

Basic Packing of Arborescences

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Abstract

We provide the directed counterpart of a slight extension of Katoh and Tanigawa's result [8] on rooted-tree decompositions with matroid constraints. Our result characterises digraphs having a packing of arborescences with matroid constraints. It is a proper extension of Edmonds' result [1] on packing of spanning arborescences and implies – using a general orientation result of Frank [4] – the above result of Katoh and Tanigawa.

We also give a complete description of the convex hull of the incidence vectors of the basic packings of arborescences and prove that the minimum cost version of the problem can be solved in polynomial time.

1 Introduction

Let $G = (V, E)$ be a graph. For a vertex set X , $E(X)$ denotes the set of edges of G with both extremities in X . We say that G is a *rooted-tree* or more precisely a *tree rooted at r* if G is connected and cycle free and r is a vertex of G . We note that a tree rooted at r may consist of only the vertex r and no edges. Note also that a tree can be rooted at any vertex of its.

Our starting point is the result of Tutte [10] and Nash-Williams [9] on packing of spanning trees. For a partition \mathcal{P} of V , $e_G(\mathcal{P})$ denotes the number of edges of G between the different members of \mathcal{P} . We always suppose that the members of \mathcal{P} are not empty. Following Frank [5], G is called *k -partition-connected* if

$$e_G(\mathcal{P}) \geq k(|\mathcal{P}| - 1) \quad \text{for every partition } \mathcal{P} \text{ of } V. \quad (1)$$

Theorem 1.1 (Tutte [10], Nash-Williams [9]). *A graph $G = (V, E)$ contains k edge-disjoint spanning trees (rooted at a vertex r of G) if and only if G is k -partition-connected.*

Let $D = (V, A)$ be a digraph. For a vertex set X , $D[X]$ denotes the induced subgraph of D on X . We say that a vertex v is *reachable* from a vertex u in D if there exists a directed path from u to v in D . For convenience, we will not distinguish the vertex v from the set $\{v\}$. For a vertex set X , we denote by $\varrho_D(X)$ the set of arcs entering X and we define $\rho_D(X) = |\varrho_D(X)|$. We say that D is an *arborescence rooted at r* if D is a directed tree, r is a vertex of D of in-degree 0 and all the other vertices of D are of in-degree 1. We note that an arborescence rooted at r may consist of only the vertex r and no arcs. Note also that an arborescence has a unique root. It is well-known that D contains a spanning arborescence rooted at a vertex r of D if and only if every non-empty vertex set not containing r has in-degree at least 1.

The directed counterpart of Theorem 1.1 is the result of Edmonds [1] on packing of spanning arborescences.

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Theorem 1.2 (Edmonds [1]). *A digraph $D = (V, A)$ contains k arc-disjoint spanning arborescences rooted at a vertex r of D if and only if*

$$\rho_D(X) \geq k \quad \text{for all non-empty } X \subseteq V \setminus r. \quad (2)$$

Frank [2] showed how to deduce Theorem 1.1 from Theorem 1.2. He proved that (1) is the necessary and sufficient condition for the undirected graph G to have an orientation D that satisfies (2). Then, by Theorem 1.2, D contains k arc-disjoint spanning arborescences rooted at r that provide the k edge-disjoint spanning trees rooted at r in G .

Let $S = \{s_1, \dots, s_t\}$ be a set and π a map from S to V . We may think of π as a placement of the elements of S at vertices of V and different elements of S may be placed at the same vertex. In this paper t will always denote the size of S . The triplet (G, S, π) (respectively (D, S, π)) is called a *graph* (resp. *digraph*) *with roots*. For $X \subseteq V$, we denote by S_X the set $\pi^{-1}(X)$.

A function $b : 2^\Omega \rightarrow \mathbb{Z}$ is called *submodular* (respectively *intersecting submodular*) if for all $X, Y \subseteq \Omega$ (resp. for all $X, Y \subseteq \Omega$ that are intersecting),

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y).$$

A function $p : 2^\Omega \rightarrow \mathbb{Z}$ is called *supermodular* if $-p$ is submodular. Note that the in-degree function ρ_D of a digraph D is submodular.

