Minimizing the Bandwidth of a Matrix

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0	x	0	0	0	x	0	x	0	0
x	0	x	0	0	x	0	x	0	x
0	x	0	x	0	0	x	x	0	0
0	0	x	0	0	0	0	x	x	x
0	0	0	0	0	x	x	0	x	0
x	x	0	0	x	0	x	0	0	0
0	0	x	0	x	x	0	0	x	0
x	x	x	x	0	0	0	0	0	0
0	0	0	x	x	0	x	0	0	0
0	x	0	x	0	0	0	0	0	0

input matrix
(bandwidth $= \psi = 8$)

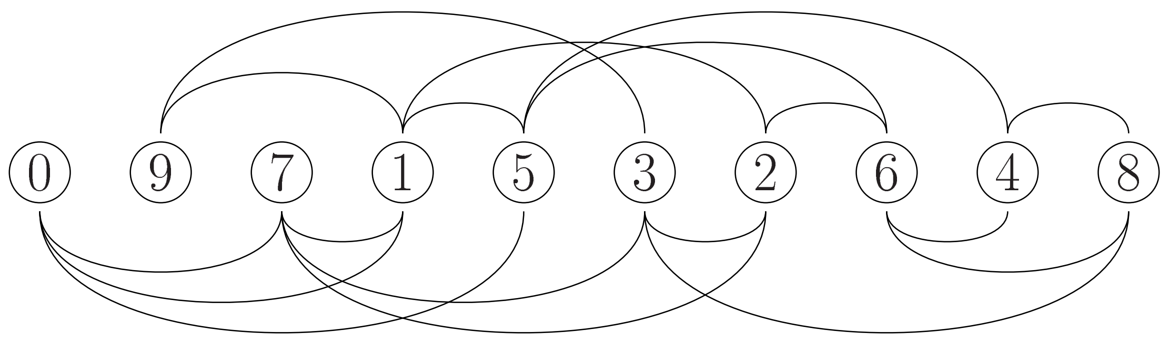
$\Longrightarrow$

$n = 10$
$m = 18$

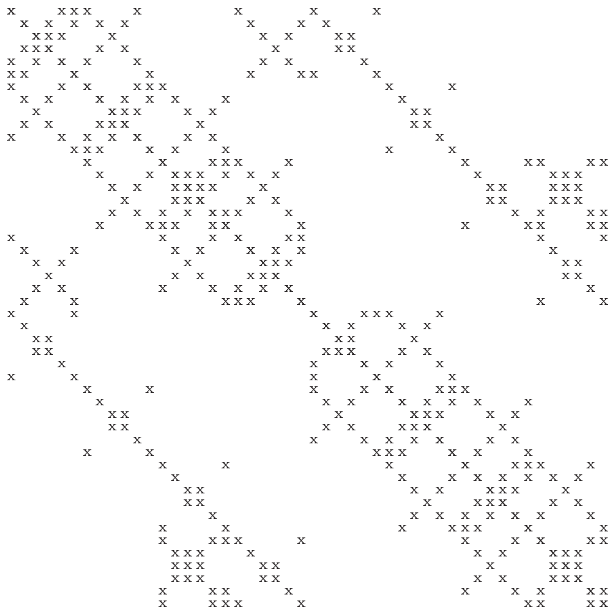
0	0	x	x	x	0	0	0	0	0
0	0	0	x	0	x	0	0	0	0
x	0	0	x	0	x	x	0	0	0
x	x	x	0	x	0	x	0	0	0
x	0	0	x	0	0	0	x	x	0
0	x	x	0	0	0	x	0	0	x
0	0	x	x	0	x	0	x	0	0
0	0	0	0	x	0	x	0	x	x
0	0	0	0	x	0	0	x	0	x
0	0	0	0	0	x	0	x	x	0

output matrix
(bandwidth $= \phi = 4$)

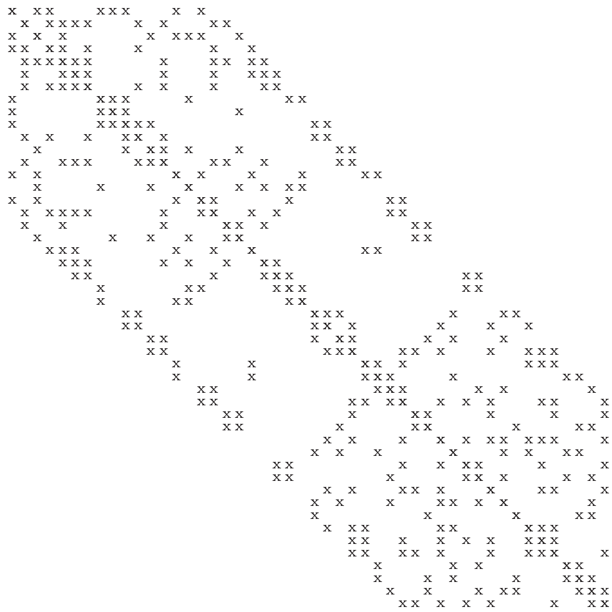
Graph Representation:



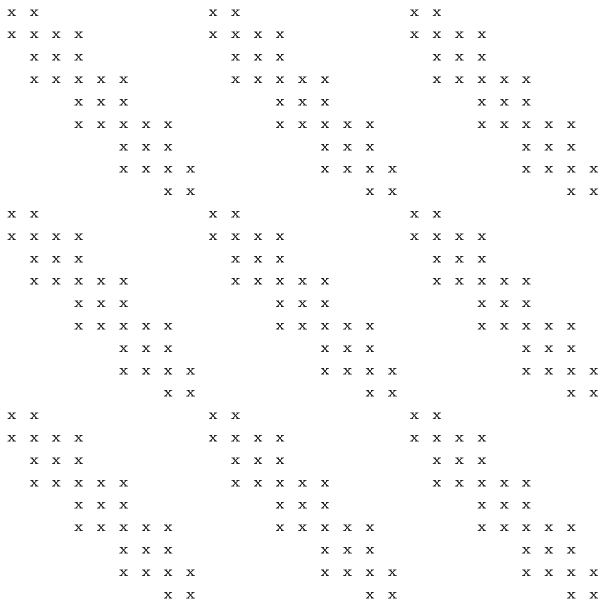
bcsstk01: n=48, $m = 224$, ψ=35, $\phi = 16$



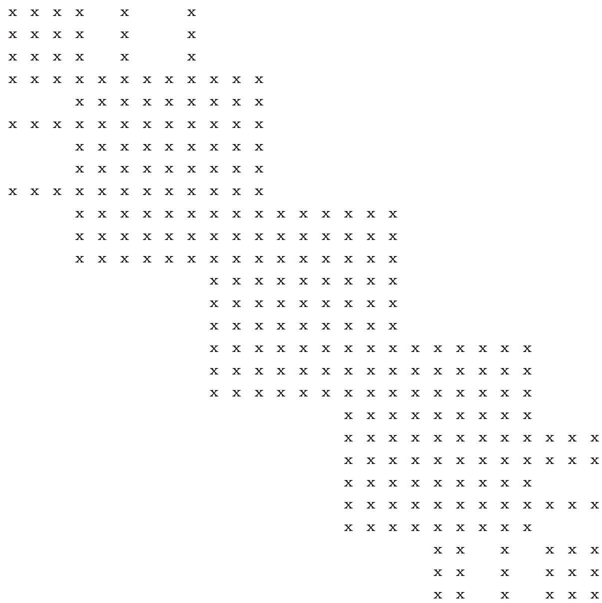
$\Longrightarrow$



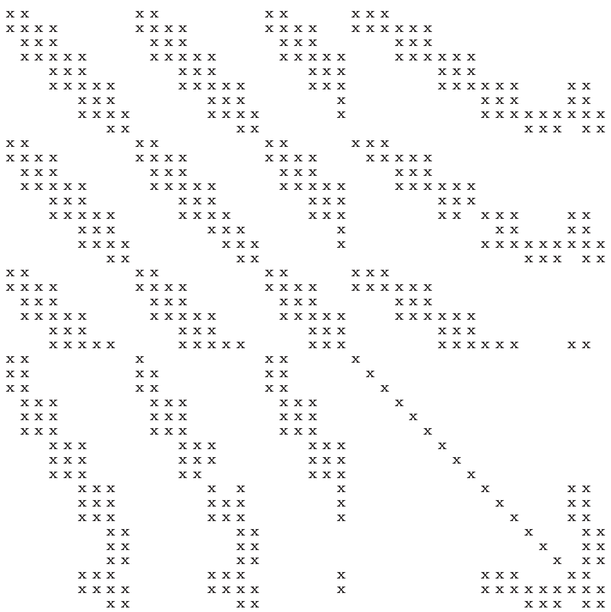
fidap005: n=27, $m = 279$, ψ=20, $\phi = 8$



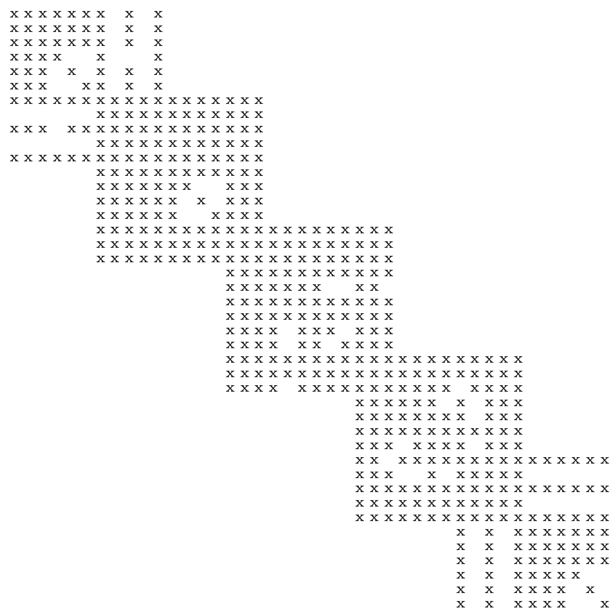
$\Longrightarrow$



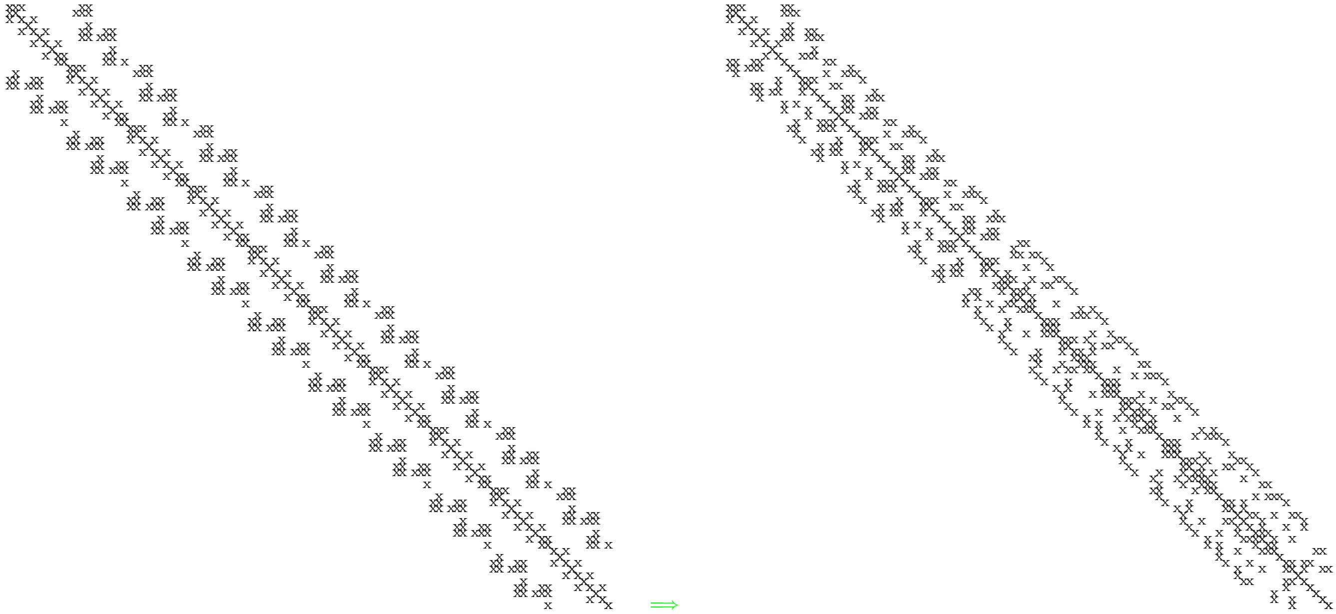
fidapm05: n=42, $m = 520$, ψ=35, $\phi = 11$



