Experiments with Linear and Semidefinite Relaxations for Solving the Minimum Graph Bisection Problem

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- Node Weighted Graph Bisection
- Knapsack Tree Inequalities
- Bisection Knapsack Walk Inequalities
- Connections to Odd Cycle Inequalities
- Strengthenings
- The Cluster Weight Polytope
- Some Numerical Results
- Conclusion and Problems

(supported by the DFG)
The Node Weighted Bisection Problem

- simple undirected Graph $G = (V, E)$, 
  $V = \{1, \ldots, n\}$, $E \subseteq \{ij : i, j \in V, i \neq j\}$
  node weights $f_i \in \mathbb{N}_0$ for $i \in V$,
  capacity $F \in \mathbb{N}_0$,
- Find a bisection $(S, V \setminus S)$ with
  $f(S) := \sum_{i \in S} f_i \leq F$ and $f(V \setminus S) \leq F$
  and $\delta(S)$ minimal (weights $w_{ij}$)

\[ P_B = \text{conv}\{y = \delta(S) : S \subseteq V, f(S) \leq F, f(V \setminus S) \leq F\} \subseteq P_{\text{CUT}} \]

**IP-formulation:** (suppose $G$ contains a spanning star rooted at $s$)

\[ y_{ij} = \begin{cases} 
  1 & \text{if } ij \text{ is in the cut} \\
  0 & \text{otherwise}
\end{cases} \]

\[
\begin{align*}
\min & \quad \sum_{ij} w_{ij} y_{ij} \\
\text{s.t.} & \quad f_s + \sum_{v \neq s} f_v (1 - y_{sv}) \leq F & i \in V \\
& \quad \sum_{v \neq s} f_v y_{sv} \leq F \\
& \quad \sum_{i \in C \setminus U} y_{ij} + \sum_{i \in U} (1 - y_{ij}) \geq 1 & \text{cycle } C \subseteq E, \text{odd } U \subseteq C \\
& \quad y \in \{0, 1\}^E
\end{align*}
\]
Related Polyhedral Investigations/Surveys in the Literature

- Cut Polytope: [DezaLaurent97]
- Node Capacitated Graph Partitioning: [FMdSWW96]
- Equipartition: [ConfortiRaoSassano90], [deSouza93]
- Knapsack: [Weismantel97]
Bounding the size of the side belonging to some root \( r \in V \)

Given \( y = \delta(S) \), does \( i \) belong to the same side as \( r \)?

**Knapsack Tree Inequalities:** [FMdSWW96]

Yes, if the shortest path \( P_{ri} \) from \( r \) to \( i \) consists of edges \( y_e = 0 \):

\[
1 - \sum_{e \in P_{ri}} y_e \text{ positive } \Rightarrow \ i \text{ belongs to the same side as } r \\
\]

\[
\rightarrow \quad f_r + \sum_{i \in V \setminus \{r\}} f_i \left[ 1 - \sum_{e \in P_{ri}} y_e \right] \leq F
\]

for some tree \( T = (V_T, E_T) \) and its paths \( P_{ri} \) from \( r \) to \( i \).

Collecting the weights of the edges along the paths from root \( r \) for each edge and applying “trivial” strengthening yields a **truncated knapsack tree ineq.**

If \( \sum a_v z_v \leq a_0 \) is valid for \( P_K := \text{conv}\{z \in \{0, 1\}^V : \sum_{v \in V} f_v z_v \leq F\} \) then

\[
\sum_{e \in E_T} \alpha^r_{e} y_e \geq \alpha_0 \quad \text{with} \quad \alpha_0 := \sum_{v \in V_T} a_v - a_0 \quad \text{and} \quad \alpha^r_e := \min\{\sum_{v: e \in P_v} a_v, \alpha_0\}
\]

is valid for \( P_B \). [FMdSWW96]
The Choice of the Root in Knapsack Tree Inequalities

![Diagram](image-url)
Knapsack Tree Inequalities for a tree $T = (V_T, E_T)$

If $\sum a_v z_v \leq a_0$ is valid for $P_K := \text{conv}\{z \in \{0, 1\}^V : \sum_{v \in V} f_v z_v \leq F\}$ then

$$\sum_{e \in E_T} \alpha_e^r y_e \geq \alpha_0$$

with $\alpha_0 := \sum a_v - a_0$ and $\alpha_e^r := \min \{ \sum_{v \in V_T} a_v, \alpha_0 \}$

is valid for $P_B$.

