity issues

Bridge between computational geometry and math programming

Make use of the empirically excellent LP bound

Lower bounds for the stabbing number for triangulations?