Strong Formulations for the Survivable Network Design with Hop Constraints Problem

A. Ridha Mahjoub$^1$, Luidi Simonetti$^2$, Eduardo Uchoa$^2$

$^1$Université Paris-Dauphine
mahjoub@lamsade.dauphine.fr

$^2$Universidade Federal Fluminense
luidi@ic.uff.br
uchoa@producao.uff.br

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The Survivable Network Design with Hop Constraints (SNDH) Problem

**Instance:** Undirected graph $G = (V, E)$ with $n$ vertices and $m$ edges, edge costs $c_e$, a set of demands (pairs of vertices) $D$, integers $K \geq 1$ and $H \geq 2$.

**Solution:** A minimum cost subgraph $T$ containing $K$ edge-disjoint paths of length at most $H$ joining the pairs of vertices in each demand.

- $K$ controls the desired level of Network Survivability,
- $H$ controls the Quality of Service requirements.

Instances where all the demands have a common vertex (the root) are called *rooted*, the other instances are *unrooted*.

A vertex that does not belong to any demand is a *Steiner vertex*. Instances without Steiner vertices are *spanning*. 
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Example of a rooted spanning instance with $K = 3$ and $H = 3$; complete graph, Euclidean costs.
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A more general version considers potentially distinct values $K(d)$ and $H(d)$ for each $d \in D$ in order to model demand importance.

There is an even more general version where each demand has its required profile of Survivability $\times$ QoS.

- For example, an important demand may require a primary path of length $\leq 2$ and two secondary paths of length $\leq 3$.
- A less important demand may require a primary path of length $\leq 3$ and a secondary path of length at $\leq 4$. 
Even some very particular cases are already NP-hard.

- Case $|D| = 1$ (single demand):
  - Polynomial for $H = 2$ or $3$;
  - NP-hard for $H \geq 4$.

- Case $K = 1$, rooted and spanning (equivalent to the Spanning Tree with Hop Constraints Problem):
  - NP-hard for $H \geq 2$. 
Some recent algorithmic work on the SNDH Problem

- Case $K = 2$:

- Case $K = 3$:

- General SNDH:

Significant gaps, some instances with only 20 demands can be very challenging.
Some recent algorithmic work on the SNDH Problem

- **Case $K = 2$:**

- **Case $K = 3$:**

- **General SNDH:**

Significant gaps, some instances with only 20 demands can be very challenging.
Each edge \((i, j) \in E\) defines a binary design variable \(x_{ij}\).

Each demand \(d = (u, v) \in D\) defines an auxiliary network with \(H\) layers, with associated binary variables \(f_{ij}^{dh}\) (a path serving demand \(d\) goes from \(i\) to \(j\) at hop \(h\)). There must be \(K\) units of flow in each network.

The \(f\) variables are coupled to the \(x\) variables.

**Figure:** Example of network with \(d = (0, 4), H = 3\).
Hop Multi-Commodity Flow Formulation (Hop-MCF), BFGP10

\[
\min \sum_{(i,j) \in E} c_{ij} x_{ij}
\]

\[
\text{s.t.}
\sum_{[j,i,h] \in \delta^-(i,h)} f_{ji}^{dh} - \sum_{[i,j,h+1] \in \delta^+(i,h)} f_{ij}^{d(h+1)} = 0 \quad d \in D; (i, h) \in V_H^d, i \notin \{o_d, d_d\}
\]

\[
\sum_{[o_d,j,1] \in \delta^+(o_d,0)} f_{o_dj}^{d1} = K \quad d \in D
\]

\[
\sum_{h=1}^H \sum_{[j,d_d,h] \in \delta^-(d_d,h)} f_{jd_d}^{dh} = K \quad d \in D
\]

\[
f_{o_dj}^{d1} \leq x_{o_dj} \quad d \in D; (o_d, j) \in \delta(o_d)
\]

\[
\sum_{h=2}^{H-1} (f_{ji}^{dh} + f_{ij}^{dh}) \leq x_{ij} \quad d \in D; (i, j) \in E \setminus (\delta(o_d) \cup \delta(d_d))
\]

\[
\sum_{h=2}^H f_{jd_d}^{dh} \leq x_{jd_d} \quad d \in D; (j, d_d) \in \delta(o_d)
\]
Hop Multi-Commodity Flow Formulation (Hop-MCF), BFGP10

- Only known formulation for the most general versions of the SNDH.
- Quite large size: $O(|D| \cdot H \cdot m)$ variables and $O(|D| \cdot H \cdot n)$ constraints.
- Typical duality gaps: 5% – 25%.
Introduce formulations significantly stronger than Hop-MCF for the general SNDH problem.

- It is well-known that extending a formulation may yield smaller gaps. Even automatic extension schemes (e.g. Sherali and Adams' RLT) do exist.
- However we do not want to increase the formulation size by a large factor that may depend on $n$ or $m$, but only by a small constant factor, that can be even controlled.
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Spanning instance rooted at 0 with $K = 2$ and $H = 3$; complete graph, Euclidean costs.

Linear relaxation of Hop-MCF (cost 641).

Optimal integral solution (cost 682).
How Hop-MCF is cheating?