Best root in $T$ is a minimal root $r \in R = \text{Argmin}_{s \in V_T} \sum_{e \in E_T} \alpha_e^s$ (easy to find)

**Theorem 1** If $r \in R$ then for all $s \in V_T$, $y \in P_B$  
$$\sum_{e \in E_T} \alpha_e^s y_e \geq \sum_{e \in E_T} \alpha_e^r y_e \geq \alpha_0$$

For $f_v = 1 \ (v \in V)$ and root $r$ each unreduced edge has weight $|V_e^r| = |\{v : e \in P_r\}|$ (nodes “below” $e$) → minimal roots are “centered”

**Theorem 2** Assume $G = (V, E)$ is a tree, $f_v = 1 \ (v \in V)$, $\frac{|V|}{2} + 1 \leq F \leq |V|$, and for root $r$ all edges of branchless paths have reduced knapsack weight.  
$$\sum_{e \in E} \min\{|V_e^r|, |V| - F\} y_e \geq |V| - F$$
is facet-defining iff $r$ is a minimal root.
Bounding the size of the side belonging to some root $r \in V$

Given $y = \delta(S)$, does $i$ belong to the same side as $r$?

**Knapsack Tree Inequalities:**
Yes, if the shortest path $P_{ri}$ from $r$ to $i$ consists of edges $y_e = 0$:

$$1 - \sum_{e \in P_{ri}} y_e \text{ positive } \Rightarrow \text{ } i \text{ belongs to the same side as } r$$

$$\rightarrow f_r + \sum_{i \in V \setminus \{r\}} f_i \left[1 - \sum_{e \in P_{ri}} y_e\right] \leq F$$

**Bisection Knapsack Walk Inequalities:** (exploit bipartition)
Yes, if there is a path $P_{ri}$ with an **even** set $H_{ri} \subseteq P_{ri}$ of cut edges:

$$1 - \sum_{e \in P_{ri} \setminus H_{ri}} y_e - \sum_{e \in H_{ri}} (1 - y_e) \text{ positive } \Rightarrow \text{ } i \text{ belongs to the same side as } r$$

$$f_r + \sum_{i \in V \setminus \{r\}} f_i \left[1 - \sum_{e \in P_{rv} \setminus H_{ri}} y_e - \sum_{e \in H_{ri}} (1 - y_e)\right] \leq F$$
Finding the best path $P_{ri}$ with even set $H_{ri} \subseteq P_{ri}$

For $y \in [0,1]^E$, root $r$ and node $i$ the goal is to maximize

$$1 - \sum_{e \in P_{ri} \setminus H_{ri}} y_e - \sum_{e \in H_{ri}} (1 - y_e)$$

simultaneously for all $i$ by a shortest path tree in an auxiliary graph:

Note: • best walk $P_{ri}$ and $H_{ri}$ can be found in polynomial time
• they do not depend on the knapsack inequality $f(S) \leq F$
  → find paths first, then use knapsack separator → $a(S) \leq a_0$

Alg. almost identical to odd cycle separation for $P_{\text{CUT}}$ [BM86]
Connections of odd cycle to bisection knapsack walk inequalities

**Odd cycle inequalities:** Each cycle must be cut an even number of times
For \( C \) a cycle and odd \( U \subseteq C \):

\[
\sum_{e \in C \setminus U} y_e + \sum_{e \in U} (1 - y_e) \geq 1
\]

Suppose \( y \in [0, 1]^E \) and \( ri \in E \), then \( i \) belongs to the same side as \( r \) if

\[1 - y_{ir}\]

is close to one.

Let \( P_{ri} \) with even \( H_{ri} \) be the best bisection path from \( r \) to \( i \) with \( ri \notin P_{ri} \),

Set \( C = P_{ri} \cup \{ri\} \) and \( U = H_{ri} \cup \{ri\} \) (odd) for the odd cycle inequality

\[
1 - y_{ir} \geq 1 - \sum_{e \in P_{ri} \setminus H_{ri}} y_e - \sum_{e \in H_{ri}} (1 - y_e)
\]

- In the presence of odd cycle ineqs. a direct edge gives the best bound!
- Without a direct edge, one may derive the bound using odd cycle ineqs.

**Theorem 3** Let \( G = K_n \) and \( y \) in the metric polytope, then the strongest bisection knapsack walk inequalities are stars.
Strengthenings of Bisection Knapsack Walk Inequalities

- Trivial strengthening by rounding down coefficients that are too large

- If it is not possible/worthwhile to reach a part of $G$ from root $r$:

Consider, e.g., a bisection cut $y$ and connected components $V_1$ and $V_2$ with partitions $(S_1(y), V_1 \setminus S_1(y))$, $S_1(y) \subseteq V_1$ and $(S_2(y), V_2 \setminus S_2(y))$, $S_2(y) \subseteq V_2$, then

$$
\sum_{i \in S_1(y)} f_i \leq F - \min\{ \sum_{i \in S_2(y)} f_i, \sum_{i \in V_2 \setminus S_2(y)} f_i \}
$$

