A new characterization of Seymour graphs

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Outline

1. Motivation
2. Definitions: complete packing of cuts, joins
3. Seymour Graphs
4. Around Seymour graphs
5. Co-NP characterization of Seymour graphs
6. New Co-NP characterization of Seymour graphs
7. Proof
8. Algorithmic aspects
9. Open problem
Motivation

Edge-disjoint paths problem

Given a graph $H = (V, E)$ and $k$ pairs of vertices $\{s_i, t_i\}$, decide whether there exist $k$ edge-disjoint paths connecting the $k$ pairs $s_i, t_i$. 
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Reformulation by adding the set $F$ of edges $s_i t_i$. 
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**Complete packing of cycles**

Given a graph $H' = (V, E + F)$, decide whether there exist $|F|$ edge-disjoint cycles in $H'$, each containing exactly one edge of $F$. 
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Suppose $H'$ is planar. The problem in the dual :
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Suppose $H'$ is planar. The problem in the dual:

Complete packing of cuts

Given a graph $G = (V', E' + F')$, decide whether there exist $|F'|$ edge-disjoint cuts in $G$, each containing exactly one edge of $F'$. 
An example

Edge-disjoint paths problem
Complete packing of paths

An example
An example

Adding the edges

Characterization of Seymour graphs

January 2009
The graph $H'$
An example

Complete packing of cycles
An example

$H'$ is planar
An example

\[ H' \text{ and his dual} \]
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Complete packing of cycles and cuts
Complete packing of cuts

The graphs are not planar anymore!
Complete packing of cuts

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Necessary condition

If the graph $G = (V, E + F)$ admits a complete packing of cuts, then $F$ is a join: for every cycle $C$, $|C \cap F| \leq |C \setminus F|$.
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Given a graph \( G = (V, E + F) \), decide whether there exist \(|F| \) edge-disjoint cuts in \( G \), each containing exactly one edge of \( F \).

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\[ \text{NOT: } K_4 \]
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Theorem (Middendorf, Pfeiffer)
Given a join in a graph, decide whether there exists a complete packing of cuts is an NP-complete problem.
Theorem (Seymour)

If $G$ is a bipartite graph, then for every join there exists a complete packing of cuts.
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If $G$ is a bipartite graph, ($\iff$ no odd cycle)
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$G$ is a Seymour graph $\iff$ if for every join there exists a complete packing of cuts.
Around Seymour graphs

subclasses

1. **Seymour**: Graphs without odd cycle,
2. **Seymour**: Graphs without subdivision of $K_4$,
3. **Gerards**: Graphs without odd $K_4$ and without odd prism,
4. **Szigeti**: Graphs without non-Seymour odd $K_4$ and without non-Seymour odd prism.
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$K_4$, prism, odd $K_4$, odd prism
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Superclass

Seymour graph \(\implies\) no even subdivision of $K_4$ and of prism.
Attention!

Seymour property is not inherited to subgraphs!
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non-Seymour odd $K_4$  Seymour graph
Definition

Given a join $F$, a cycle $C$ is $F$-tight if $|C \cap F| = |C \setminus F|$.
Remarks

Given a join $F$, an $F$-complete packing of cuts $Q$, two $F$-tight cycles $C_1$ and $C_2$ and a cycle $C$ in $C_1 \cup C_2$, then

- each edge of $C_i$ (and hence of $C$) belongs to a cut $Q \in Q$,
- $\{C \cap Q : Q \in Q, C \cap Q \neq \emptyset\}$ partitions $C$ and $|C \cap Q|$ is even,
- $|C|$ is even so $C_1 \cup C_2$ is bipartite.
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Preliminaries

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- $|C|$ is even so $C_1 \cup C_2$ is bipartite.

Lemma (Sebő)

If for a join $F$ of $G$ there exist two $F$-tight cycles whose union is not bipartite, then $G$ is not Seymour.
Co-NP characterization of Seymour graphs

**Theorem (Ageev, Kostochka, Szigeti)**

*G* is not Seymour if and only if *G* admits a join *F* and two *F*-tight cycles whose union is an odd *K*_4 or an odd prism.
Theorem (Ageev, Kostochka, Szigeti)

\( G \) is not Seymour if and only if \( G \) admits a join \( F \) and two \( F \)-tight cycles whose union is an odd \( K_4 \) or an odd prism.