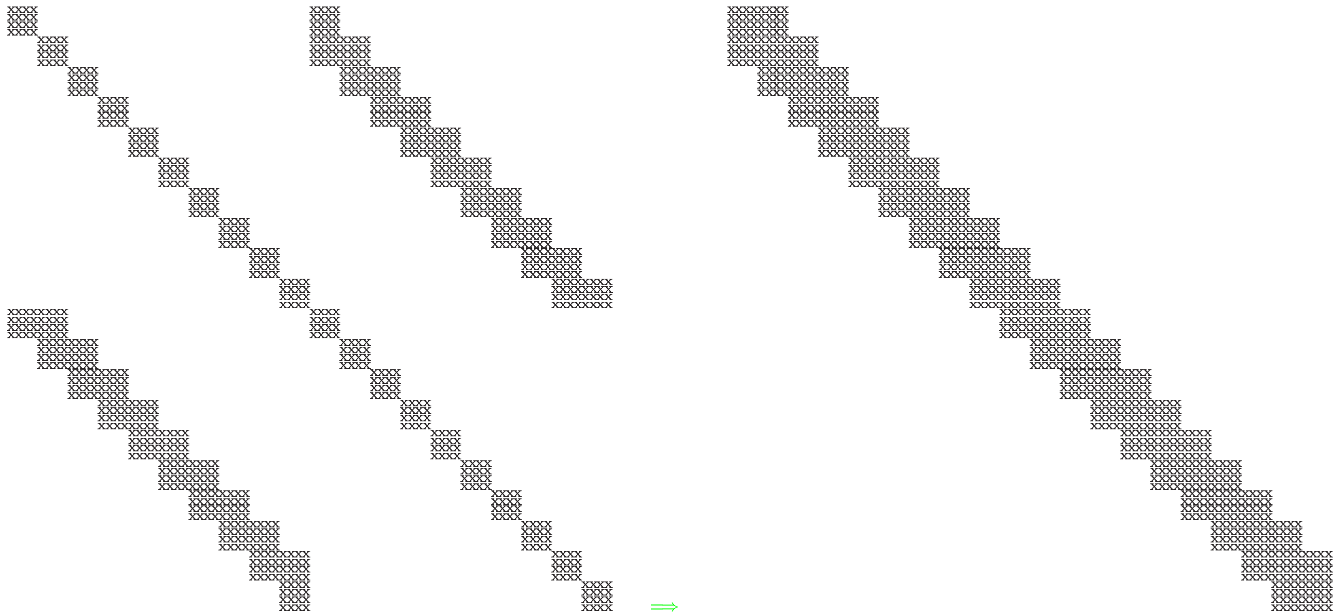
$\Longrightarrow$



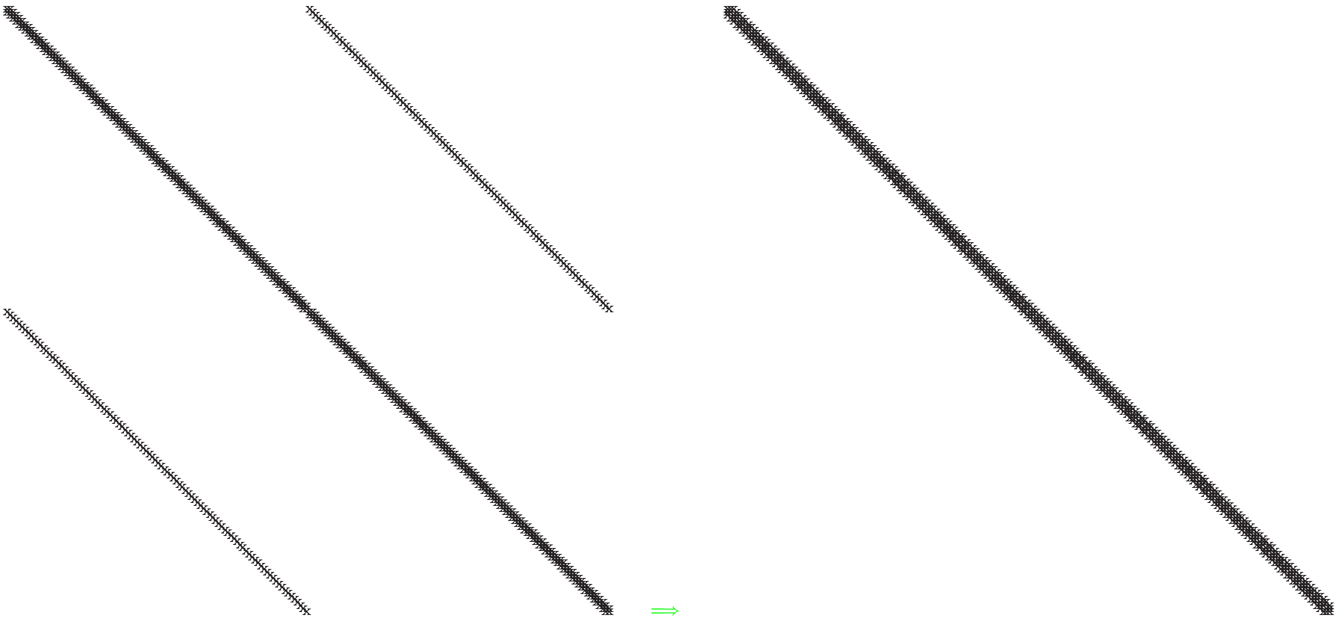
nos4: n=100, m = 347, ψ=13, ϕ = 10



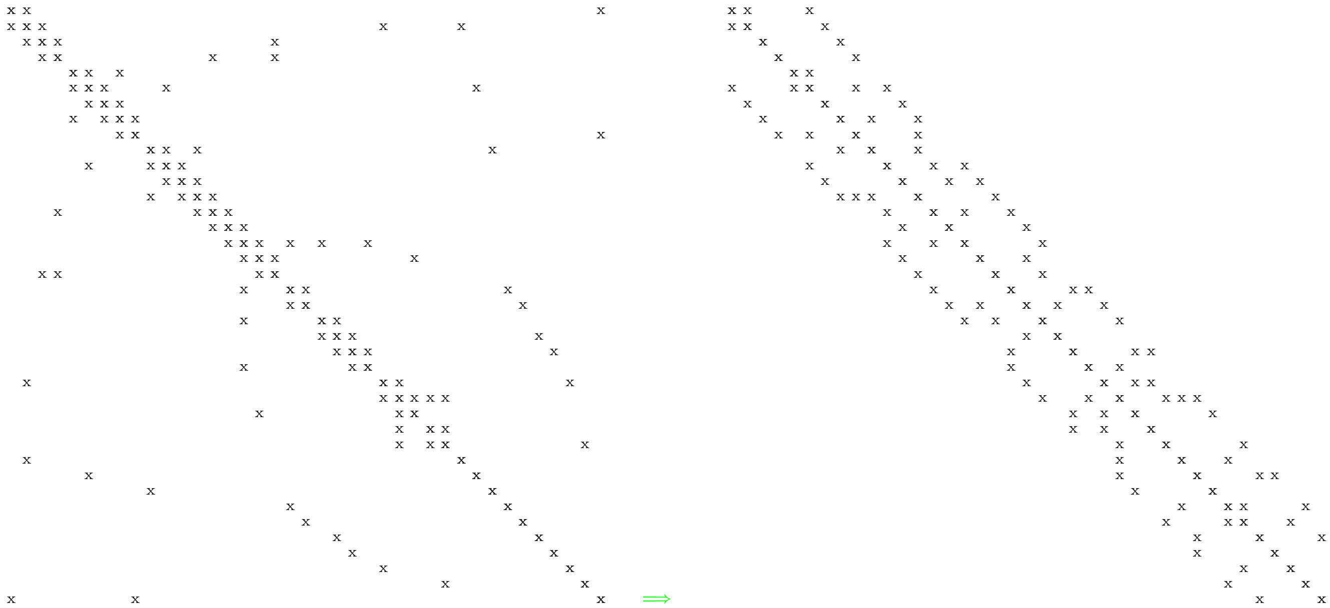
steam3: n=80, m = 928, ψ=43, ϕ = 7



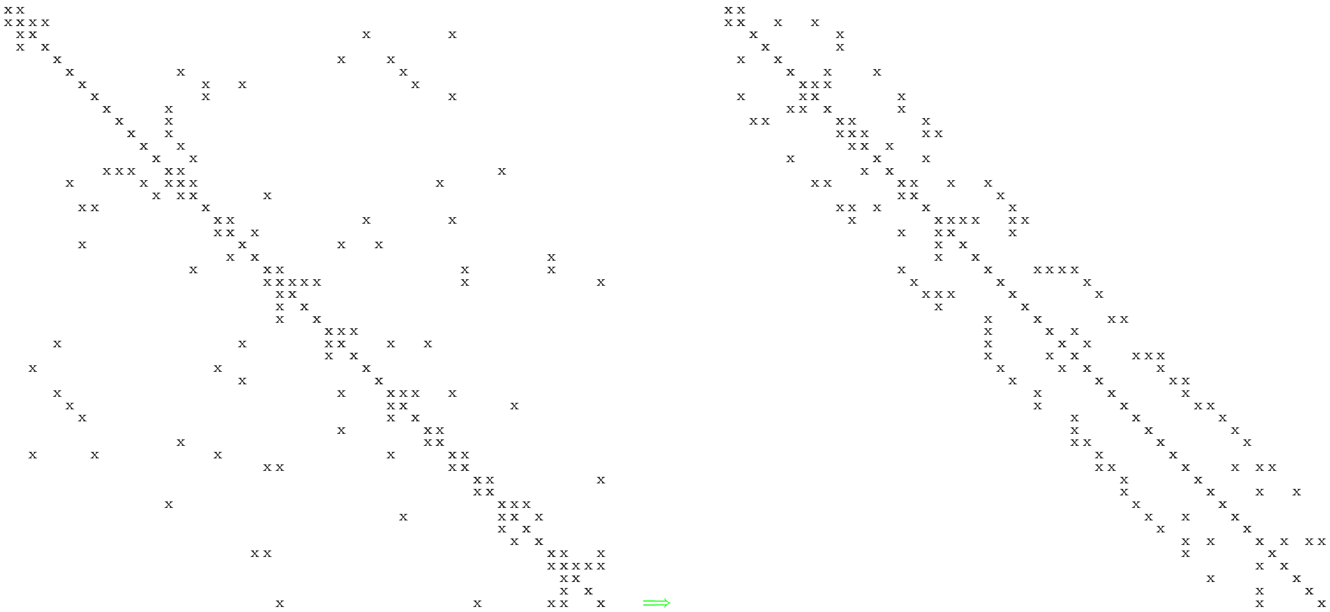
bwm200: n=200, m = 796, ψ=100, ϕ = 2



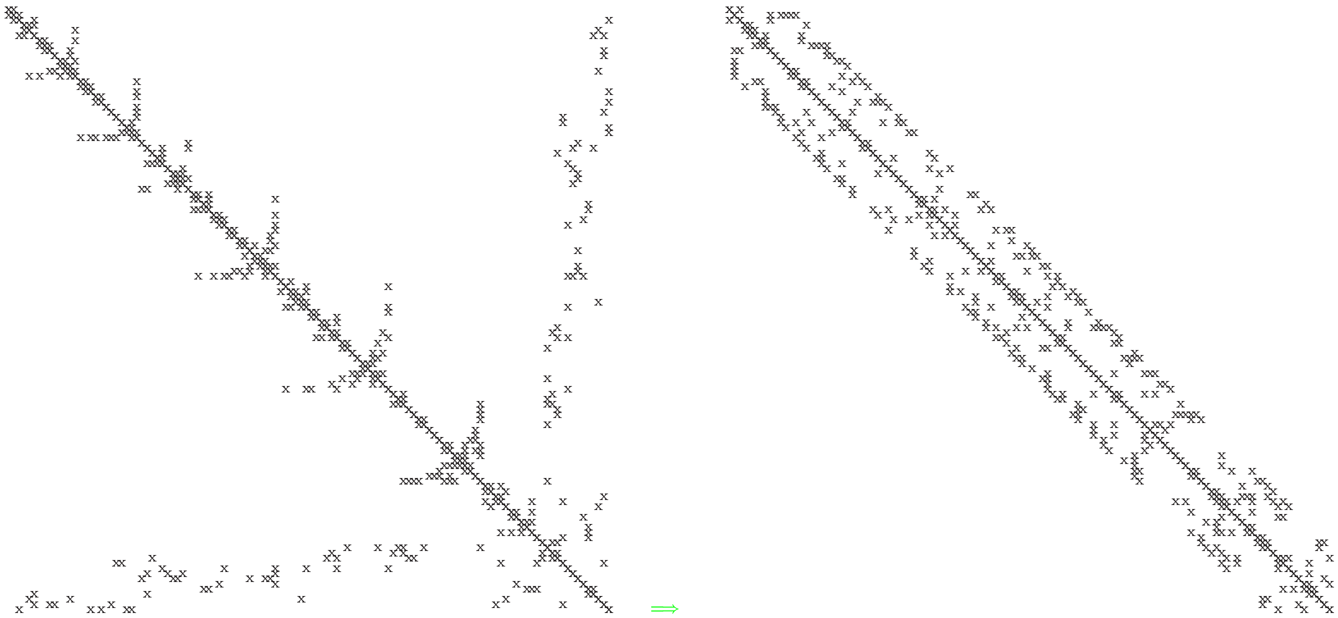
bcspwr01: n=39, m = 85, ψ=38, ϕ = 5