→ try to bound $\min\{ \sum_{i \in S_2(y)} f_i, \sum_{i \in V_2 \setminus S_2(y)} f_i \}$ from below

→ Study, for the knapsack inequality $a^T x \leq a_0$, $a \geq 0$, and subgraph $\bar{G}$ the polyhedron of the convex lower envelope of the function

$$
\beta_{\bar{G}}(y) = \inf\{a(S), a(\bar{V} \setminus S) : S \subseteq \bar{V}, \max\{a(S), a(\bar{V} \setminus S)\} \leq a_0, y = \chi_{\delta_0(S)}\}$$
The Cluster Weight Polytope

**Definition:**
For $G(V,E)$ and $a \in \mathbb{R}_+^V$, $a_0 \in \mathbb{R}_+$, set for $S \subseteq V$

$$h(S) = \left( \begin{array}{c} a(S) \\ \chi^{\delta(S)} \end{array} \right) =: \left( \begin{array}{c} y_0 \\ y \end{array} \right)$$

then the cluster weight polytope is the set

$$P_{CW} = \text{conv} \left\{ h(S) : S \subseteq V, a(S) \leq a_0, a(V \setminus S) \leq a_0 \right\}$$

Observe that for feasible $S$,

$$\frac{1}{2} h(S) + \frac{1}{2} h(V \setminus S) = \left( \begin{array}{c} \frac{1}{2} a(V) \\ \chi^{\delta(S)} \end{array} \right)$$

**Observation 4** $P_{CW}$ is symmetric with respect to the hyperplane $y_0 = \frac{1}{2} a(V)$

We will give a complete description of $P_{CW}$ for stars with $a_0 \geq a(V)$. 
(Stars can be separated reasonably well within the bisection setting)
The nontrivial facets of $P_{CW}$ for stars with $a_0 \geq a(V)$

Call a triple $(V_p, \bar{v}, V_n)$ feasible if it is a partition of $V \setminus \{r\}$ with

$$a(V_p) \leq \frac{1}{2} a(V) \quad \text{and (if } \bar{v} \text{ exists)} \quad a(V_p \cup \{\bar{v}\}) > \frac{1}{2} a(V).$$

For each feasible triple the following inequality is facet defining:

$$y_0 + \sum_{i \in V_p} a_i y_{ri} + [a(V) - 2a(V_p) - a_{\bar{v}}] y_{r\bar{v}} - \sum_{i \in V_n} a_i y_{ri} \leq a(V)$$

Its symmetric version is

$$y_0 - \sum_{i \in V_p} a_i y_{ri} - [a(V) - 2a(V_p) - a_{\bar{v}}] y_{r\bar{v}} + \sum_{i \in V_n} a_i y_{ri} \geq 0$$
The complete description of \( P_{CW} \) for stars with \( a_0 \geq a(V) \)

**Theorem 5** \( G = (V, E) \) a star with root \( r \in V \), \( a \in \mathbb{R}_+^E \setminus \{0\} \), \( a_0 \geq a(V) \).

- If \( a_r < \frac{1}{2}a(V) \) then \( P_{CW} \) is the set of points satisfying
  
  \[
  0 \leq y_{ri} \leq 1 \quad \forall ri \in E
  \]
  
  \[
  y_0 + \sum_{i \in V_p} a_i y_{ri} + [a(V) - 2a(V_p) - a_{\bar{v}}] y_{r\bar{v}} - \sum_{i \in V_n} a_i y_{ri} \leq a(V) \quad \forall \text{feasible } (V_p, \bar{v}, V_n)
  \]
  
  \[
  y_0 - \sum_{i \in V_p} a_i y_{ri} - [a(V) - 2a(V_p) - a_{\bar{v}}] y_{r\bar{v}} + \sum_{i \in V_n} a_i y_{ri} \geq 0 \quad \forall \text{feasible } (V_p, \bar{v}, V_n)
  \]

- If \( a_r \geq \frac{1}{2}a(V) \) then \( P_{CW} \) is the set of points satisfying
  
  \[
  0 \leq y_{ri} \leq 1 \quad \forall ri \in E
  \]
  
  \[
  y_0 + \sum_{i \in V \setminus \{r\}} a_i y_{ri} \leq a(V)
  \]
  
  \[
  y_0 - \sum_{i \in V \setminus \{r\}} a_i y_{ri} \geq 0
  \]

The decisive step in the proof:

All inequalities bounding \( y_0 \) from above are of this form.
Sketch of proof: We use

Lemma 6 Suppose \( y_0 + \sum_{i \in V \setminus \{r\}} \gamma_i y_i \leq \gamma_0 \) is a facet of \( P_{CW} \). Then

\[-a_i \leq \gamma_i \leq a_i \quad i \in V \setminus \{r\}\]
\[
\gamma_0 = a(V)\]
\[
\sum_{i \in V \setminus \{r\}} \gamma_i \leq a_r
\]

