On the Node Flow Cone of an Acyclic Directed Network

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The Problem

Given:

- an acyclic digraph **$G = (V, A)$** (no directed cycles)
- nonempty node subsets **S** $\subseteq$ V of **sources**
  T $\subseteq$ V of **sinks**

let $\mathcal{P}$ be the set of all directed paths **P** $= (v_1, v_2, \ldots, v_k)$ in G with $v_1 \in S$ and $v_k \in T$, called the ***S*-*T*-paths** in G

A vector $y \in \mathbf{R}^{\mathcal{P}}_{+}$ is a **path flow vector**, where $y_P \geq 0$ is the flow on path $P \in \mathcal{P}$

To every path flow vector $y \in \mathbf{R}^{\mathcal{P}}_{+}$ associate its **node flow vector $j\ (y)$**$\in \mathbf{R}^{V}_{+}$ defined by

$$j\,(y)_u \;=\; \sum \{\, y_P :\ \text{all } P \in \mathcal{P} \text{ with } u \in P \,\} \quad \text{for all } u \in V.$$

thus $n\,(y)_u$ is the **total flow through node *u***

The **node flow cone *X*** of (G, S, T) is

$$X \;=\; \{\, x \in \mathbf{R}^{V}_{+} \;:\; x = j\,(y) \text{ for some } y \in \mathbf{R}^{\mathcal{P}}_{+} \,\}\ .$$

Motivation: production planning models for make-to-order systems

inspired by: Michael O. Ball, Chien-Yu Chen & Zhen-Ying Zhao, "Material Compatibility Constraints for Make-to-Order Production Planning," University of Maryland, 2001.

Nodes $v \in V$ correspond to **components**

S to the set of all possible first components (e.g., in assembly order)

T to the set of all possible last components

Assume that **component compatibility constraints** may be represented by the acyclic digraph $G = (V, A)$ in such a way that component sequence $(v_1, v_2, \ldots, v_k)$ defines a (feasible) component **configuration** if and only if $(v_1, v_2, \ldots, v_k)$ is a directed path in G.

(That is, component compatibility constraints only arise between "consecutive pairs" of components $v_j \, v_{j+1} \in A$)

A vector $y \in \mathbf{R}^{P}{}_{+}$ represents a **production plan** for the configurations

Its node flow vector $x = j\,(y)$ represents the amounts of each component required for that production plan.

As P may be very large and the y variables appear only in connection with the x variables, we want to "project away" the y variables and use instead a system of linear inequalities defining the corresponding set X of all node flow vectors.

Remark:

Let $\boldsymbol{g(P)} \in \mathbf{R}^V$ denote the node-path incidence vector of path $P \in \mathcal{P}$, that is,

$$g(P)_u = \begin{cases} 1 & \text{if } u \in P \\ 0 & \text{otherwise} \end{cases}$$

then X is the cone generated by the vectors $g(P)$ for all $P \in \mathcal{P}$

Thus, knowing this "internal description" (or "extreme ray description") of the cone X, we seek its "external description" (or "linear inequality description")

Notation: For $\boldsymbol{U} \subseteq V$ let

$\boldsymbol{x(U)} = \sum_{v \in U} x_u$

$\boldsymbol{A^+(U)} = \{ v \in V : uv \in A \text{ for some } u \in U \}$ the set of (immediate) **successors** of U

$\boldsymbol{A^-(U)} = \{ v \in V : vu \in A \text{ for some } u \in U \}$ the set of (immediate) **predecessors** of U

Known Results

1) Bipartite graphs

Let $G = (V_1+V_2, A)$ where $A \subseteq V_1 \times V_2$ be a bipartite graph; $S = V_1$ and $T = V_2$

Theorem 1 (Ball & al.): When $G = (V_1+V_2, A)$ is bipartite with $S = V_1$ and $T = V_2$ its node flow cone is

$$X = \{ x \in \mathbf{R}^V_+ : x(V_1) = x(V_2),\ x(U) \leq x(A^+(U)) \text{ for all } U \subseteq V_1 \}$$

$$= \{ x \in \mathbf{R}^V_+ : x(V_1) = x(V_2),\ x(W) \leq x(A^-(W)) \text{ for all } W \subseteq V_2 \}.$$

Remark: Let X_1 denote the first cone above, and X_2 the second cone.