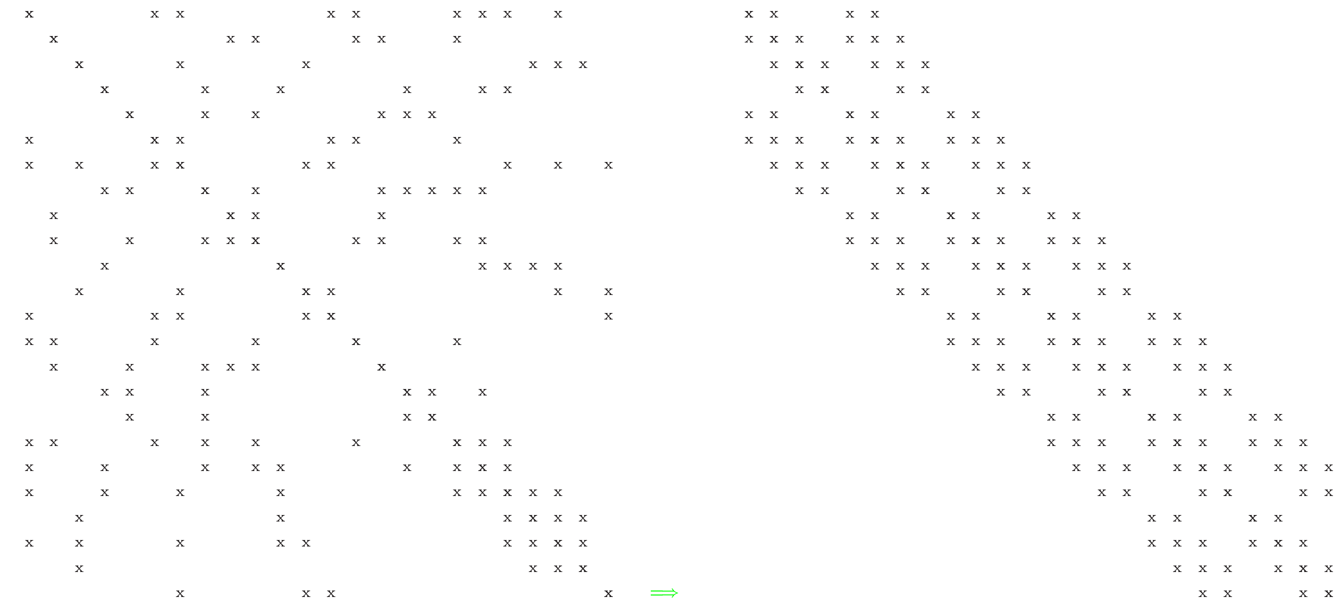
bcspwr02: n=49, m = 108, ψ=34, ϕ = 7



bcspwr03: n=118, m = 297, ψ=115, ϕ = 12

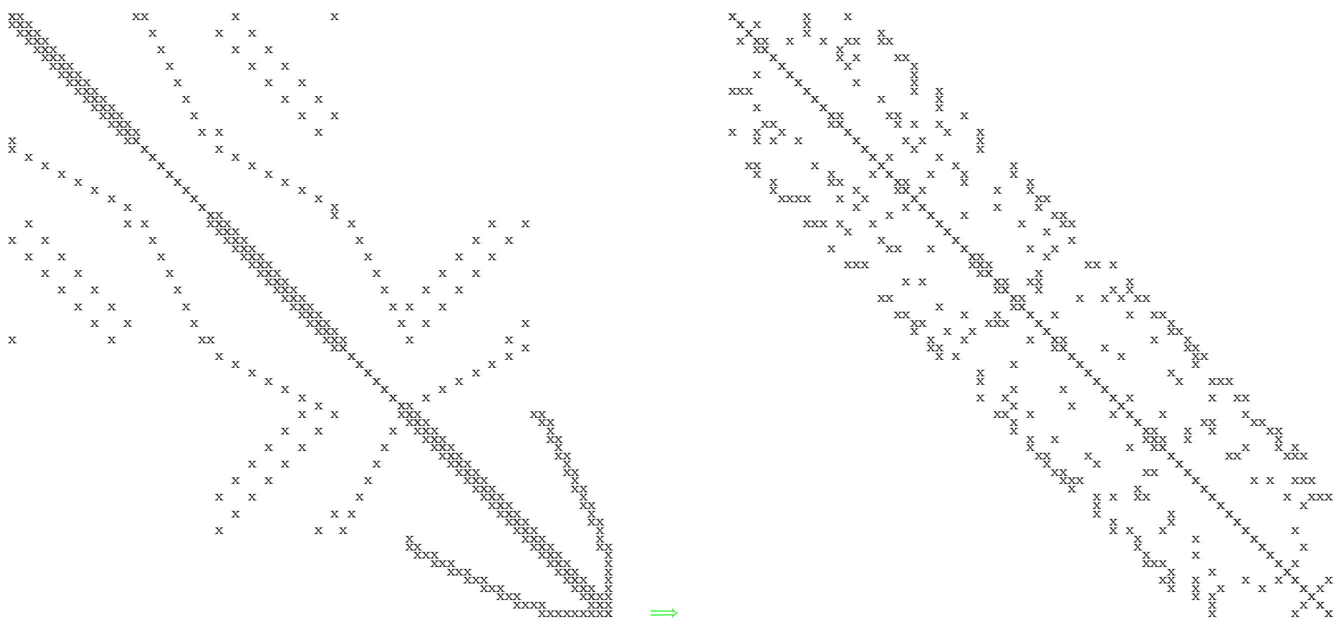


can___24: n=24, $m = 92$, ψ=21, $\phi = 5$

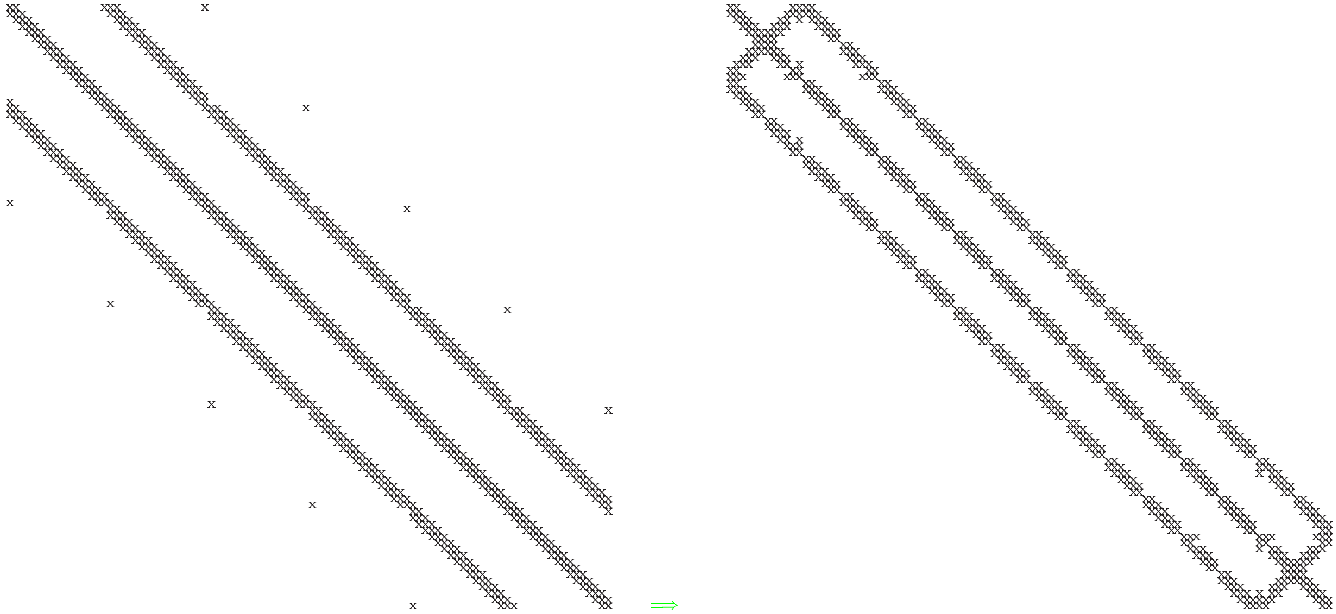


can___61: n=61, $m = 309$, ψ=50, $\phi = 13$

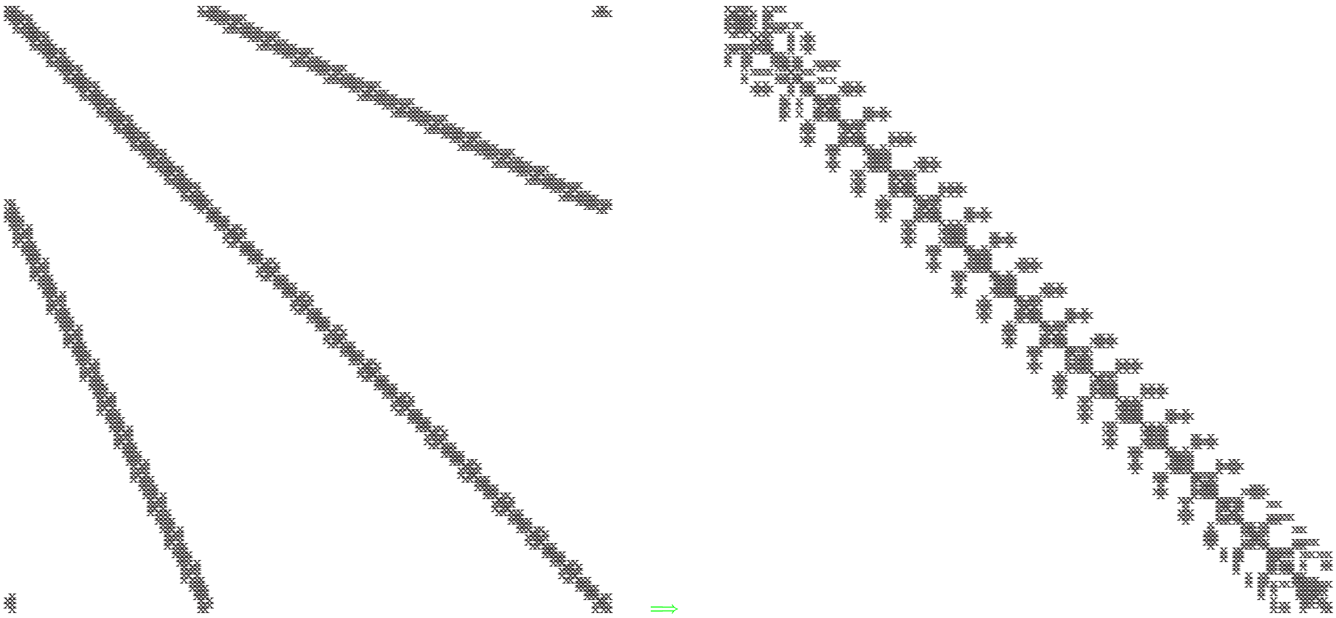
can___73: n=73, $m = 225$, ψ=39, $\phi = 16$