For a given \( \bar{y} \in [0, 1]^E \) find the best inequality bounding \( y_0 \) by solving

\[
\begin{align*}
\min \quad & a(V) - \sum_{i \in V \setminus \{r\}} \bar{y}_i \gamma_i \\
\text{s.t.} \quad & \sum_{i \in V \setminus \{r\}} \gamma_i \leq a_r \\
& -a_i \leq \gamma_i \leq a_i \quad i \in V \setminus \{r\}
\end{align*}
\]

\[
\Longleftrightarrow \quad \begin{align*}
\max \quad & \sum_{i \in V \setminus \{r\}} \bar{y}_i \xi_i \\
\text{s.t.} \quad & \sum_{i \in V \setminus \{r\}} \xi_i \leq a(V) \\
& 0 \leq \xi_i \leq 2a_i \quad i \in V \setminus \{r\}
\end{align*}
\]

Continuous knapsack with greedy solution: Set \( \xi_i \) to max sorted by \( \bar{y}_i \):

\[
\Rightarrow \quad \xi_i = 2a_i \quad \text{for } \quad i \in V_p \subseteq V \quad \text{with } \min_{i \in V_p} \bar{y}_i \geq \max_{i \in V \setminus (V_p \cup \{r\})} \bar{y}_j \quad \text{and } \sum_{i \in V_p} 2a_i \leq a(V)
\]

and the fractional \( \xi_{\bar{v}} = a(V) - 2a(V_p) \)

\[
\Rightarrow \quad y_0 + \sum_{i \in V_p} a_i y_i + [a(V) - 2a(V_p) - a_{\bar{v}}] y_{r_{\bar{v}}} - \sum_{i \in V_n} a_i y_i \leq a(V)
\]
Finding Stars for Knapsack Walk Inequalities when \( f_i = 1 \ \forall i \)

Given

\[
\pi_i := \sum_{i \in V \setminus \{r\}} \sum_{i \in V \setminus \{r\}} \left[ 1 - \sum_{e \in P_r \setminus H_i} y_e - \sum_{e \in H_i} (1 - y_e) \right] \leq F - 1,
\]

find the best star rooted at some \( s \in V \setminus \{r\} \) to strengthen this inequality.

---

\( V_s \ldots \) candidate nodes for the star rooted at \( s \) \hspace{1cm} (r excluded)

decide on \( \overline{v} \), \( |V_p| = |V_n| \) or do not include

Build the star by adding pairs of nodes, one to \( V_p \) and one to \( V_n \) with gain

\[
|y_{si} - y_{sj}| - \pi_i - \pi_j
\]

(and a pair consisting of the root \( s \) and a node \( j \) with gain \( y_{sj} - \pi_j \))

\( \rightarrow \) construct an auxiliary graph and find a maximum weight matching

---

\( \Rightarrow \) if \( f_i = 1 \ \forall i \) the most violated star strengthened knapsack walk inequalities can be found in polynomial time.
The Setting of the Numerical Experiments

Used LP and SDP-relaxation in the same Branch&Cut-framework SCIP
[thanks to Tobias Achterberg from ZIB Berlin]

LP-relaxation
- basic relaxation: add a star into $G$ if necessary and separate odd cycles
- solve LPs using CPLEX

SDP-relaxation
- use same graph as LP
- canonical max-cut relaxation in $\{-1, 1\}$-variables ($\text{diag}(X) = e, X \succeq 0$)
- capacity constraint by $\langle ff^T, X \rangle \leq (2F - f(V))^2$,
- solve dual by Spectral Bundle Method with primal aggregation
- separate on primal aggregate w.r.t. the support, possibly enlarge the support

Separation routines
except for odd cycles, both use the same separation routines for
knapsack star and bisection knapsack walk inequalities
## Root Node Value and Computation Time (≤ 4h)

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* lower bound, n number of nodes, m number of edges
## Linear vs. Semidefinite Relaxation - Branch & Cut

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Summary

• In Knapsack Tree Ineqs. the root should be chosen carefully

• Knapsack Walk Ineqs. are a specialization of Knapsack Trees to Bisection, they are closely related to Odd Cycle Inequalities, the best paths from the root to each node can be found in polynomial time

• We introduced the Cluster Weight Polytope and gave its complete description for stars without capacity limit.

• For $f_v = 1$, Star Strengthened Bisection Knapsack Walk Inequalities can be separated in pol. time

• For general bisection problems, current SDP-relaxation approaches are competitive if not superior to current LP-techniques in practice

Problems

• Complete description of the Cluster Weight Pol. for trees and $a_0 \geq a(V)$? (would allow to span entire components; no hope for $a_0 < a(V)$)

• Strengthen Knapsack Walk Inequalities by stars for general weights