Let \mathcal{M} be a matroid on S with rank function $r_{\mathcal{M}}$. It is well-known that $r_{\mathcal{M}}$ is monotone non-decreasing and submodular. A set $Q \subseteq S$ is *independent* if $r_{\mathcal{M}}(Q) = |Q|$. Recall that every subset of an independent set is independent. A maximal independent set is a *base* of \mathcal{M} . Each base has the same size, namely $r_{\mathcal{M}}(S)$. \mathcal{M} is called a *free matroid* if each subset of S is independent. For a set $Q \subseteq S$, we define $\text{Span}_{\mathcal{M}}(Q) = \{s \in S : r_{\mathcal{M}}(Q \cup \{s\}) = r_{\mathcal{M}}(Q)\}$. The set Q is called a *spanning set* of \mathcal{M} if $\text{Span}_{\mathcal{M}}(Q) = S$.

The following definition was introduced by Katoh and Tanigawa [8]. An \mathcal{M} -*basic packing of rooted-trees* is a set $\{T_1, \dots, T_t\}$ of pairwise edge-disjoint trees such that for $i = 1, \dots, t$, T_i is rooted at $\pi(s_i)$ and, for each $v \in V$, the set $\{s_i \in S : v \in V(T_i)\}$ forms a base of \mathcal{M} . For the sake of convenience, we say that T_i is rooted at s_i . Note that the trees are not necessarily spanning and each vertex of G belongs to exactly $r_{\mathcal{M}}(S)$ trees.

The following result characterizes graphs with roots that have a basic packing of rooted-trees. It will be derived from its directed counterpart (Theorem 1.6) at the end of this section. We say that the map π is \mathcal{M} -*independent* if S_v is independent in \mathcal{M} for all $v \in V$. The graph with roots (G, S, π) is called \mathcal{M} -*partition-connected* if

$$e_G(\mathcal{P}) \geq r_{\mathcal{M}}(S)|\mathcal{P}| - \sum_{X \in \mathcal{P}} r_{\mathcal{M}}(S_X) \quad \text{for every partition } \mathcal{P} \text{ of } V.$$

Theorem 1.3. *Let (G, S, π) be a graph with roots and \mathcal{M} a matroid on S . There exists an \mathcal{M} -basic packing of rooted-trees in (G, S, π) if and only if π is \mathcal{M} -independent and (G, S, π) is \mathcal{M} -partition-connected.*

If \mathcal{M} is the free matroid then the problem of \mathcal{M} -basic packing of rooted-trees and that of packing of spanning trees coincide. Hence Theorem 1.3 is a proper extension of Theorem 1.1. It is not difficult to see that this theorem easily implies the following theorem of Katoh and Tanigawa [8]. A *rooted-component* of (G, S, π) is a pair (C, s) where C is a connected subgraph of G and $s \in S_{V(C)}$.

Theorem 1.4 (Katoh and Tanigawa [8]). *Let $G = (V, E)$ be a graph, $S = \{s_1, \dots, s_t\}$ a set, π a placement of S in V and \mathcal{M} a matroid on S . Then (G, S, π) can be decomposed into rooted-components $(C_1, s_1), \dots, (C_t, s_t)$ such that the set $\{s_i \in S : v \in V(C_i)\}$ is a spanning set of \mathcal{M} for every $v \in V$ if and only if (G, S, π) is \mathcal{M} -partition-connected.*

Katoh and Tanigawa deduced Theorem 1.4 from the following dual form of its. We show that Theorem 1.3 also implies Theorem 1.5.