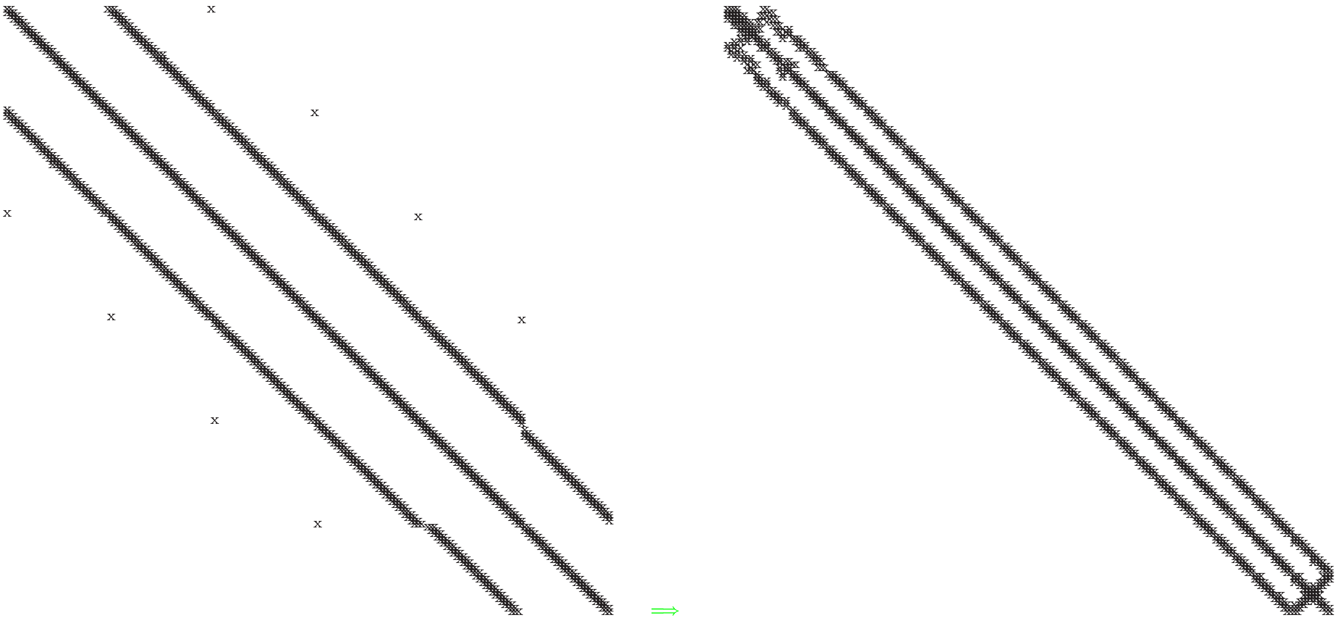
can___96: n=96, m = 432, ψ=31, ϕ = 13



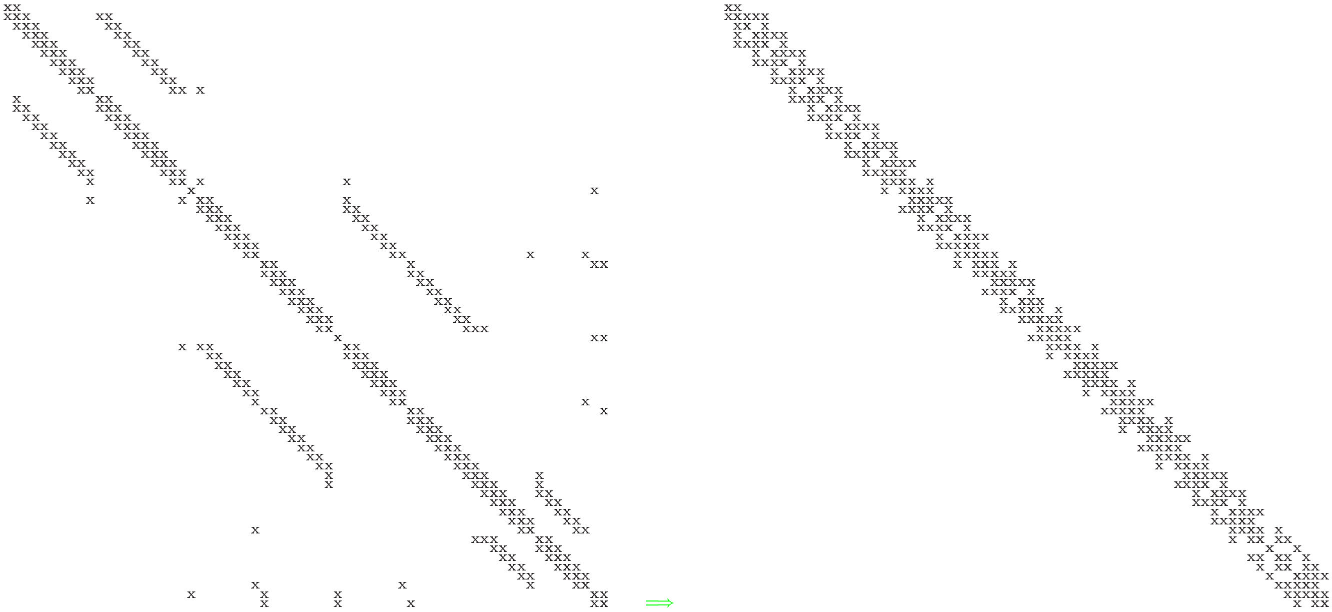
can__144: n=144, m = 720, ψ=142, ϕ = 13



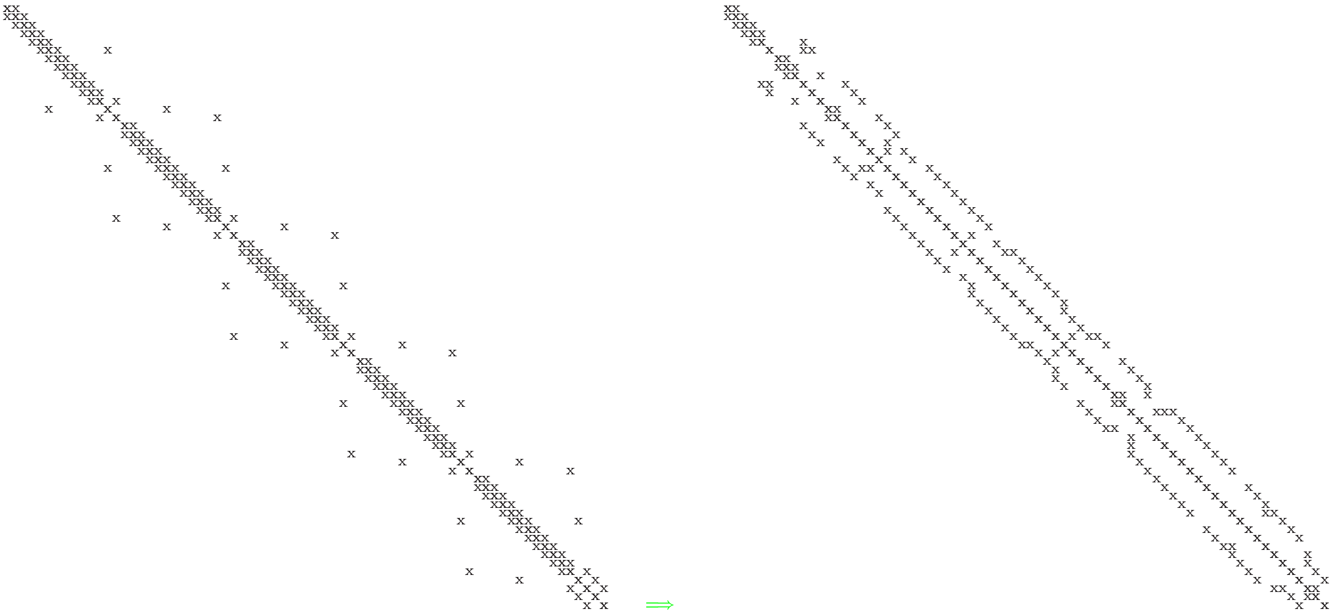
can__187: n=187, m = 839, ψ=63, ϕ = 13



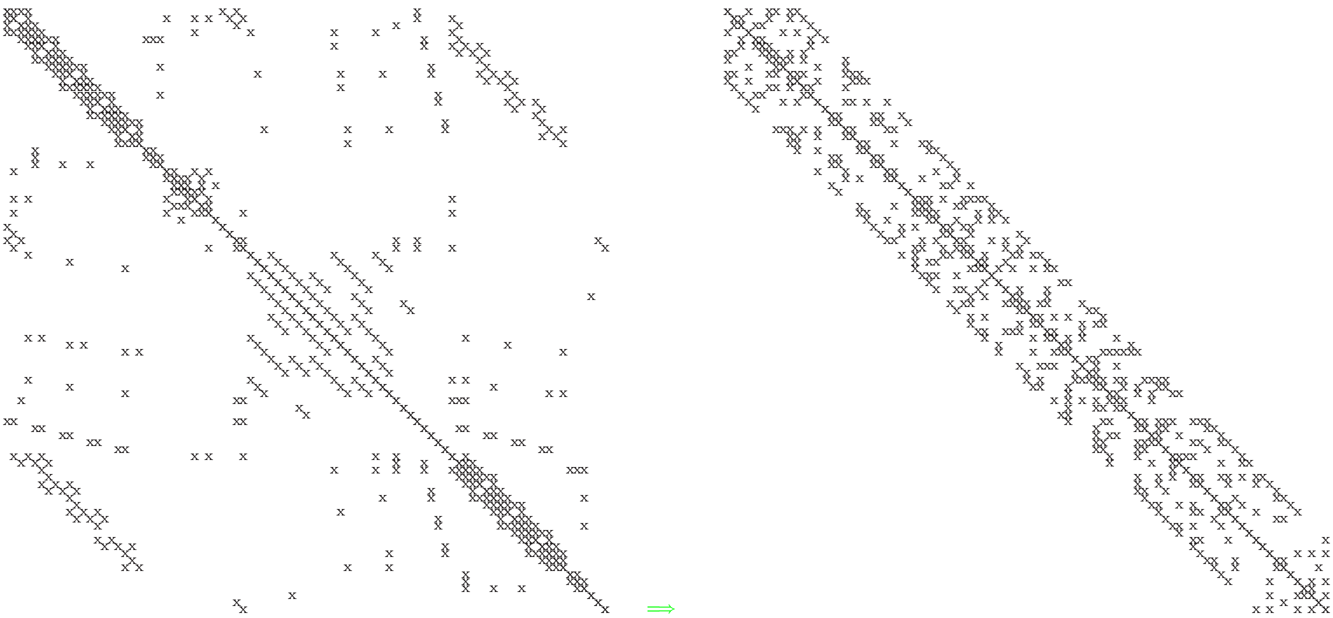
dwt___66: n=66, m = 193, ψ=44, ϕ = 3



dwt___72: n=72, m = 147, ψ=12, ϕ = 5



dwt___87: n=87, m = 314, ψ=63, ϕ = 10



Notation:

Consider an undirected connected graph $G = (V, E)$;
$n := |V|$, $m := |E|$.

A *(linear) layout* of G is a total ordering of the vertices of G, defined by a one-to-one function $\tau : V \to \{1, \ldots, n\}$.

The *bandwidth of the layout* is
$\phi_\tau(G) := \max\{|\tau_u - \tau_v| : (u, v) \in E\}$.

The *bandwidth of G* is
$\phi(G) = \min\{\phi_\tau(G) : \tau \text{ layout of } G\}$.

Given two vertices $u, v \in V$, $d(u, v)$ is the minimum number of edges in a path from u to v.

Given two (not necessarily disjoint) node sets $S, T \subseteq V$
$d(S, T) := \max\{d(u, v) : u \in S, v \in T\}$

The *diameter* of S is $d(S) := d(S, S)$.

For a node $u \in V$ and $k \leq d(u, V)$,
$N_k(u) := \{v \in V : d(u, v) \leq k\}$.

Previous works:

- Computational complexity:
  - Papadimitriou (1976) [it is $\mathcal{NP}$-hard]
  - Garey, Graham, Johnson, Knuth (1978)
  - Saxe (1980)
  - Kratsch (1987)
  - Smithline (1995)
  - Unger (1998)
- Heuristic algorithms:
  - Alway, Martin (1965)
  - Cuthill and McKee (1969)
  - Cuthill (1972)
  - Gibbs, Poole, Stockmeyer (1976)
  - Turner (1986)
  - Luo (1992)
  - Dueck, Jeffs (1995) [Simulated Annealing]
  - Esposito, Fiorenzo Catalano, Malucelli, Tarricone (1997)
  - Karpinski, Wirtgen, Zelikovosky (1997) [appr. algorithm]
  - Feige (2000) [approximated algorithm]
  - Blum, Konjevod, Ravi, Vempala (2000) [appr. algorithm]
  - Martí, Laguna, Glover, Campos (2001) [Tabu Search]
