On the integrality gap of hypergraphic Steiner tree relaxations

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Joint work with Michel Goemans, Thomas Rothvoß and Rico Zenklusen.
Steiner tree

Terminals

Steiner nodes
## Approximation results

<table>
<thead>
<tr>
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<th>Perf. guarantee</th>
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</thead>
<tbody>
<tr>
<td>Folklore</td>
<td>2</td>
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<tr>
<td>Zelikovsky [1991]</td>
<td>$11/6 \leq 1.83$</td>
</tr>
<tr>
<td>Berman &amp; Ramaiyer [1991]</td>
<td>$16/9 \leq 1.78$</td>
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<tr>
<td>Zelikovsky [1993]</td>
<td>$1 + \ln(2) + \epsilon \leq 1.70$</td>
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<tr>
<td>Karpinski &amp; Zelikovsky [1997]</td>
<td>$\leq 1.65$</td>
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<tr>
<td>Prömel &amp; Steger [2000]</td>
<td>$5/3 \leq 1.67$</td>
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<tr>
<td>Hougardy &amp; Prömel [1999]</td>
<td>$\leq 1.59$</td>
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<tr>
<td>Robins &amp; Zelikovsky [2005]</td>
<td>$1 + \ln(3)/2 \leq 1.55$</td>
</tr>
<tr>
<td>Byrka, Grandoni, Rothvoß, Sanità [2010]</td>
<td>$\ln(4) + \epsilon \leq 1.39$</td>
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</table>
Hypergraphic relaxations

- Different (but equivalent) relaxations:
  - Warme [1998] (we’ll use this one).
  - Polzin and Vahdati-Deneshmand [2003].
  - Könemann, Pritchard and Tan [2009].
- All have a variable for each component.
Hypergraphic relaxations

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Undirected component-based relaxation

$$\min \sum_{C \in \mathcal{K}} x_C \text{cost}(C)$$

$$\sum_{C \in \mathcal{K}} x_C(|S \cap R(C)| - 1)^+ \leq |S| - 1 \quad \forall \emptyset \subsetneq S \subseteq R \quad \text{(LP)}$$

$$\sum_{C \in \mathcal{K}} x_C(|R(C)| - 1) = |R| - 1$$

$$x_C \geq 0 \quad \forall C \in \mathcal{K}.$$ 

- $\mathcal{K} =$ set of components.
- Exponential # of variables—but can get a PTAS by only considering components up to a certain size.

*Warme [1998]*

*Borchers & Du [1997]*
Integrality gaps

- Directed variant used by BGRS for their algorithm—but their analysis does not bound the integrality gap!
  - They do prove a bound of 1.55 on the integrality gap by other methods.
  - Chakrabarty, Könemann and Pritchard [2010] give a simpler proof of this bound.
Our results (I)

- Deeper understanding of the component-based LP.
- Leads to simpler algorithm, performance directly comparable to LP solution:
  - Deterministic (derandomization much simpler than for BGRS algorithm).
  - Gives matching bound on the integrality gap.
  - Don’t need to re-solve LP in each iteration.
Overview of algorithm

- Solve component LP to get near-optimal solution $x$.
- Until all terminals are connected:
  - Select component $Q$ to contract; add $Q$ to solution.
  - Modify LP solution $x$ to be feasible in the new contracted instance.
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- No re-solving of LP!
Choose $N$ s.t. $x_C \cdot N \in \mathbb{N}$ for all $C \in \mathcal{K}$. 

Make $x_C \cdot N$ "copies" of $C \in \mathcal{K}$. 

Notation: $\Gamma(\mathcal{X}) :=$ set of components in $\mathcal{X}$. 