Theorem 1.5 (Katoh and Tanigawa [8]). *Let $G = (V, E)$ be a graph, $S = \{s_1, \dots, s_t\}$ a set and π a placement of S in V . Let \mathcal{M} be a matroid on S of rank k with rank function $r_{\mathcal{M}}$. Then (G, S, π) admits an \mathcal{M} -basic rooted-tree decomposition if and only if π is \mathcal{M} -independent, $|E| + |S| = k|V|$ and $|F| + |S_{V(F)}| \leq k|V(F)| - k + r_{\mathcal{M}}(S_{V(F)})$ for all non-empty $F \subseteq E$.*

Proof. The necessity of the conditions is pretty straightforward as one can see in [8].

Now suppose that the conditions hold. For every partition \mathcal{P} of V , by the inequality applied for $E(X)$ ($X \in \mathcal{P}$) and by the equality, $e_G(\mathcal{P}) = |E| - \sum_{X \in \mathcal{P}} |E(X)| \geq |E| - \sum_{X \in \mathcal{P}} (k|X| - k + r_{\mathcal{M}}(S_X) - |S_X|) = k|\mathcal{P}| - \sum_{X \in \mathcal{P}} r_{\mathcal{M}}(S_X)$. Hence, π is \mathcal{M} -independent and (G, S, π) is \mathcal{M} -partition-connected. Then Theorem 1.3 implies that (G, S, π) admits an \mathcal{M} -basic packing of rooted-trees which, by $|E| + |S| = k|V|$, must be an \mathcal{M} -basic rooted-tree decomposition of (G, S, π) . \blacksquare

The main contribution of the present paper is to mimic Frank's approach (mentioned above on packing of spanning trees) for basic packing of rooted-trees. We provide the directed counterpart of Theorem 1.3, a short proof of it and we show that it implies Theorem 1.3 (and hence Theorem 1.4 and Theorem 1.5) via an orientation theorem of Frank.

Inspired by the definition of Katoh and Tanigawa, we define an \mathcal{M} -basic packing of arborescences as a set $\{T_1, \dots, T_t\}$ of pairwise arc-disjoint arborescences such that for $i = 1, \dots, t$, T_i is rooted at $\pi(s_i)$ and, for each $v \in V$, the set $\{s_i \in S : v \in V(T_i)\}$ forms a base of \mathcal{M} . We also say that T_i is rooted at s_i . For a better understanding, let us mention that the arborescences are not necessarily spanning and each vertex of D belongs to exactly $r_{\mathcal{M}}(S)$ arborescences.

Our main result is the following theorem. The digraph with roots (D, S, π) is called \mathcal{M} -connected if

$$\rho_D(X) \geq r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \quad \text{for all non-empty } X \subseteq V. \quad (3)$$

Theorem 1.6. *Let (D, S, π) be a digraph with roots and \mathcal{M} a matroid on S . There exists an \mathcal{M} -basic packing of arborescences in (D, S, π) if and only if π is \mathcal{M} -independent and (D, S, π) is \mathcal{M} -connected.*

If \mathcal{M} is the free matroid and π places every element of S at a single vertex r of D then the problem of \mathcal{M} -basic packing of arborescences and that of packing of spanning arborescences rooted at r coincide. Hence Theorem 1.6 is a proper extension of Theorem 1.2.

Let us recall the following general orientation result of Frank [4].

Theorem 1.7 (Frank [4]). *Let $G = (V, E)$ be a graph and $h : 2^V \rightarrow \mathbb{Z}_+$ an intersecting supermodular non-negative non-increasing set-function. There exists an orientation D of G such that $\rho_D(X) \geq h(X)$ for all non-empty $X \subset V$ if and only if for every partition \mathcal{P} of V ,*

$$e_G(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} h(X).$$

Theorem 1.7 immediately implies the following corollary by taking $h(X) = r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X)$.

Corollary 1.1. *Let (G, S, π) be a graph with roots and \mathcal{M} a matroid on S . There exists an orientation D of G such that (D, S, π) is \mathcal{M} -connected if and only if (G, S, π) is \mathcal{M} -partition-connected.*

Let us show that Corollary 1.1 and Theorem 1.6 imply Theorem 1.3.