If $x \in X_2$ then for all $U \subseteq V_1$ define $W = V_2 \setminus A^+(U)$, so $A^-(W) \subseteq V_1 \setminus U$ and

$$0 \geq x(W) - x(A^-(W)) \geq x(V_2 \setminus A^+(U)) - x(V_1 \setminus U)$$
$$= x(V_2) - x(A^+(U)) - x(V_1) + x(U) = x(U) - x(A^+(U))$$

so $x \in X_1$. This shows that $X_2 \subseteq X_1$. Similarly one shows that $X_1 \subseteq X_2$ and therefore $X_1 = X_2$.

Theorem 1 (Ball & al.): When $G = (V_1+V_2, A)$ is bipartite with $S = V_1$ and $T = V_2$ its node flow cone is

$$X = \{ x \in \mathbf{R}^V_+ : x(V_1) = x(V_2)$$
$$x(U) \leq x(A^+(U)) \quad \text{for all } U \subseteq V_1 \}.$$

Proof (Ball & al.): Given rational $x¢ \in \mathbf{Q}^V_+$ rescale $x¢$ as $x¢ = l\, x$ such that x is integral and $x¢ \in X$ iff $x \in X$.

Make x_u copies $u_1, u_2, \ldots, u_k \in V_i^2$ of each node $u \in V_i$ for $i = 1,2$

and $x_u\, x_v$ copies $u_i v_j \in A^2$ of each arc $uv \in A$.

Let $V^2 = V_1^2 + V_2^2$ and invoke the **Balas & Pulleyblank** (1983) characterization of the **perfectly matchable induced subgraphs** of the resulting bipartite graph $G^2 = (V_1^2 + V_2^2, A^2)$:

$$\text{conv}\{ z \in \{0,1\}^{V^2} : z \text{ is the characteristic vector of a subset } Z \subseteq V^2$$
$$\text{such that } (Z, A(Z)) \text{ contains a perfect matching } \}$$

$$= \{ z \in \mathbf{R}^V : 0 \leq z \leq 1, \quad z(V_1^2) = z(V_2^2)$$
$$z(U) \leq z(A^+(U)) \quad \text{for all } U \subseteq V_1^2 \}$$

QED

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$$x(U) \le x(A^{+}(U)) \quad \text{for all } U \subseteq V_1 \,\}.$$

Direct Proof : To every $x \in \mathbf{R}^{V}_{+}$ associate the **capacitated network** ***N*(*x*)** = (V¢, A¢, c^x)

where **V¢** $= V + s + t$, s is a new source, t is a new sink,

A¢ consists of : the **source arcs *su*** for all $u \in V_1$ with capacity $c^{x}_{su} = x_u$

the **sink arcs *vt*** for all $v \in V_2$ with capacity $c^{x}_{vt} = x_v$

and all arcs $uv \in A$ with capacity $c^{x}_{uv} = +\infty$

Let **$z(x)$** $= x(V_1)$.

By the **Max-flow Min-cut Theorem** there exists a feasible flow with value $z(x)$ in $N(x)$ if and only if $x(V_2) = z(x)$ and the capacity of every (other) s-t-cut in $N(x)$ is at least $z(x)$.

It suffices to consider all s-t-cuts $(U+s, W+t)$ with **finite capacity**, that is, letting $U_i = U_i \cap V_i$ and $W_i = W_i \cap V_i$ for i =1,2, such that $U_2 \subseteq A^{+}(U_1)$.

Furthermore, since $c^x \ge 0$, the capacity of such finite capacity cuts $(U+s, W+t)$ satisfy

$$c^x(U+s, W+t) \;\ge\; c^x(U+s, (V_2 \setminus A^{+}(U_1))+t) \;=\; x(V_1) - x(U_1) + x(A^{+}(U_1))$$

Thus $x \in X$ if and only if $x(V_1) = x(V_2)$ and $x(V_1) - x(U_1) + x(A^{+}(U_1)) \ge x(V_1)$ for all $U_1 \subseteq V_1$