There are fractional $u - v$ paths with length $\leq 3$ summing 2 for each demand $(u, v)$. 

![Graph diagram](image-url)
For example, take demand (0, 1):

- Path 0-1 with value 1;
How Hop-MCF is cheating?

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For example, take demand \((0, 1)\):

- **Path 0-1** with value 1;
- **Path 0-2-1** with value 1/2;
- **Path 0-4-3-1** with value 1/2.
The Main Idea: Sort the vertices by their distances to a source.

Given a solution $T$, a chosen source vertex $s \in V$ and a chosen positive integer $L$, we can partition $V$ into $L + 2$ levels, according to their distance from $s$ in $T$, as follows:

- Level 0 only contains $s$;
- Level $i$, $1 \leq i \leq L - 1$, contains vertices with distance $i$;
- Level $L$ contains the vertices with finite distance $\geq L$;
- Level $L + 1$ contains the vertices with infinite distance.
The proposed formulation, besides the $m$ edge variables $x$, also has:

- $O(L.n)$ binary variables $w_i^l$ indicating if vertex $i$ is in level $l$;
- $O(L.m)$ binary variables $y_{ij}^{l1l2}$ indicating that edge $(i,j)$ belongs to $T$ and that $i$ is in level $l_1$ and $j$ in level $l_2$; (remark that $|l_1 - l_2| \leq 1$)
- $O(|D|.L.H.m)$ binary flow variables $g_{ij}^{dhl1l2}$ associated to $|D|$ auxiliary Hop-Level networks.
Translating an integral $x$ solution into $(w, y)$ variables.

- $x_{01} = x_{02} = x_{12} = x_{23} = x_{24} = x_{34} = 1$.

- $w_0^0 = w_1^1 = w_2^1 = w_3^2 = w_4^2 = 1$.

- $y_{01}^0 = y_{02}^0 = y_{12}^{11} = y_{23}^{12} = y_{24}^{12} = y_{34}^{22}$. 

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Strong Formulations for the SNDH Problem
The Hop-Level Multi-Commodity Flow Formulation (HL-MCF)

\[
\min \sum_{(i,j) \in E} c_{ij}x_{ij} \quad (10)
\]

The \(x\) and \((w, y)\) variables are linked by the following constraints:

\[
w_s^0 = 1 \quad (11)
\]

\[
\sum_{l=1}^{L+1} w_i^l = 1 \quad i \in V \setminus s \quad (12)
\]

\[
w_j^1 = y_{sj} = x_{sj} \quad (s, j) \in \delta(s) \quad (13)
\]

\[
\sum_{l=1}^{L-1} (y_{ij}^{l(l+1)} + y_{ji}^{l(l+1)}) + \sum_{l=1}^{L+1} y_{ij}^{ll} = x_{ij} \quad (i, j) \in E \setminus \delta(s) \quad (14)
\]
The Hop-Level Multi-Commodity Flow Formulation (HL-MCF)

\begin{align*}
y_{ij}^l + y_{ij}^{(l+1)} &\leq w_i^l & (i, j) \in E \setminus \delta(s); \ l = 1 \\
y_{ij}^l + y_{ji}^{(l+1)} &\leq w_i^l \\
y_{ij}^l + y_{ij}^{(l+1)} + y_{ji}^{(l-1)} &\leq w_i^l & (i, j) \in E \setminus \delta(s); \ l = 2, \ldots, L - 1 \\
y_{ij}^l + y_{ji}^{(l-1)} &\leq w_i^l \\
y_{ij}^l + y_{ij}^{(l-1)} &\leq w_j^l \\
y_{ij}^l &\leq w_i^l \\
y_{ij}^l &\leq w_j^l \\
w_i^l &\leq \sum_{j \in \delta(i), j \neq s} y_{ji}^{(l-1)} & i \in V \setminus s; \ l = 2, \ldots, L - 1 \\
w_i^l &\leq \sum_{j \in \delta(i), j \neq s} (y_{ji}^{(l-1)} + y_{ij}^l) & i \in V \setminus s; \ l = L
\end{align*}
Translating a fractional $x$ solution into $(w, y)$ variables.

- $x_{01} = x_{34} = 1$; $x_{02} = x_{04} = x_{12} = x_{13} = x_{23} = x_{24} = 1/2$.

\[
\begin{align*}
\bullet \quad & w_0^0 = w_1^1 = w_3^1 = 1; \quad w_2^1 = w_2^2 = w_4^1 = w_4^2 = 1/2. \\
\bullet \quad & y_{01}^{01} = 1; \\
\bullet \quad & y_{02}^{01} = y_{04}^{01} = y_{12}^{12} = y_{13}^{12} = y_{24}^{12} = y_{43}^{12} = y_{23}^{22} = y_{34}^{22} = 1/2.
\end{align*}
\]
What is wrong with that \((w, y)\) solution?