Proof. (of Theorem 1.3) First suppose that there exists an \mathcal{M} -basic packing $\{T_1, \dots, T_t\}$ of rooted-trees in (G, S, π) . Let D be an orientation of G where each tree T_i rooted in s_i becomes an arborescence T'_i rooted in s_i . Then $\{T'_1, \dots, T'_t\}$ is an \mathcal{M} -basic packing of arborescences in (D, S, π) . By Theorem 1.6, π is \mathcal{M} -independent and (D, S, π) is \mathcal{M} -connected. Hence, by Corollary 1.1, G is \mathcal{M} -partition-connected.

Now suppose that π is \mathcal{M} -independent and (G, S, π) is \mathcal{M} -partition-connected. By Corollary 1.1, there exists an orientation D of G such that (D, S, π) is \mathcal{M} -connected. Then, by Theorem 1.6, there exists an \mathcal{M} -basic packing of arborescences in (D, S, π) which provides, by forgetting the orientation, an \mathcal{M} -basic packing of rooted-trees in (G, S, π) . \blacksquare

2 Proof of the main theorem

First we prove the necessity of the conditions.

Proof. (of necessity in Theorem 1.6) Suppose that there exists an \mathcal{M} -basic packing $\{T_1, \dots, T_t\}$ of arborescences in (D, \mathcal{S}, π) . Let v be an arbitrary vertex of V and X a vertex set containing v . Then $\mathcal{B} := \{s_i \in \mathcal{S} : v \in V(T_i)\}$ forms a base of \mathcal{M} . Let $\mathcal{B}_1 = \mathcal{B} \cap \mathcal{S}_X$ and $\mathcal{B}_2 = \mathcal{B} \setminus \mathcal{S}_X$. Then, since \mathcal{B}_1 is independent in \mathcal{M} and $\mathcal{S}_v \subseteq \mathcal{B}_1$, π is \mathcal{M} -independent. Moreover, since $r_{\mathcal{M}}$ is monotone, $|\mathcal{B}_1| = r_{\mathcal{M}}(\mathcal{B}_1) \leq r_{\mathcal{M}}(\mathcal{S}_X)$. For each root $s_i \in \mathcal{B}_2$, there exists an arc of T_i that enters X and the arborescences are arc-disjoint, so we have $\rho_D(X) \geq |\mathcal{B}_2| = |\mathcal{B}| - |\mathcal{B}_1| \geq r_{\mathcal{M}}(\mathcal{S}) - r_{\mathcal{M}}(\mathcal{S}_X)$ that is (D, \mathcal{S}, π) is \mathcal{M} -connected. \blacksquare

Before proving the sufficiency of the conditions we establish two technical claims.

Claim 2.1. *Let \mathcal{M} be a matroid on \mathcal{S} with rank function $r_{\mathcal{M}}$ and $P, Q \subseteq \mathcal{S}$ such that $r_{\mathcal{M}}(P \cap Q) + r_{\mathcal{M}}(P \cup Q) = r_{\mathcal{M}}(P) + r_{\mathcal{M}}(Q)$ and $s \in \text{Span}_{\mathcal{M}}(P) \cap \text{Span}_{\mathcal{M}}(Q)$. Then $s \in \text{Span}_{\mathcal{M}}(P \cap Q)$.*

Proof. By the monotonicity and submodularity of the rank function and by the assumptions, $r_{\mathcal{M}}(P \cap Q) + r_{\mathcal{M}}(P \cup Q) \leq r_{\mathcal{M}}((P \cap Q) \cup s) + r_{\mathcal{M}}((P \cup Q) \cup s) \leq r_{\mathcal{M}}(P \cup s) + r_{\mathcal{M}}(Q \cup s) = r_{\mathcal{M}}(P) + r_{\mathcal{M}}(Q) = r_{\mathcal{M}}(P \cap Q) + r_{\mathcal{M}}(P \cup Q)$. Hence equality holds everywhere, in particular $r_{\mathcal{M}}(P \cap Q) = r_{\mathcal{M}}((P \cap Q) \cup s)$, that is $s \in \text{Span}_{\mathcal{M}}(P \cap Q)$. \blacksquare