  - Piñana, Plana, Campos, Martí (2002) [GRASP]
- Exact approaches:
  - Corso, Manzini (2000)

Related problems:

1. Linear Arrangement Problem:

   it minimizes

$$\sum\{|\tau_u - \tau_v| : (u, v) \in E\}$$

   instead of

$$\max\{|\tau_u - \tau_v| : (u, v) \in E\}.$$

2. Profile-width Minimization Problem:

3. Cut-width Minimization Problem:

Known Lower Bounds:

The density lower bound (Chvátal, 1970):

$$\beta(G) := \max_{S \subseteq V} \left\lceil \frac{|S|-1}{d(S)} \right\rceil .$$

- For every graph G, $\phi(G) \geq \beta(G)$.
- Computing $\beta(G)$ is $\mathcal{NP}$-hard.

A recent lower bound (Blum, Konjevod, Ravi, Vempala, 2000):

$$\alpha(G) := \max_{v \in V} \max_{S \subseteq V : v \in S} \left\lceil \frac{|S|-1}{2d(v,S)} \right\rceil .$$

- For every graph G, $\beta(G)/2 \leq \alpha(G) \leq \beta(G)$.
- $\alpha(G)$ can be computed in $O(nm)$ time.

Another recent lower bound (Corso, Manzini, 2000):

$$\delta(G) := \min_{v \in V} |N_1(v)| - 1.$$

New Lower Bound:

$$\gamma(G) := \min_{v \in V} \max_{S \subseteq V, v \in S} \left\lceil \frac{|S|-1}{d(v,S)} \right\rceil$$

- For every graph G, $\phi(G) \geq \gamma(G)$.
- $\gamma(G)$ can be computed in $O(nm)$ time.

Matrix Representation:

0	x	0	0	0	x	0	x	0	0
x	0	x	0	0	x	0	x	0	x
0	x	0	x	0	0	x	x	0	0
0	0	x	0	0	0	0	x	x	x
0	0	0	0	0	x	x	0	x	0
x	x	0	0	x	0	x	0	0	0
0	0	x	0	x	x	0	0	x	0
x	x	x	x	0	0	0	0	0	0
0	0	0	x	x	0	x	0	0	0
0	x	0	x	0	0	0	0	0	0

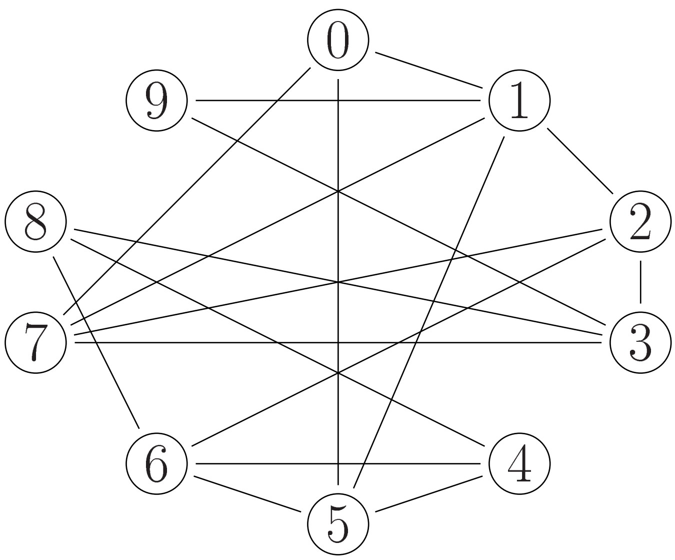
input matrix
(bandwidth = 8)

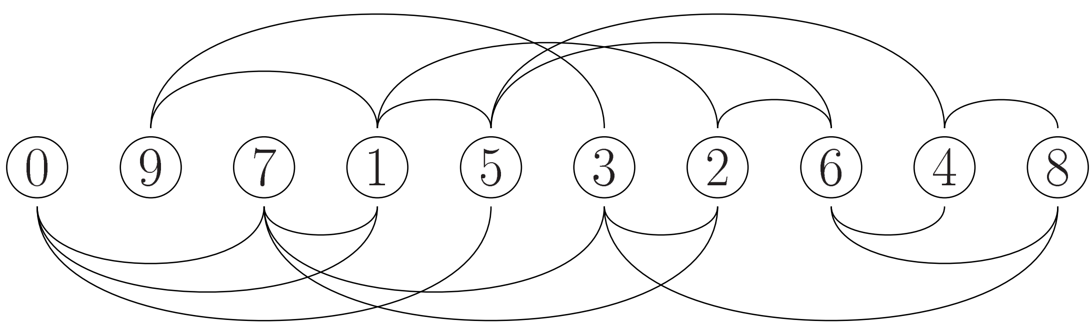
$\rightarrow$

0	0	x	x	x	0	0	0	0	0
0	0	0	x	0	x	0	0	0	0
x	0	0	x	0	x	x	0	0	0
x	x	x	0	x	0	x	0	0	0
x	0	0	x	0	0	0	x	x	0
0	x	x	0	0	0	x	0	0	x
0	0	x	x	0	x	0	x	0	0
0	0	0	0	x	0	x	0	x	x
0	0	0	0	x	0	0	x	0	x
0	0	0	0	0	x	0	x	x	0

output matrix
(bandwidth = 4)

Graph Representation:





Summarize:

$$\phi_\tau(G) := \max\{|\tau_u - \tau_v| : (u,v) \in E\}$$

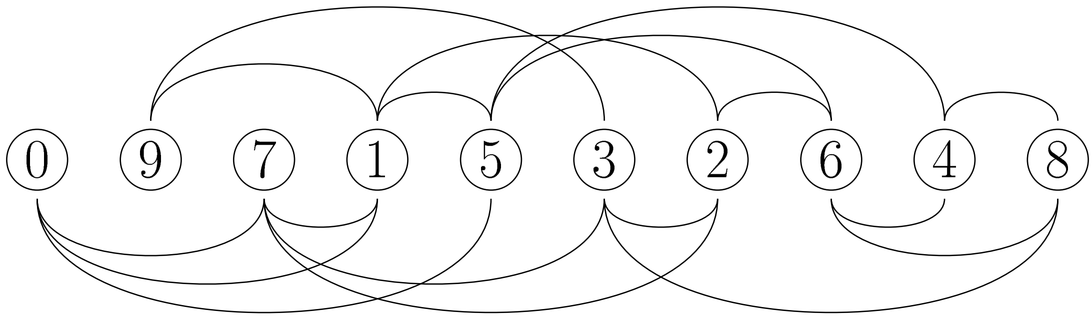
$$\beta(G) := \max_{S\subseteq V} \left\lceil \frac{|S|-1}{d(S)} \right\rceil = \max_{S\subseteq V} \min_{v\in S} \left\lceil \frac{|S|-1}{d(v,S)} \right\rceil$$

$$\alpha(G) := \max_{v\in V} \max_{S\subseteq V: v\in S} \left\lceil \frac{|S|-1}{2d(v,S)} \right\rceil = \max_{v\in V} \max_{k=1}^{d(v,V)} \left\lceil \frac{|N_k(v)|-1}{2k} \right\rceil$$

$$\varepsilon(G) = \max_{v\in V} \left\lceil \frac{|N_1(v)|-1}{2} \right\rceil$$

$$\gamma(G) := \min_{v\in V} \max_{S\subseteq V, v\in S} \left\lceil \frac{|S|-1}{d(v,S)} \right\rceil = \min_{v\in V} \max_{k=1}^{d(v,V)} \left\lceil \frac{|N_k(v)|-1}{k} \right\rceil$$

$$\delta(G) := \min_{v\in V} |N_1(v)| - 1$$



$$\phi_\tau(G) = \phi(G) = 4$$

$\beta(G) = 3$ for $S = \{0,1,2,5,7,9\}$

$\alpha(G) = 3$ for $S = \{0,1,2,5,7,9\}$ and $v = 1$ with $d(v,S) = 1$

$\gamma(G) = 4$ for $v = 4$ and $S = V \setminus \{7,9\}$ with $d(v,S) = 2$

$\delta(G) = 2$ for $v = 9$ with $N_1(v) = \{1,3,9\}$

$\varepsilon(G) = 3$ for $v = 1$.

Lower bounds for partial layouts:

Suppose enumeration has already fixed the, say, k leftmost vertices in the layout $u_1\ u_2\ \ldots\ u_k$ with $\tau_{u_i} = i$ for $i = 1, \ldots, k$.

Let $L := \{u_1, \ldots, u_k\}$ and $F := V \setminus L$ the set of free vertices.

Let $\phi_\tau(G)$ denote the minimum bandwidth of a layout which is obtained by extending this partial layout.

For $u_i \in L$, let $s_i := \min\{h : N_h(u_i) \cap F \neq \emptyset\}$.

$$\gamma_\tau(G) := \max_{i=1}^{k} \max_{h=s_i}^{d(u_i,F)} \left\lceil \frac{|N_h(u_i) \cap F| + k - i}{h} \right\rceil .$$

$$\delta_\tau(G) := \max_{i=1}^{k} |N_1(u_i) \cap F| + k - i.$$

- $\phi_\tau(G) \geq \gamma_\tau(G)$
- $\phi_\tau(G) \geq \delta_\tau(G)$

Integer Linear Programming Model:

The set Π of all permutations of $\{1, \ldots, n\}$, each one corresponding to a layout τ of G, can be linearly modelled by:

$$\begin{array}{c} \sum_{u \in V} \tau_u = \frac{n(n+1)}{2}, \\ \sum_{u \in S} \tau_u \geq \frac{|S|(|S|+1)}{2}, \quad S \subset V, S \neq \emptyset, \\ \tau \text{ integer.} \end{array}$$

An ILP formulation for the Bandwidth Problem is:

$$\begin{array}{c} \min \phi \\ \phi \geq \tau_v - \tau_u \quad \forall [u, v] \in E \\ \tau \in \Pi. \end{array}$$

If a partial layout is given, the previous model can be adapted: let $\tau_{u_i} = i$ for $i = 1, \ldots, k$, $L := \{u_1, \ldots, u_k\}$, $F := V \setminus L$. Then, for each $i = 1, \ldots, k$, let $\delta_\tau(i)$ the value of the relaxation:

$$\begin{array}{rl} \min \phi, & \\ \phi \geq \tau_v - \tau_{u_i}, & \forall v \in F, \forall [u_i, v] \in E, \\ \tau \in \Pi, & \\ \tau_{u_h} = h, & \forall h = 1, \ldots, k. \end{array}$$

Proposition: $\max_{i=1,\ldots,k} \delta_\tau(i) = \delta_\tau(G)$.

For each $i = 1, \ldots, k$, let $\gamma_\tau(i)$ the value of the relaxation:

$$\begin{array}{rl} \min \phi, & \\ d(u_i, v) \cdot \phi \geq \tau_v - \tau_{u_i}, & \forall v \in F \\ \tau \in \Pi, & \\ \tau_{u_h} = h, & \forall h = 1, \ldots, k. \end{array}$$

Proposition: $\max_{i=1,\ldots,k} \gamma_\tau(i) = \gamma_\tau(G)$.

ILP Model with **stronger** LP-relaxations:

$$\begin{aligned} \min \phi, & \\ \phi \geq \tau_v - \tau_u, & \quad \forall u \in L, v \in F, [u, v] \in E, \\ \tau \in \Pi, & \\ \tau_{u_i} = i, & \quad \forall i = 1, \ldots, k, \end{aligned}$$

Proposition: It can be solved in $O(m)$ time.

$$\begin{aligned} \min \phi, & \\ d(u, v) \cdot \phi \geq \tau_v - \tau_u, & \quad \forall u \in L, v \in F, \\ \tau \in \Pi, & \\ \tau_{u_i} = i, & \quad \forall i = 1, \ldots, k. \end{aligned}$$

Proposition: It can be solved in $O(n^2 \log n)$ time.

Generalization:

Given:

- $\tau_{u_i} = i$ for $i = 1, \ldots, k$
- $\tau_{v_i} = n - i + 1$ for $i = 1, \ldots, q$.

Let $L := \{u_1, \ldots, u_k\}$, $R := \{v_1, \ldots, v_q\}$ and $F := V \setminus (L \cup R)$. The relaxation is now:

$$\begin{aligned} \min \phi, & \\ d(u, v) \cdot \phi \geq \tau_v - \tau_u, & \quad \forall u \in L, v \in F, \\ d(u, v) \cdot \phi \geq \tau_u - \tau_v, & \quad \forall u \in R, v \in F, \\ \tau \in \Pi, & \\ \tau_{u_i} = i, & \quad \forall i = 1, \ldots, k, \\ \tau_{v_i} = n - i + 1, & \quad \forall i = 1, \ldots, q. \end{aligned}$$

Proposition: It can be solved in $O(n^2 \log n)$ time.

Enumerative algorithm:

function FEASIBLE_RELAX();
begin
    for $i := k+1$ **to** $n-q$ **do**
        for each $v \in F$ **do** $f_v := \max\{f_v, i\}$;
        if $\{v \in F : f_v \leq i\} = \emptyset$ **then return**(FALSE);
        $w := \arg\min\{\ell_v : v \in F, f_v \leq i\}$;
        if $\ell_w < i$ **then return**(FALSE);
        $\tau_w := i$; $F := F \setminus \{w\}$;
    end for;
    return(TRUE);
end.

procedure UPDATE(v,DIR);
begin
    for $h := 1$ **to** $d(v, F)$ **do**
        for each $w \in N_h(v)$ **do**
            if DIR = LEFT **and** $\ell_w > k + h \cdot \phi$ **then** $\ell_w := k + h \cdot \phi$;
            if DIR = RIGHT **and** $f_w < (n-q+1) - h \cdot \phi$ **then** $f_w := (n-q+1) - h \cdot \phi$;
end.

procedure FIND_LAYOUT();
begin
    if $F = \emptyset$ **then** SUCCESS;
    if not FEASIBLE_RELAX() **return**;
    $A := \{v \in F : f_v = k\}$; **comment:** nodes that may occupy the first available position;
    $B := \{v \in F : \ell_v = n-q+1\}$; **comment:** nodes that may occupy the last available position;
    if $|A| \leq |B|$ **then**
        comment: expand layout on the left;
        for each $v \in A$ **do**
            $F := F \setminus \{v\}$; $\tau_k := v$; $k := k+1$; SAVE(ℓ, f);
            UPDATE(v,LEFT);
            FIND_LAYOUT();
            $F := F \cup \{v\}$; $k := k-1$; RESTORE(ℓ, f);
        end for each;
    else
        comment: expand layout on the right;
        for each $v \in B$ **do**
            $F := F \setminus \{v\}$; $\tau_{n-q+1} := v$; $q := q+1$; SAVE(ℓ, f);
            UPDATE(v,RIGHT);
            FIND_LAYOUT();
            $F := F \cup \{v\}$; $q := q-1$; RESTORE(ℓ, f);
        end for each;
    end if;
end.

Initially called with $F := V$; $\ell_v := n, f_v := 1$ $(v \in V)$; $k := 1$; $q := 1$.

Computing $\beta(G)$:

$$\beta(G) := \max_{S \subseteq V} \left\lceil \frac{|S|-1}{d(S)} \right\rceil .$$

REMARK: Let S^* be the set leading to the maximum. If $d(S^*) = 1$, computing $\beta(G)$ is the same as finding the size $\omega(G)$ of a largest clique of G.

We propose an enumerative scheme for the computation of $\beta(G)$ which is based on guessing the value t of $d(S^*)$ and searching for a set S with $d(S) = t$ and $|S|$ as large as possible.

Proposition: For every set S with $d(S) = t$, $|S| \leq \omega(A(t, V))$.

where, for $W \subseteq V$ and $t \in \{1, \ldots, n-1\}$, $A(t, W) := (W, D)$ defines the *auxiliary graph* with $D := \{(u, v) : u, v \in W, d(u, v) \leq t\}$.