Examples

- Seymour odd \( K_4 \)
- non-Seymour odd prism
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### Examples

- **Seymour odd $K_4$**
- **non-Seymour odd prism**

### Important remark

If a graph $G$ contains as a subgraph an even subdivision of $K_4$ or of prism then $G$ is not Seymour.
Forbidden minors?

Attention!

Contraction of an edge does not keep Seymour property.
A new notion of contraction

Definitions

1. \( G \) is factor-critical if \( \forall v \in V, \ G - v \) admits a perfect matching.

2. The contraction of a factor-critical subgraph and its neighbors is a factor-contraction.
A new notion of contraction

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\[ G \xrightarrow{\text{factor-contraction}} (X \cup N(X)) \]
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Important lemma

Factor-contraction keeps the Seymour property!
Theorem (Ageev, Sebő, Szigeti)

$G$ is not Seymour if and only if

- $G$ can be factor-contracted to a graph
- that contains as a subgraph an even subdivision of $K_4$ or of the prism.
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Examples

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Proof of sufficiency:

1. Factor-contraction keeps the Seymour property,
2. If the contracted graph $H$ contains as a subgraph an even subdivision of $K_4$ or of prism then $H$ is not Seymour.
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Complete 2-packing of cuts

Complete 2-packing of cuts (for $G$ and $F \subseteq E(G)$)

1. $2|F|$ cuts so that
2. every edge of $G$ belongs to $\leq 2$ cuts and
3. every cut contains exactly one edge of $F$. 
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Characterization of Seymour graphs

Z. Szigeti (G-SCOP, Grenoble)

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1. $2|F|$ cuts so that
2. every edge of $G$ belongs to $\leq 2$ cuts and
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**Example:** If $Q$ is a CPC, then $2Q$ is a C2PC.
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Theorem (Edmonds-Johnson, Lovász)

$F$ is a join $\iff$ there exists a complete 2-packing of cuts.
Complete 2-packing of cuts

<table>
<thead>
<tr>
<th>Complete 2-packing of cuts (for $G$ and $F \subseteq E(G)$)</th>
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<tbody>
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<td>1. $2</td>
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**Theorem (Sebő)**

Let $G$ be a graph, $F \neq \emptyset$ a join, $v \in V(F)$.

(a) $\exists$ an $F$-complete 2-packing of cuts $\{\delta(X) : X \in C\}$ and $C \in C$ st

- $G[C]$ is factor-critical,
- $\{c\} \in C \ \forall c \in C$ (if $|C| = 1$, then $C$ is contained twice in $C$),
- $v \notin C$. ($C \subseteq V(F) - v$.)

(b) If there exists an $F$-complete packing of cuts then there is one containing a star different of $\delta(v)$. 
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Z. Szigeti (G-SCOP, Grenoble)  Characterization of Seymour graphs  January 2009  15 / 18
Complete 2-packing of cuts

**Complete 2-packing of cuts (for \( G \) and \( F \subseteq E(G) \))**

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Proof of necessity:

1. **Minimal counter-example:**
   1. $G$ non-Seymour graph,
   2. any factor-contraction results in a Seymour graph,
   3. $F$ a join without $F$-complete packing of cuts.

2. **Application of Sebő’s Theorem:**
   1. No C2PC for $(G, F)$ contains a star twice.
   2. Let $v \in V(F)$. Let $C$ and $C' \in C$.
   3. Factor-contracting $C$, $F_C$ is a join and $G_C$ is Seymour.
   4. $\exists$ CPC $Q'$ for $(G_C, F_C)$ containing a star different of $\delta(v_C)$.
   5. $2Q' \cup \delta(C) \cup \{\delta(c) : c \in C\}$ is a C2PC for $(G, F)$.
   6. By (2.1), $F_C = \emptyset$, that is $C = V(F) - v$.

3. **Subgraph:**
   1. $G' - v = C$ is factor-critical $\forall v \in V(F)$, that is $G'$ is bicritical (and non-trivial).

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   1. non-trivial bicritical graphs contain as a subgraph an even subdivision of $K_4$ or of the prism.
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What we can not do

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NP characterization?
Open problem

NP characterization?

Find a construction for Seymour graphs!