Let us introduce the following definitions. A vertex set X is called *tight* if $\rho_D(X) = r_{\mathcal{M}}(\mathcal{S}) - r_{\mathcal{M}}(\mathcal{S}_X)$. For vertex sets X and Y , we say that Y *dominates* X if $\mathcal{S}_X \subseteq \text{Span}_{\mathcal{M}}(\mathcal{S}_Y)$. Note that since, for $Q \subseteq \mathcal{S}$, $\text{Span}_{\mathcal{M}}(\text{Span}_{\mathcal{M}}(Q)) = \text{Span}_{\mathcal{M}}(Q)$, domination is a transitive relation. We say that an arc uv is *good* if v dominates u , otherwise it is *bad*.

Claim 2.2. *Suppose that (D, \mathcal{S}, π) is \mathcal{M} -connected. Let X be a tight set and v a vertex of X .*

- (a) *If Y is a tight set that contains v , then $X \cap Y$ and $X \cup Y$ are tight and $r_{\mathcal{M}}(\mathcal{S}_X \cap \mathcal{S}_Y) + r_{\mathcal{M}}(\mathcal{S}_X \cup \mathcal{S}_Y) = r_{\mathcal{M}}(\mathcal{S}_X) + r_{\mathcal{M}}(\mathcal{S}_Y)$.*
- (b) *If Y is the set of vertices of X from which v is reachable in $D[X]$, then $v \in Y \subseteq X$, Y is tight and dominates X .*
- (c) *If Y is the set of vertices of X from which v is reachable in $D[X]$ using only good arcs, then v dominates Y .*

Proof. (a) By the submodularity of $r_{\mathcal{M}}$, tightness of X and Y , the submodularity of ρ_D , $X \cap Y \neq \emptyset$ and (3), $r_{\mathcal{M}}(\mathcal{S}_{X \cap Y}) + r_{\mathcal{M}}(\mathcal{S}_{X \cup Y}) = r_{\mathcal{M}}(\mathcal{S}_X \cap \mathcal{S}_Y) + r_{\mathcal{M}}(\mathcal{S}_X \cup \mathcal{S}_Y) \leq r_{\mathcal{M}}(\mathcal{S}_X) + r_{\mathcal{M}}(\mathcal{S}_Y) = r_{\mathcal{M}}(\mathcal{S}) - \rho_D(X) + r_{\mathcal{M}}(\mathcal{S}) - \rho_D(Y) \leq r_{\mathcal{M}}(\mathcal{S}) - \rho_D(X \cap Y) + r_{\mathcal{M}}(\mathcal{S}) - \rho_D(X \cup Y) \leq r_{\mathcal{M}}(\mathcal{S}_{X \cap Y}) + r_{\mathcal{M}}(\mathcal{S}_{X \cup Y})$. Hence equality holds everywhere and (a) follows.

(b) By the definition of Y , $v \in Y \subseteq X$ and every arc that enters Y enters X as well. Then, by (3), the tightness of X and the monotonicity of $r_{\mathcal{M}}$, we have $r_{\mathcal{M}}(\mathcal{S}) - r_{\mathcal{M}}(\mathcal{S}_Y) \leq \rho_D(Y) \leq \rho_D(X) = r_{\mathcal{M}}(\mathcal{S}) - r_{\mathcal{M}}(\mathcal{S}_X) \leq r_{\mathcal{M}}(\mathcal{S}) - r_{\mathcal{M}}(\mathcal{S}_Y)$. Thus equality holds everywhere and (b) follows.