QED

2) Multipartite graphs

Let $G = (V, A)$ be a **multipartite graph**,

where $V = V_1 + V_2 + \ldots + V_L$ consists of L **layers**, with $S = V_1$ and $T = V_L$ and

$A \subseteq (V_1 \times V_2) \cup (V_2 \times V_3) \cup \ldots \cup (V_{L\text{-}1} \times V_L)$ so arcs only connect successive layers

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Theorem 2 (Ball & al.): When G is multipartite as described, then its node flow cone is

$$X = \{ x \in \mathbf{R}^V_+ : x(V_i) = x(V_{i+1}) \quad \text{for all } i = 1, 2, \ldots, L-1$$

$$x(U) \leq x(A^+(U)) \quad \text{for all } U \subseteq V_i \text{ and all } i = 1, 2, \ldots, L-1 \}$$

$$= \{ x \in \mathbf{R}^V_+ : x(V_i) = x(V_{i+1}) \quad \text{for all } i = 1, 2, \ldots, L-1$$

$$x(W) \leq x(A^-(W)) \quad \text{for all } W \subseteq V_i \text{ and all } i = 2, 3, \ldots, L \}$$

Proof: Let $G_i = (V_i + V_{i+1}, A \cap (V_i \times V_{i+1}))$ be the subgraph induced by layers V_i and V_{i+1}

X_i denote the node flow cone of the bipartite graph G_i (with $S = V_i$ and $T = V_{i+1}$)

$X¢_i = \{ x \in \mathbf{R}^V : (x_{V_i}, x_{V_{i+1}}) \in X_i \}$ the **cylinder** of $\mathbf{R}^V$ with base X_i

Then $X = X¢_1 \cap X¢_2 \cap \ldots \cap X¢_{L\text{-}1}$ since the restriction of x to each $V_i + V_{i+1}$ must be in X_i .

Conversely, if $x \in X¢_1 \cap X¢_2 \cap \ldots \cap X¢_{L\text{-}1}$ then there exist **arc flows y^i** in each G_i such that $j\,(y^i) = (x_{V_i}, x_{V_{i+1}})$. We may paste these arc flows into path flows, and therefore $x \in X$.

QED

General Acyclic Digraph:

Let $G = (V, A)$ be an acyclic digraph, with source set $\emptyset \neq S \subseteq V$ and sink set $\emptyset \neq T \subseteq V$

- Are the constraints $x(U) \leq x(A^{+}(U))$ for all $U \subseteq V$

  and $x(W) \leq x(A^{-}(W))$ for all $W \subseteq V$ **valid** for the node flow cone?

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No, because $x(U)$ "double-count" the flow on the paths that visit U more than once.

Node set $U \subseteq V$ is **path-independent** if no path $P \in \mathcal{P}$ visits U more than once.

If U is path-independent, and $y \in \mathbf{R}^{\mathcal{P}}_+$ is a path flow vector with $x = j(y)$, then

$$x(U) = \Sigma \{ y_P : \text{ all } P \in \mathcal{P} \text{ with } P \cap U \neq \emptyset \},$$

the **total flow** through node set U.

Further if $U \cap T = \emptyset$ then these $x(U)$ units of flow must traverse the successors $A^+(U)$ therefore:

(1) $x(U) \leq x(A^+(U))$ for all $U \subseteq V$ that are path independent and satisfy $U \cap T = \emptyset$

are valid inequalities for the node flow cone X.

…therefore:

(1)	$x(U) \leq x(A^{+}(U))$	for all $U \subseteq V$ that are path independent and satisfy $U \cap T = \emptyset$

are valid inequalities for the node flow cone X.

Similarly,

(2)	$x(W) \leq x(A^{-}(W))$	for all $W \subseteq V$ that are path independent and satisfy $W \cap S = \emptyset$

are valid inequalities for the node flow cone X.

Node set $U \subseteq V$ is an **exact cut** if every path $P \in \mathcal{P}$ visits U exactly once.

If U is an exact cut, and $y \in \mathbf{R}^{\mathcal{P}}_{+}$ is a path flow vector with total flow value $z(y)$, and if $x = j(y)$, then $x(U) = z(y)$.

Therefore,

(3)	$x(U) = x(W)$	for all exact cuts U and W

are valid equalities for the node flow cone X.

Remark: Assume G is multipartite.