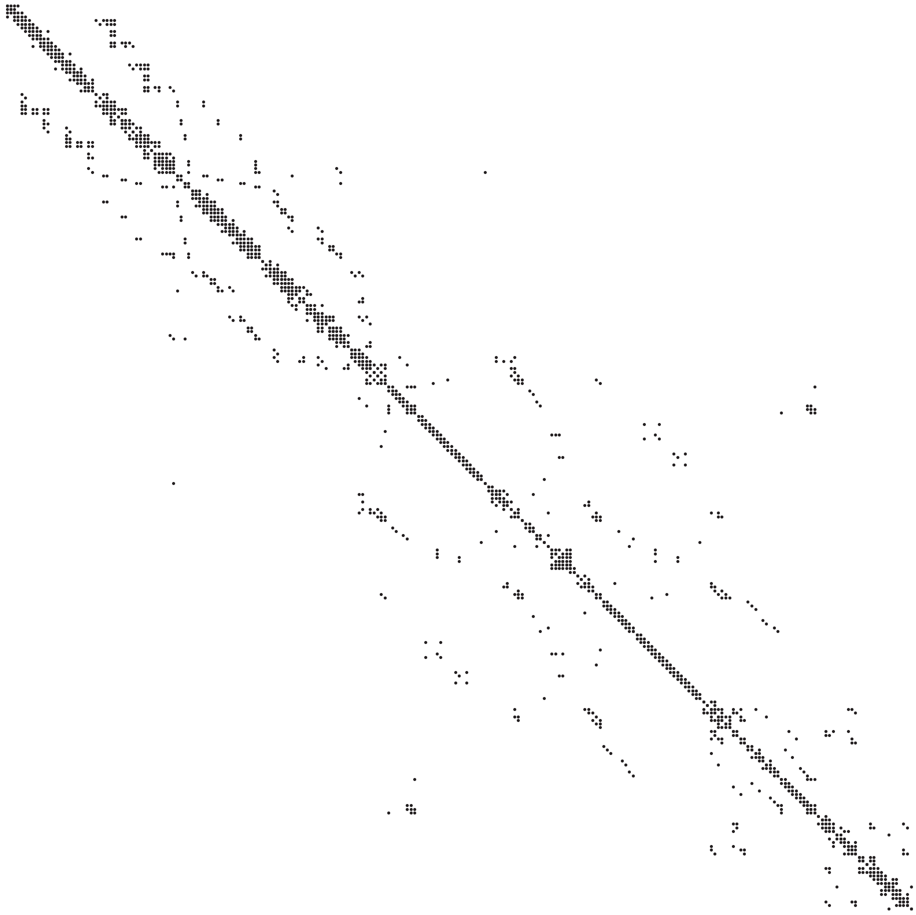
procedure FIND_MAX_S();
begin
  if $F = \emptyset$ **then**
    if $|S| > |S^*|$ **then** $S^* := S$;
    return;
  end if;
  if $\omega(A(t, S \cup F)) \leq |S^*|$ **then return**;
  for each $v \in F$ **do**
    $F := F \setminus \{v\}$; $S := S \cup \{v\}$;
    $F' := F$; $F := F \setminus \{w \in F : d(v, w) > t\}$;
    FIND_MAX_S();
    $F := F'$; $S := S \setminus \{v\}$;
    FIND_MAX_S();
    $F := F \cup \{v\}$;
  end for each;
end.

Initially called with $F := V$, $S := \emptyset$ and $S^* := \emptyset$.

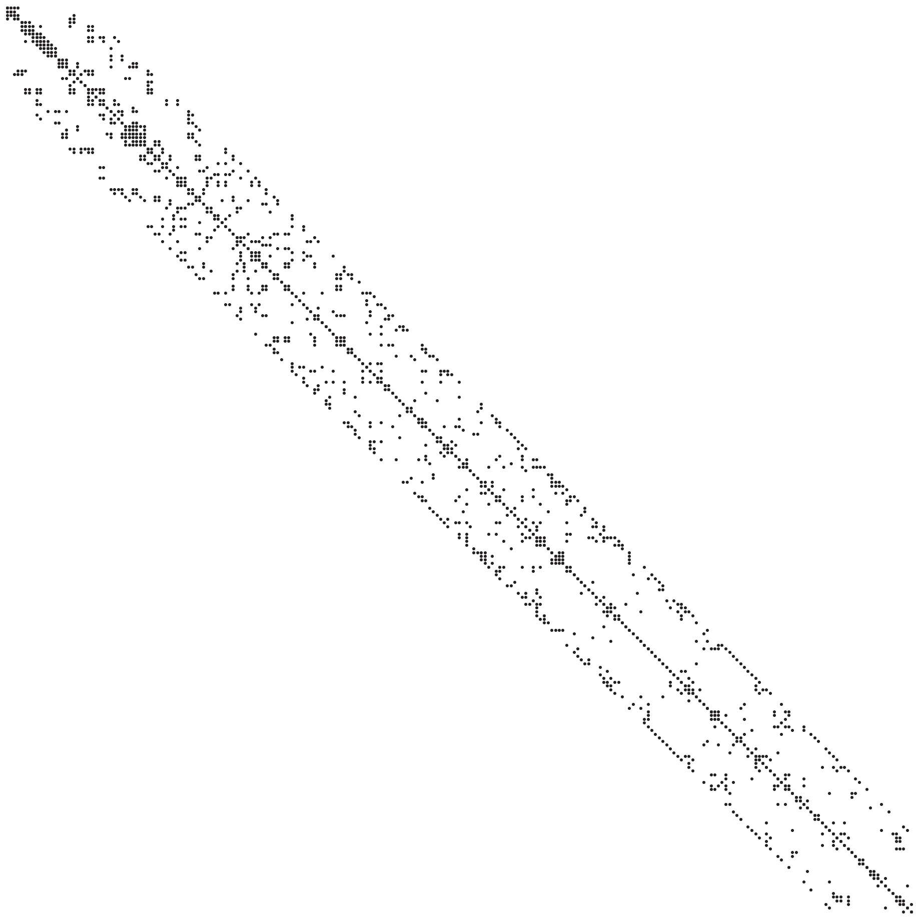
Computational Experiments:

On a personal computer AMD 1333 Mhz.

- All symmetric matrix inducing a connected graph, from *Matrix Market* (including *Harwell-Boeing Collection*) and with $n \leq 200$.
- Random generator of Corso & Manzini (2000)



(a) Initial layout with bandwidth $\phi(G) = 115$



(b) Optimal layout with bandwidth $\phi(G) = 21$

Figure 1: Initial layout and minimum-bandwidth layout for Matrix *dwt_245* with $n = 245$ from the MatrixMarket collection.

name	n	m	dens	$\delta(G)$	$\varepsilon(G)$	$\alpha(G)$	$\gamma(G)$	$\beta(G)$	time	subpr	ψ	$\phi(G)$
bcspwr01	39	46	0.06	1	3	4	4	4	0.00	96	38	5
bcspwr02	49	59	0.05	1	3	5	6	6	0.00	225	34	7
bcspwr03	118	179	0.03	1	5	9	9	10	0.22	568	115	10-11
can__144	144	576	0.06	5	7	7	12	11	2462.09	559	142	13
can__161	161	608	0.05	5	8	12	16	18	7235.50	782	79	18
can__187	187	652	0.04	4	5	8	12	8	>5 hours	—	63	13
can__229	229	888	0.03	2	6	15	21	19	5630.36	2429	172	24-29
can___24	24	68	0.25	3	4	5	5	5	0.00	62	21	5
can___61	61	248	0.14	4	12	13	11	13	0.00	179	50	13
can___62	62	78	0.04	1	3	5	5	5	0.00	160	48	6
can___73	73	152	0.06	2	4	9	14	12	1.82	442	39	16
can___96	96	336	0.07	5	4	7	13	12	10.33	380	31	13
dwt__162	162	510	0.04	1	4	9	8	9	1.15	464	156	10-13
dwt__193	193	1650	0.09	7	15	30	29	30	2.42	1026	62	31-38
dwt__209	209	871	0.04	3	8	20	18	21	3.51	2116	184	21-26
dwt__221	221	814	0.03	3	6	11	9	12	3.62	848	187	11-13
dwt__245	245	730	0.02	1	6	21	16	21	4.12	2052	115	21
dwt___59	59	104	0.06	1	3	5	5	5	0.00	121	25	6
dwt___66	66	127	0.06	1	3	3	3	3	0.16	83	44	3
dwt___72	72	75	0.03	1	2	4	4	4	0.00	123	12	5
dwt___87	87	227	0.06	1	6	10	8	10	0.06	293	63	10
bcsstk01	48	176	0.16	4	6	9	15	14	0.00	147	35	16
bcsstk02	66	2145	1.00	65	33	33	65	65	0.05	133	65	65
bcsstk04	132	1758	0.20	14	23	32	28	34	1.70	580	47	37
bcsstk05	153	1135	0.10	3	12	16	14	18	0.99	569	28	20
bwm200	200	298	0.01	2	2	2	2	2	144.45	111	100	2
e05r0000	236	2928	0.11	7	31	44	47	47	188.18	1770	66	51-53
fidap001	216	2187	0.09	7	25	40	38	40	10.39	1295	64	40-41
fidap005	27	126	0.36	5	7	7	6	8	0.00	33	20	8
fidapm05	42	239	0.28	5	10	10	9	10	0.00	81	35	11
lund_a	147	1151	0.11	4	10	17	14	17	0.94	731	23	19-23
lund_b	147	1147	0.11	4	10	17	14	17	0.93	731	23	19-23
nos4	100	247	0.05	1	3	9	9	9	0.11	390	13	10
pde225	225	532	0.02	2	2	10	10	10	92.55	1179	15	11-15
rdb200	200	460	0.02	3	3	12	13	12	22.74	1086	20	15-16
saylr1	238	564	0.02	2	2	10	10	10	84.26	1178	14	12-14
steam1	240	1881	0.07	8	10	26	26	26	23.57	2181	146	32-45
steam3	80	424	0.13	7	6	6	7	7	0.33	27	43	7
tub100	100	148	0.03	2	2	2	2	2	7.64	61	2	2

Table 1: Characteristics of the real-world instances.

	Corso & Manzini			HYBRID				
name	$\underline{\phi}$	time	subpr	$\underline{\phi}$	$\overline{\phi}$	heu-time	tot-time	subprob
bcspwr01	5	41.91	14858401	5	5	0.05	1.75	89955
bcspwr02	6	t.l.	1111508124	7	7	0.88	0.88	77098
bcspwr03	9	t.l.	745964464	10	11	60.03	t.l.	66410000
can__144	11	t.l.	720547084	13	13	0.00	0.00	146
can__161	16	t.l.	995371667	18	18	0.00	0.00	164
can__187	12	t.l.	888556396	13	13	0.00	0.00	205
can__229	19	t.l.	523991741	24	29	331.91	t.l.	11760000
can___24	5	0.00	145	5	5	0.00	0.00	25
can___61	12	t.l.	981671303	13	13	0.00	0.00	62
can___62	5	t.l.	1011543437	6	6	0.00	0.00	549
can___73	12	t.l.	1242189524	16	16	0.60	0.60	5012
can___96	13	t.l.	1579396857	13	13	0.00	0.00	97
dwt__162	8	t.l.	591233876	11	13	316.37	t.l.	34120000
dwt__193	28	t.l.	1047040262	31	38	1031.61	t.l.	14680000
dwt__209	18	t.l.	368405422	21	28	421.39	t.l.	13170000
dwt__221	9	t.l.	871272016	11	13	301.43	t.l.	9710000
dwt__245	15	t.l.	365660902	21	21	2616.03	2616.03	10604348
dwt___59	6	13.57	4024432	6	6	60.03	60.03	3122523
dwt___66	3	0.00	66	3	3	0.00	0.00	67
dwt___72	4	t.l.	997042066	5	5	0.16	0.16	7146
dwt___87	8	t.l.	1026957908	10	10	0.00	0.00	88
bcsstk01	15	t.l.	1371830047	16	16	0.00	0.00	106
bcsstk02	65	0.00	2146	65	65	0.00	0.00	0
bcsstk04	28	t.l.	1533739977	37	37	34.16	34.16	47776
bcsstk05	15	t.l.	453246149	20	20	0.06	0.06	169
bwm200	2	0.00	201	2	2	0.00	0.00	201
e05r0000	47	t.l.	755374321	50	53	277.43	t.l.	2060000
fidap001	36	t.l.	1000415412	40	41	245.08	t.l.	6040000
fidap005	8	0.00	688	8	8	0.00	0.00	29
fidapm05	11	126.39	60356522	11	11	0.00	60.75	1521913
lund_a	14	t.l.	1357107291	19	23	0.06	t.l.	14650000
lund_b	14	t.l.	1357257996	19	23	0.00	t.l.	13540000
nos4	9	t.l.	582865781	10	10	0.00	66.29	1002861
pde225	8	t.l.	315172726	11	15	0.00	t.l.	12480000
rdb200	11	t.l.	476306794	15	20	4.34	t.l.	14520000
saylr1	9	t.l.	273666777	11	14	0.00	t.l.	8640000
steam1	24	t.l.	1703461772	32	45	788.35	t.l.	6050000
steam3	7	0.00	159	7	7	0.00	0.00	81
tub100	2	0.00	100	2	2	0.00	0.00	0

Table 2: Corso and Manzini's algorithm [?].