(c) For all $y \in Y$, there exists a directed path $y = v_l, \dots, v_1 = v$ from y to v in $D[X]$ using only good arcs. Then $\mathcal{S}_y = \mathcal{S}_{v_l} \subseteq \dots \subseteq \text{Span}_{\mathcal{M}}(\mathcal{S}_{v_1}) = \text{Span}_{\mathcal{M}}(\mathcal{S}_v)$. Hence $\mathcal{S}_Y = \bigcup_{y \in Y} \mathcal{S}_y \subseteq \text{Span}_{\mathcal{M}}(\mathcal{S}_v)$ and (c) follows. \blacksquare

Now we can prove the main result.

Proof. (of sufficiency in Theorem 1.6) We start by proving the following claim.

Claim 2.3. *If there is no bad arc then taking $|\mathcal{S}_v|$ times each vertex v gives an \mathcal{M} -basic packing of arborescences in (D, \mathcal{S}, π) .*

Proof. For every vertex v , let us denote by Z_v the set of vertices from which v is reachable in D . Since V is tight, Claim 2.2(b) implies that Z_v dominates V . Moreover, since every arc is good, by Claim 2.2(c), v dominates Z_v and hence, since π is \mathcal{M} -independent, \mathcal{S}_v is a base of \mathcal{M} for all $v \in V$. \blacksquare

We now prove the sufficiency by induction on $|A|$. If A is empty, then there is no bad arc, and, by Claim 2.3, the theorem is proved.

So we may assume that A is not empty and there exists at least one bad arc.

For a bad arc $uv \in A$ and $\mathfrak{s} \in \mathcal{S}_u \setminus \text{Span}(\mathcal{S}_v)$, let $D' = D - uv$, S' the set obtained by adding a new element \mathfrak{s}' to S , \mathcal{M}' the matroid on S' obtained from \mathcal{M} by considering \mathfrak{s}' as an element parallel to \mathfrak{s} and π' the placement of S' in V obtained from π by placing the new element \mathfrak{s}' at v .

By choice of \mathfrak{s} , π' is \mathcal{M}' -independent. If the digraph with roots (D', S', π') is \mathcal{M}' -connected, then, by induction, there exists an \mathcal{M}' -basic packing \mathcal{P}' of arborescences in (D', S', π') . Since \mathfrak{s} and \mathfrak{s}' are parallel in \mathcal{M}' , the arborescences T and T' of \mathcal{P}' rooted at \mathfrak{s} and \mathfrak{s}' are vertex disjoint, so $T'' = T \cup T' \cup uv$ is an arborescence rooted at \mathfrak{s} . Then $(\mathcal{P}' \cup \{T''\}) \setminus \{T, T'\}$ is an \mathcal{M} -basic packing of arborescences in (D, S, π) . Hence the proof of the theorem is reduced to the proof of the following claim.

Claim 2.4. *There exist a bad arc uv and $\mathfrak{s} \in \mathcal{S}_u \setminus \text{Span}(\mathcal{S}_v)$ such that (D', S', π') is \mathcal{M}' -connected.*