Then each V_i is an exact cut and the equations $x(V_i) = x(V_{i+1})$ for all $i = 1, 2, \ldots, L-1$ are instances of (3).

If $U \subseteq V_i$ for $i \leq L-1$ then U is path independent and $U \cap T = \varnothing$.

Then the node flow cone is defined by the equations (3) and the inequalities (1) .

Similarly, if $W \subseteq V_i$ for $i \geq 2$ then W is path independent and $W \cap S = \varnothing$.

Then the node flow cone is defined by the equations (3) and the inequalities (2) .

- Are the constraints (1)–(3) **sufficient** to define the node flow cone of an acyclic directed network?

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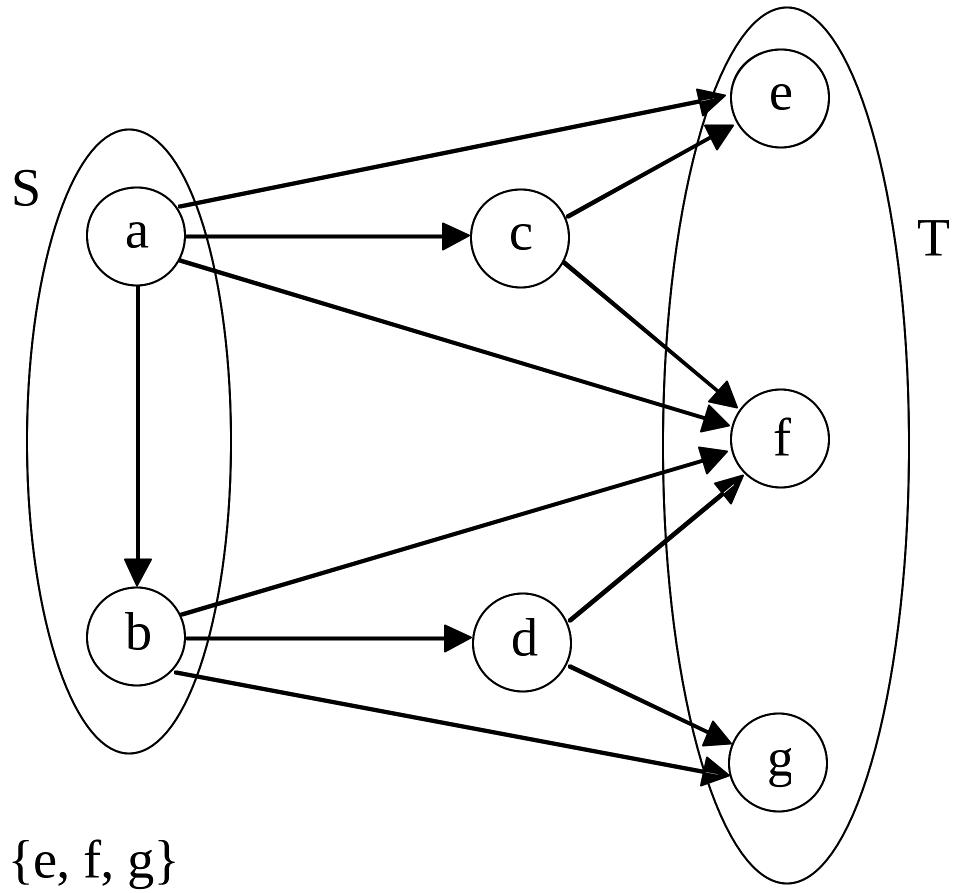
No, for example, let G =



There is a unique exact cut {e, f, g}
The node flow cone X is full-dimensional (…so (3) is OK)

The constraints $x_g \leq x_b$ and $x_b \leq x_f + x_g$ are facet-defining for X but neither is of type (1) or (2).

Path Independent and Cover inequalities:

Node set $C \subseteq V$ is a **path-cover** for node subset $U \subseteq V$ if every path $P \in \mathcal{P}$ which visits U visits also C.