For example, take demand \((0, 1)\):

- **Path** 0-1 with value 1;
- **Path** 0-4-3-1 with value 1/2.
- The only remaining **path** 0-2-4-3-2-1 has length 5. Wrong!
  - The splitting of vertex 2 removed path 0-2-1.
- Cutting the \((w, y)\) solution indirectly cuts the \(x\) solution.
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- Cutting the \((w, y)\) solution indirectly cuts the \(x\) solution.
The HL-MCF is completed by enforcing, for each demand $d = (u, v)$, the existence of $K (u, v)$-paths with length $\leq H$ in the network induced by the $(w, y)$ solution.

This is done by building $|D|$ auxiliary hop-level indexed networks.

- One variable for each arc in those networks: $g_{ij}^{dhl_1l_2}$ indicates that a path serving demand $d$ goes from $i$ at level $l_1$ to $j$ at level $l_2$ in its hop $h$. 
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- One variable for each arc in those networks: \( g_{ij}^{dhl_1l_2} \) indicates that a path serving demand \( d \) goes from \( i \) at level \( l_1 \) to \( j \) at level \( l_2 \) in its hop \( h \).
Hop-Level Network, source is one of the vertices of the demand: \( d = (0, 4), H = 3, L = 3, \) and \( s = 0. \)

- There must be \( K \cdot w^l \) units of flow from \((0, 0, 0)\) to vertices \((4, h, l)\);
- The \( g \) variables are constrained by the corresponding \( y \) variables.
Hop-Level Network, general case: \( d = (0, 4), H = 3, L = 3, \) and \( s = 1. \)

- Level \( L + 1 \) usually not necessary.
**Hop-MCF \times HL-MCF: 72 rooted instances, complete graphs with \( n = 21 \), Euc. cost, 5 to 20 demands.**

**Table:** Rooted instances: average percentual duality gaps.

<table>
<thead>
<tr>
<th></th>
<th>( H )</th>
<th>( K = 1 )</th>
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<td>25.82</td>
<td>13.05</td>
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<td></td>
<td>0.83</td>
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In the HL-MCF, \( L \) is taken as min\{\( H, 4 \)\}, the source is the root.
Hop-MCF × HL-MCF: 72 rooted instances, complete graphs with $n = 21$, Euc. cost, 5 to 20 demands.

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- **Gap reduction > 75%**
- **50 ≤ Gap reduction ≤ 75**
- **Gap reduction < 50%**
Explaining the results of HL-MCF.

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HL-MCF works by splitting vertices, cutting shorter paths in the fractional solution; the remaining paths have length $> H$.

- As $K$ increases, the solution gets denser, vertices are concentrated on levels 1 and 2 and are not sufficiently split.
- As $H$ increases, the hop-constraints are looser and it is easier to find alternative paths with length $\leq H$. 
Hop-MCF $\times$ HL-MCF: 38 unrooted instances, sparse graphs, 9 to 48 demands.

Table: Unrooted instances: average percentual duality gaps.

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In the HL-MCF, $L = 5$, the source is vertex 0.
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Less satisfactory results.
The gap reductions obtained by HL-MCF are very significant in some cases, specially for the rooted instances, allowing dramatic reductions in the overall time taken by a B&B algorithm.

In other cases, specially for non-rooted instances, the reductions are not enough to compensate for the increase in formulation size.
Spanning instance rooted at 0 with $K = 2$ and $H = 3$; complete graph, Euclidean costs.

Linear relaxation of Hop-MCF (cost 641).

Linear relaxation of HL-MCF (cost 672).

Optimal integral solution (cost 683).
The fractional edges are arranged in order to avoid vertex splitting. For example, $w_2^2 = 1$ because 2 is connected to source 0 by paths $0 - 1 - 2$ and $0 - 4 - 2$ with value $1/2$. 
By setting $d(0, 1) = 2$ (other distances remain 1), the same $x$ solution would force $w_2^2 = w_3^2 = 1/2$, which would lead to infeasibility.

The relaxation of the modified HL-MCF with that non-unitary distance is integral in this instance.
Translating an integral solution into \((w, y)\) variables, in case of non-unitary distances

\[
\begin{align*}
     &x_{01} = x_{02} = x_{12} = x_{23} = x_{24} = x_{34} = 1. \\
w_0 = w_1 = w_2 = w_3 = w_4 = 1. \\
y_{01} = y_{02} = y_{12} = y_{23} = y_{24} = y_{34}.
\end{align*}
\]
In this generalization, one can choose not only $s$ and $L$, but also the distances: an edge-vector of (small) integers between 0 and $L$.

The choice of the distance makes a lot of difference:

- It is typical that a good choice (at the moment, a lucky choice) closes half of the gap, while several poor choices are worse than unitary HL-MCF.
We are now trying to find systematic ways of choosing good distance vectors.

- Some amount of trial and error is not unreasonable.
- Perhaps one can even use a few distinct distance vectors at once.
  - For each vector there would be a distinct set of variables \((w, y, g)\), the \(x\) solution should be compatible with all of them.
Conclusions

- The proposed unitary distance HL-MCF already proved to yield significant algorithmic improvements upon existing methods for solving some kinds of SNDH instances.
- The non-unitary version of HL-MCF is currently being investigated.
- The idea of trying to extend an existing formulation by only multiplying its size by a constant factor may be useful on other problems.
Thank you!