Proof. Assume that the claim is false. Let $uv \in A$ be a bad arc and $\mathfrak{s} \in \mathcal{S}_u \setminus \text{Span}(\mathcal{S}_v)$, by assumption, there exists $\emptyset \neq X_{\mathfrak{s}} \subseteq V$ such that $\rho_{D'}(X_{\mathfrak{s}}) < r_{\mathcal{M}}(S) - r_{\mathcal{M}'}(S'_{X_{\mathfrak{s}}})$. Hence, by (3) and the monotonicity of $r_{\mathcal{M}'}$, $\rho_{D'}(X_{\mathfrak{s}}) + 1 \geq \rho_{D'}(X_{\mathfrak{s}}) + \rho_{uv}(X_{\mathfrak{s}}) = \rho_D(X_{\mathfrak{s}}) \geq r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_{X_{\mathfrak{s}}}) \geq r_{\mathcal{M}}(S) - r_{\mathcal{M}'}(S'_{X_{\mathfrak{s}}}) \geq \rho_{D'}(X_{\mathfrak{s}}) + 1$, so equality holds everywhere and hence uv enters $X_{\mathfrak{s}}$, $X_{\mathfrak{s}}$ is tight and $\mathfrak{s} \in \text{Span}_{\mathcal{M}}(S_{X_{\mathfrak{s}}})$. Hence, by Claim 2.2, $X = \cup_{\mathfrak{s} \in \mathcal{S}_u \setminus \text{Span}(\mathcal{S}_v)} X_{\mathfrak{s}}$ is tight and, by $v \in X$, $\mathcal{S}_u = (\mathcal{S}_u \setminus \text{Span}(\mathcal{S}_v)) \cup (\mathcal{S}_u \cap \text{Span}(\mathcal{S}_v)) \subseteq \text{Span}(S_X) \cup \text{Span}(S_X) = \text{Span}(S_X)$. So we proved that

$$\text{every bad arc } uv \text{ enters a tight set } X \text{ that dominates } u. \quad (4)$$

Among all pairs (uv, X) satisfying (4) choose one with X minimal.

Suppose that every arc in $D[X]$ is good. Note that, by Claim 2.2(b) and the minimality of X , v can be reached from all vertices of X in $D[X]$. Then, by (4), X dominates u and, by Claim 2.2(c), v dominates X so v dominates u which contradicts the fact that uv is bad.

Hence there exists a bad arc $u'v'$ in $D[X]$. Then, by (4), $u'v'$ enters a tight set Y that dominates u' . By $v' \in X \cap Y$, the tightness of X and Y , $u' \in X$, $\mathcal{S}_{u'} \subseteq \text{Span}_{\mathcal{M}}(S_Y)$, Claim 2.2(a) and Claim 2.1, we have that $X \cap Y$ is tight and $\mathcal{S}_{u'} \subseteq \text{Span}_{\mathcal{M}}(S_X \cap S_Y) = \text{Span}_{\mathcal{M}}(S_{X \cap Y})$. Since the bad arc $u'v'$ enters the tight set $X \cap Y$ that dominates u' and $X \cap Y$ is a proper subset of X (since $u' \in X \setminus Y$), this contradicts the minimality of X . ■

3 Polyhedral aspects

In this section we study a polyhedron describing the basic packings of arborescences.

We need the following general result of Frank [3].

Theorem 3.1 (Frank [3]). *Let $D = (V, A)$ be a digraph, $p : 2^V \rightarrow \mathbb{Z}_+$ a non-negative intersecting supermodular set-function such that $\rho_D(Z) \geq p(Z)$ for every $Z \subseteq V$. Then the polyhedron defined by the following linear system is integer:*

$$\begin{aligned} 1 &\geq x(a) \geq 0 && \text{for all } a \in A, \\ x(\varrho_D(X)) &\geq p(X) && \text{for all non-empty } X \subseteq V. \end{aligned}$$

This following theorem is a corollary of Theorem 1.6 and Theorem 3.1.

Theorem 3.2. *Let $(D = (V, A), \mathcal{S}, \pi)$ be a digraph with roots and \mathcal{M} a matroid on \mathcal{S} of rank k with rank function $r_{\mathcal{M}}$. There exists an \mathcal{M} -basic packing of arborescences in (D, \mathcal{S}, π) if and only if the polyhedron $P_{\mathcal{M}, D}$ defined by the linear system*

$$1 \geq x(a) \geq 0 \quad \text{for all } a \in A, \quad (5)$$

$$x(\varrho_D(X)) \geq k - r_{\mathcal{M}}(\mathcal{S}_X) \quad \text{for all non-empty } X \subseteq V, \quad (6)$$

$$x(A) = k|V| - |\mathcal{S}| \quad (7)$$

is not empty. In this case, $P_{\mathcal{M}, D}$ is integer and its vertices are the characteristic vectors of the arc sets of the \mathcal{M} -basic packings of arborescences in (D, \mathcal{S}, π) .