$\blacksquare$ 

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\( \mathcal{X} \) contains all information about \( x \); in particular, can determine feasibility.

\[
\text{cost}(\mathcal{X}) := \sum_{e \in E(\mathcal{X})} c(e) = N \cdot \text{cost}(x).
\]

For any \( F \subseteq E(\mathcal{X}) \), \( \mathcal{X} - F \) is another blowup graph.
Edge removals

Crucial question

What are the possible edge removals after contracting a set of terminals $Q$?
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- Important to look at *minimal* edge removals:
Matroid structure

Given a set $Q$ of terminals to contract, let

$$\mathcal{B}_Q = \{B \subseteq E(\mathcal{X}) | (\mathcal{X}/Q) - B \text{ is feasible, and } B \text{ is minimal with this property}\}.$$ 

Theorem

- $\mathcal{B}_Q$ form the bases of a matroid $M_Q$ of rank $N(|R(Q)| - 1)$. 

Can precisely describe the rank function $r_Q$. 

$M_Q$ is a gammoid.
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Thought experiment

- We want to show that there exists a component $Q$ and removal set $B \in \mathcal{B}_Q$ s.t. $\text{cost}(B)/N$ is “large” compared to $\text{cost}(Q)$.

- Try an averaging argument: pick $(Q, B)$ randomly from some distribution $\mathcal{D}$ s.t. $B \in \mathcal{B}_Q$, and $\mathbb{E}\{\text{cost}(B)/N\}$ large compared to $\mathbb{E}\{\text{cost}(Q)\}$.
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- Choose $\mathcal{D}$ s.t. $\mathbb{P}\{Q = C\} = 1/|\Gamma(\mathcal{X})|$ $\forall C \in \Gamma(\mathcal{X})$.
  - So $\mathbb{E}\{\text{cost}(Q)\} = \text{cost}(\mathcal{X})/|\Gamma(\mathcal{X})|$.
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- So if we can also ensure that $\mathbb{P}\{e \in B\} \geq \alpha \cdot N/|\Gamma(\mathcal{X})|$, we get

$$\mathbb{E}\{\text{cost}(B)/N\} = \frac{1}{N} \sum_{e} c(e) \mathbb{P}\{e \in B\} = \alpha \cdot \text{cost}(\mathcal{X})/|\Gamma(\mathcal{X})|,$$

implying an integrality gap $\leq \alpha$. 

Unfortunately we can't get better than $g = 13/29$. 

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implying an integrality gap $\leq \alpha$.

Unfortunately we can’t get better than $\alpha = 2$. 
Removal probabilities

- Let $B(M_Q)$ be the base polytope associated with $M_Q$.
- $p \in B(M_Q)$ iff there exists a distribution $\mathcal{D}_Q$ on $B_Q$ s.t.
  \[ P_{\mathcal{D}_Q} \{ e \in B_Q \} = p_e. \]
- Consider the scaled Minkowski sum
  \[ B_{rem} = \frac{1}{|\Gamma(\mathcal{X})|} \sum_{Q \in \Gamma(\mathcal{X})} B(M_Q). \]
- $p \in B_{rem}$ iff there exists $\mathcal{D}$ s.t. for random $(Q, B)$, $B \in B_Q$, $P\{Q = C\} = 1/|\Gamma(\mathcal{X})|$ $\forall C$ and
  \[ P_{\mathcal{D}} \{ e \in B \} = p_e. \]
Polymatroids for random contraction

- $B_{rem}$ is a polymatroid, with rank function

$$r = \frac{1}{|\Gamma(X)|} \sum_{Q \in \Gamma(X)} r_Q.$$

- But $r_Q$ has an implicit description:

$$r_Q(F) = \min_{S \supseteq Q} h_{\chi-F}(S).$$
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**Theorem**

$$B_{rem} \supseteq \frac{N}{|\Gamma(X)|} B(M_R).$$

Equivalently:

$$\sum_{Q \in \Gamma(X)} B(M_Q) \supseteq N \cdot B(M_R).$$
Core edges

- The set of extreme points of $B(M_R)$ is precisely $B_R$.
- So for any $K \in B_R$, the vector $p$ of marginals given by

$$p_e = \begin{cases} 
N/|\Gamma(X)| & e \in K \\
0 & e \notin K 
\end{cases}$$

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- What must remain after removing some $B \in B_R$? Precisely a spanning tree on $\mathcal{X}/R$. 