If U is path-independent,
C is a path-cover for U,
and $y \in \mathbf{R}^{\mathcal{P}}_{+}$ is a path flow vector with $x = j\,(y)$,

then all $x(U)$ units of flow which traverse U must also traverse its path-cover C, therefore:

(4) $x(U) \leq x(C)$ for all $U \subseteq V$ that are path independent and all path-covers C for U

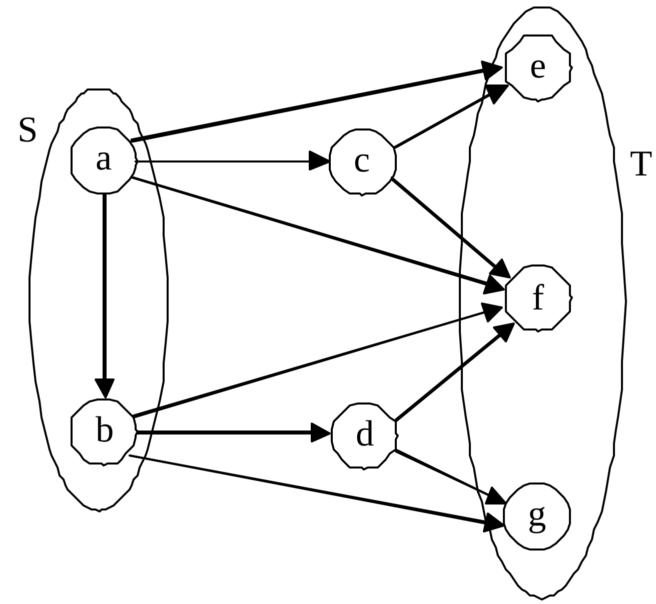
are valid inequalities for the node flow cone X.

Note that these inequalities (4) generalize (1) and (2).

(3)	$x(U) = x(W)$	for all exact cuts U and W
(4)	$x(U) \leq x(C)$	for all $U \subseteq V$ that are path independent and all path-covers C for U

are valid for the node flow cone X.

Example (continued):



- for $U = \{g\}$ and $C = \{b\}$, (4) gives $x_g \leq x_b$
- for $U = \{b\}$ and $C = \{f, g\}$, (4) gives $x_b \leq x_f + x_g$
- for $U = \{a\}$ and $C = \{e, f, g\}$,
  (4) gives $x_a \leq x_e + x_f + x_g$ (also facet-defining)
- for $U = \{b, c\}$ and $C = \{e, f, g\}$,
  (4) gives $x_b + x_c \leq x_e + x_f + x_g$ (also facet-defining)
- for $U = \{e, f, g\}$ and $C = \{a, b\}$,
  (4) gives $x_e + x_f + x_g \leq x_b + x_c$ (also facet-defining)
- the type (2) inequality with $W = \{c\}$ and $C = A^{-}(W) = \{a\}$,
  $x_c \leq x_a$ is also of type (4) and facet-defining

…in fact, these 6 inequalities, plus the nonnegativity constraints $x_u \geq 0$ (for $u \neq$ f), are all the facet-defining inequalities for this example.

- Are the constraints (3)–(4) **sufficient** to define the node flow cone of an acyclic directed network?

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Conjecture: Yes…(?)

A Weaker Conjecture:

The cone *X* is defined by a system of linear inequalities with left-hand-side coefficients in **{0, ±1}** (?)

Possible Extensions

1) Digraphs with Cycles

If G contains (directed) cycles, should the set $\mathcal{P}$ consist

- of all S-T-paths in G? or
- of all S-T-paths in G without any directed cycle?

2) Capacitated Networks

Given **arc capacities c_{uv}** ≥ 0 and path set $\mathcal{P}$ the path flows $y \in \mathbf{R}^{\mathcal{P}}_{+}$ must now satisfy the **arc capacity constraints**

(5) $\qquad \sum \{ y_P : \text{ all } P \in \mathcal{P} \text{ with arc } uv \in P \} \leq c_{uv} \quad$ for all $uv \in A$.

Let

$$X = \{ x \in \mathbf{R}^{V}_{+} : x = j(y) \text{ for some } y \in \mathbf{R}^{\mathcal{P}}_{+} \text{ satisfying (5) } \}$$

For which of these cases can the node flow cone X be defined by
- an **explicit** system of linear inequalities?
- a system of linear inequalities with left-hand-side coefficients in **{0, ±1}**?

Also, how about the **separation problem** for these node flow cones?