						Corso & Manzini			HYBRID			
φ	n	dens	$\alpha(G)$	$\gamma(G)$	ψ	succ	tot-time	$\underline{\phi}$	succ	tot-time	$\underline{\phi}$	$\overline{\phi}$
10	50	0.111	8.3	7.8	45.7	4	64.76	8.5	9	49.99	8.0	9.0
	100	0.059	9.2	8.7	95.1	1	194.77	8.8	7	0.02	9.0	10.0
	150	0.040	9.6	8.7	142.9	0	-	8.8	8	3.42	9.5	10.5
	200	0.030	9.7	9.0	192.5	0	-	8.8	8	4.42	9.0	10.0
	250	0.024	9.7	9.0	241.1	0	-	9.0	8	13.50	9.5	11.0
20	50	0.198	12.7	15.4	46.6	0	-	17.1	8	9.25	18.5	19.5
	100	0.110	18.4	18.0	95.3	0	-	16.9	8	0.29	19.0	20.0
	150	0.076	19.4	18.4	146.8	0	-	17.2	5	11.04	19.2	20.4
	200	0.057	19.6	18.5	196.7	0	-	18.0	7	1.81	19.7	20.7
	250	0.046	19.7	18.7	243.4	0	-	17.9	7	5.08	19.0	20.7
30	100	0.156	24.5	25.9	96.8	0	-	26.2	5	7.77	28.8	29.8
	150	0.108	28.1	27.1	147.2	0	-	25.6	5	1.11	29.0	30.6
	200	0.083	29.0	28.0	196.1	0	-	26.1	2	12.55	29.0	30.3
	250	0.068	29.4	28.1	245.8	0	-	27.1	4	0.06	29.0	30.0
40	100	0.194	25.0	33.2	96.9	0	-	34.6	1	16.31	37.9	39.4
	150	0.138	36.5	37.1	147.4	0	-	35.9	1	0.06	38.7	40.1
	200	0.108	38.1	37.5	196.5	0	-	34.8	1	0.11	39.0	40.2
	250	0.088	39.1	37.8	245.6	0	-	36.0	2	145.63	39.0	40.3
50	100	0.227	25.0	41.2	97.3	0	-	43.7	4	45.09	47.0	49.2
	150	0.166	38.0	45.9	146.8	0	-	43.0	2	0.05	48.1	49.8
	200	0.131	47.3	47.1	196.9	0	-	42.9	5	2.01	49.0	51.0
	250	0.108	48.3	47.6	246.9	0	-	43.8	2	7.53	49.1	50.5
60	150	0.192	38.0	51.5	147.3	0	-	51.9	0	-	57.9	59.9
	200	0.153	50.0	56.0	197.1	0	-	51.6	0	-	58.6	60.5
	250	0.127	57.6	57.3	246.7	0	-	51.7	1	185.48	58.9	60.6
70	150	0.215	38.0	62.2	146.9	0	-	60.0	6	35.77	68.0	69.3
	200	0.174	50.0	65.8	196.5	0	-	62.0	0	-	68.0	69.9
	250	0.145	62.5	66.2	247.3	0	-	63.0	1	91.89	68.2	70.4
80	200	0.192	50.0	72.0	196.7	0	-	68.0	0	-	78.0	80.3
	250	0.162	63.0	76.1	246.8	0	-	66.8	1	2.85	78.3	80.2
90	200	0.210	50.0	81.6	197.6	0	-	74.2	1	8.62	87.4	90.4
	250	0.178	63.0	83.1	246.8	0	-	81.5	1	291.55	87.8	89.7
100	200	0.226	50.0	92.9	198.1	0	-	82.9	4	221.91	97.3	99.0
	250	0.193	63.0	92.5	247.3	0	-	83.6	1	278.14	97.8	99.8

Table 3: Results on randomly generated instances for $p = 0.3$ (10 trials for each line)

φ	n	dens	$\alpha(G)$	$\gamma(G)$	ψ	Corso & Manzini succ	tot-time	$\underline{\phi}$	HYBRID succ	tot-time	$\underline{\phi}$	$\overline{\phi}$
10	50	0.182	9.9	9.4	47.3	10	7.92	-	10	0.80	-	-
	100	0.096	10.0	9.5	95.9	10	30.02	-	10	2.16	-	-
	150	0.065	10.0	9.8	144.0	9	4.92	10.0	10	0.55	-	-
	200	0.049	10.0	10.0	193.5	0	-	10.0	10	2.15	-	-
	250	0.039	10.0	10.0	241.0	8	7.42	10.0	10	4.72	-	-
20	50	0.326	13.2	17.7	47.2	4	131.12	19.5	9	25.36	19.0	20.0
	100	0.181	19.9	19.3	96.8	1	253.75	19.6	9	0.03	20.0	21.0
	150	0.125	20.0	19.8	146.8	2	77.58	19.4	10	1.14	-	-
	200	0.095	20.0	19.9	197.4	0	-	19.4	9	1.06	20.0	21.0
	250	0.076	20.0	19.9	244.5	0	-	20.0	10	2.08	-	-
30	100	0.257	25.0	28.6	98.0	0	-	29.5	7	1.76	30.0	31.0
	150	0.180	30.0	29.3	147.5	0	-	29.2	6	2.13	30.0	31.0
	200	0.138	30.0	29.2	197.3	0	-	29.1	6	17.00	30.0	31.3
	250	0.113	30.0	29.3	246.9	0	-	29.4	7	4.51	30.0	31.3
40	100	0.321	25.5	38.3	97.9	0	-	39.0	7	27.74	39.7	40.7
	150	0.230	38.0	39.2	147.9	0	-	39.1	7	2.17	40.0	41.3
	200	0.180	40.0	39.3	197.1	0	-	38.9	9	1.29	40.0	41.0
	250	0.147	40.0	39.4	247.8	0	-	38.9	3	0.09	40.0	41.0
50	100	0.376	28.9	47.6	97.9	0	-	49.3	5	0.27	49.0	50.4
	150	0.277	38.0	48.7	147.6	0	-	49.2	7	2.61	49.7	51.0
	200	0.219	49.3	49.1	197.3	0	-	49.0	7	0.07	49.7	51.0
	250	0.180	50.0	49.0	247.6	0	-	49.0	5	0.09	50.0	51.2
60	150	0.320	38.1	57.8	147.7	0	-	59.1	5	0.08	59.6	60.8
	200	0.255	50.0	58.8	197.6	0	-	58.8	8	4.09	60.0	61.0
	250	0.211	59.7	59.1	247.4	0	-	59.0	6	12.34	60.0	61.5
70	150	0.358	40.6	67.7	147.8	0	-	68.8	7	7.40	70.0	71.3
	200	0.289	50.0	67.9	197.7	0	-	68.8	9	21.62	70.0	71.0
	250	0.241	63.0	68.6	248.4	0	-	68.8	5	4.84	70.0	72.0
80	200	0.320	50.0	77.5	198.3	0	-	79.0	6	5.86	79.8	80.8
	250	0.269	63.0	78.8	247.9	0	-	78.8	6	22.06	80.0	81.0
90	200	0.349	51.2	87.4	198.2	0	-	88.5	8	3.21	90.0	92.0
	250	0.295	63.0	87.8	248.0	0	-	88.4	5	1.02	90.0	91.2
100	200	0.375	55.0	97.8	198.3	0	-	98.8	2	94.33	99.3	100.8
	250	0.320	63.0	97.5	247.9	0	-	98.4	3	3.92	99.7	101.4

Table 4: Results on randomly generated instances for $p = 0.5$ (10 trials for each line).

φ	n	dens	$\alpha(G)$	$\gamma(G)$	ψ	HYBRID succ	tot-time	$\underline{\phi}$	$\overline{\phi}$
50	400	0.072	49.6	48.2	396.5	3	94.56	49.7	51.0
	600	0.049	50.0	48.4	595.2	1	15.99	50.0	51.4
	800	0.037	50.0	48.7	793.5	3	34.12	50.0	51.4
	1000	0.030	50.0	48.9	991.0	1	2.09	50.0	52.2
100	400	0.134	97.7	97.7	396.6	0	-	98.9	100.5
	600	0.094	99.1	98.0	597.2	1	16.15	99.1	101.1
	800	0.072	99.8	98.3	796.5	1	1.65	99.8	101.6
	1000	0.059	99.9	98.2	995.6	1	3.08	99.9	102.8
150	400	0.188	100.0	142.6	397.4	0	-	148.0	151.0
	600	0.135	147.5	146.8	597.8	0	-	149.1	151.6
	800	0.105	148.9	147.5	795.8	0	-	149.2	152.0
	1000	0.085	149.8	147.8	996.3	1	4.31	149.8	152.6
200	400	0.231	100.0	192.4	397.3	1	409.44	197.2	206.9
	600	0.171	150.0	196.2	596.6	0	-	198.3	200.6
	800	0.135	197.5	197.1	797.6	0	-	199.0	201.5
	1000	0.111	198.8	197.4	996.2	0	-	199.0	202.5

Table 5: Results on large randomly generated instances for $p = 0.3$ (10 trials for each line).

φ	n	dens	$\alpha(G)$	$\gamma(G)$	ψ	HYBRID succ	tot-time	$\underline{\phi}$	$\overline{\phi}$
50	400	0.119	50.0	49.3	397.4	2	6.10	50.0	51.5
	600	0.081	50.0	49.9	595.6	3	126.33	50.0	51.3
	800	0.061	50.0	49.9	795.5	2	39.10	50.0	51.3
	1000	0.049	50.0	49.9	993.7	0	-	50.0	51.7
100	400	0.222	99.6	99.0	397.4	3	4.34	100.0	101.1
	600	0.155	100.0	99.5	597.6	3	6.45	100.0	101.0
	800	0.119	100.0	99.6	797.0	3	18.79	100.0	101.7
	1000	0.096	100.0	99.7	997.0	0	-	100.0	101.5
150	400	0.309	100.0	147.7	397.8	4	3.85	149.8	151.5
	600	0.222	149.5	149.0	597.8	3	1.93	150.0	151.0
	800	0.172	150.0	149.1	796.8	2	18.37	150.0	151.3
	1000	0.140	150.0	149.3	997.0	0	-	150.0	151.8
200	400	0.380	110.5	197.8	397.9	3	20.79	199.0	200.9
	600	0.281	150.0	199.1	597.4	4	14.04	200.0	201.2
	800	0.221	199.6	199.2	797.6	3	17.07	200.0	201.3
	1000	0.182	200.0	199.2	996.9	0	-	200.0	202.0

Table 6: Results on large randomly generated instances for $p = 0.5$ (10 trials for each line).