![Diagram](image-url)
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  is in $B_{rem}$.
- What must remain after removing some $B \in B_R$? Precisely a spanning tree on $\mathcal{X}/R$. 

![Diagram of a spanning tree on $\mathcal{X}/R$.]
Witness sets
Witness sets
Witness sets
Witness sets
Weights

- Charge cost of cleanup edges to edges in $K$:

$$w(f) = c(f) + \sum_{e \in K : f \in W(e)} \frac{c(e)}{|W(e)|}.$$ 

Theorem

There exists a component $Q$ and $B \in \mathcal{B}_Q$ so that $w(B)/N \geq \text{cost}(Q)$.

- $(Q, B)$ can be found algorithmically (for each $Q$, find a maximum weight basis in $\mathcal{B}_Q$; the gammoid structure of $M_Q$ is helpful here).
Algorithm

- Start with blowup graph $\mathcal{X}$ and $K \in \mathcal{B}_R$.
- $N = 2$ in this example.
Algorithm

- Find $Q$ and max weight basis $B \in \mathcal{B}_Q$ s.t. $\text{cost}(Q) \leq w(B)/N$. 

![Graph diagram](image-url)
Algorithm

- Remove $B$
Remove $B$ and cleanup edges: $F = \{ e \notin K \mid W(e) \subseteq B \}$. 
Algorithm

- Remove $B$ and cleanup edges: $F = \{ e \notin K \mid W(e) \subseteq B \}$.  
- Contract $Q$. 

![Graph Diagram]
▶ Find $Q$ and $B \in \mathcal{B}_Q$ s.t. $\text{cost}(Q) \leq \frac{w(B)}{N}$. 
Algorithm

- Find $Q$ and $B \in \mathcal{B}_Q$ s.t. $\text{cost}(Q) \leq \frac{w(B)}{N}$.
- Remove $B$
Find $Q$ and $B \in \mathcal{B}_Q$ s.t. $\text{cost}(Q) \leq \frac{w(B)}{N}$.

Remove $B$ and cleanup edges.
Algorithm

- Contract.
Algorithm

- Contract.
- Final tree.
Full algorithm

For a given blowup graph \( \mathcal{X} \) and \( K \in \mathcal{B}_R \):

**Algorithm**

1. \( T \leftarrow \emptyset \)
2. While \( T \) is not a Steiner tree:
   - Find \( Q \) and \( B \in \mathcal{B}_Q \) s.t. \( \text{cost}(Q) \leq w(B)/N \).
   - Cleanup: Let \( F = \{ e \notin K \mid W(e) \subseteq B \} \).
   - Update: \( T \leftarrow T \cup Q \), \( \mathcal{X} \leftarrow (\mathcal{X} - B - F)/Q \), \( K \leftarrow K \setminus B \).

**Theorem**

*Algorithm returns solution of cost at most \( \Phi_K(\mathcal{X})/N \), where*

\[
\Phi_K(\mathcal{X}) = \sum_{e \in K} H(|W(e)|).
\]
Potential analysis

- Can find $K$ minimizing $\Phi_K(\mathcal{X})$ by a dynamic program.

- For best $K$, can show that
  - $\Phi_K(\mathcal{X}) \leq \ln(4) \text{cost}(\mathcal{X})$.
  - If $G$ is quasi-bipartite, $\Phi_K(\mathcal{X}) \leq \frac{73}{60} \text{cost}(\mathcal{X})$. 
Our results (II)

- Quasi-bipartite:
  - 73/60 bound on integrality gap (previous best of 1.28 by Chakrabarty, Könemann and Pritchard [2010]).
  - Can show how to obtain optimal solution to hypergraphic LP by solving bidirected relaxation.

- Separation of undirected component-based LP by $|R|$ max-flow computations.
  - Rank computations with one max-flow.

- Easy proof of sparsity of basic solutions (first proven by Chakrabarty et al. [2010]).
Open problems

- Improve the bound!

- Practical (non-galactic) approximation algorithms?

- Better than 2 bound on integrality gap of bidirected cut relaxation?