Proof. Suppose there exists an \mathcal{M} -basic packing of arborescences in (D, \mathcal{S}, π) and call $A' \subseteq A$ its arc set. Let x be the characteristic vector of A' . We have $x(A) = |A'| = \sum_{v \in V} \rho_{A'}(v) = \sum_{v \in V} (k - |\mathcal{S}_v|) = k|V| - |\mathcal{S}|$ and $x(\varrho_D(X)) = \rho_{A'}(X) \geq k - r_{\mathcal{M}}(\mathcal{S}_X)$ for all non-empty $X \subseteq V$ by (3). So $x \in P_{\mathcal{M}, D}$.

Now suppose that $P_{\mathcal{M}, D}$ is not empty. Since the function $k - r_{\mathcal{M}}(\mathcal{S}_X)$ is non-negative intersecting supermodular and, by (5) and (6), $\rho_D(X) \geq k - r_{\mathcal{M}}(\mathcal{S}_X)$ for all non-empty $X \subseteq V$, Theorem 3.1 implies that the polyhedron P described by (5) and (6) is integer. By (6), for all $x \in P$,

$$x(A) = \sum_{v \in V} x(\varrho_D(v)) \geq \sum_{v \in V} (k - r_{\mathcal{M}}(\mathcal{S}_v)) \geq \sum_{v \in V} (k - |\mathcal{S}_v|) = k|V| - |\mathcal{S}|, \quad (8)$$

that is, $x(A) \geq k|V| - |\mathcal{S}|$ is a valid inequality for P . Then, by (7), $P_{\mathcal{M}, D}$ is a face of the integer polyhedron P and hence $P_{\mathcal{M}, D}$ is also integer. Furthermore, for $x \in P_{\mathcal{M}, D}$, equality holds everywhere in (8), thus, $|\mathcal{S}_v| = r_{\mathcal{M}}(\mathcal{S}_v)$ for all $v \in V$ and hence π is \mathcal{M} -independent. A vertex x of $P_{\mathcal{M}, D}$ defines an arc set $A' = \{a \in A, x(a) = 1\}$. By (6), the digraph with roots $((V, A'), \mathcal{S}, \pi)$ is \mathcal{M} -connected. Therefore, by Theorem 1.6, there exists an \mathcal{M} -basic packing of arborescences in $((V, A'), \mathcal{S}, \pi)$ whose arc set is, by (7), equal to A' , and the theorem follows. ■

4 Algorithmic aspects

We use the following theorem proved by Iwata, Fleischer and Fujishige [7] and independently by Schrijver [11].

Theorem 4.1 (Iwata, Fleischer and Fujishige [7], Schrijver [11]). *A submodular function can be minimized in polynomial time.*

In this section we assume that a matroid is given by an oracle for the rank function. The following theorem is a corollary of Theorem 4.1 and Theorem 1.6.

Theorem 4.2. *Let (D, \mathcal{S}, π) be a digraph with roots and \mathcal{M} a matroid on \mathcal{S} . An \mathcal{M} -basic packing of arborescences in (D, \mathcal{S}, π) or a vertex v certifying that π is not \mathcal{M} -independent or a vertex set X certifying that (D, \mathcal{S}, π) is not \mathcal{M} -connected can be found in polynomial time.*

Proof. By the submodularity of $\rho_D(X) + r_{\mathcal{M}}(\mathcal{S}_X)$, Theorem 4.1, using the oracle on \mathcal{M} and Theorem 1.6, we can either find a set violating (3) or a vertex certifying that π is not \mathcal{M} -independent or certify that there exists an \mathcal{M} -basic packing of arborescences.

In the latter case, an \mathcal{M} -basic packing of arborescences can be found in polynomial time following the proof of Theorem 1.6. Using the oracle, test whether each arc is good or bad. When an arc uv is bad, for each $s \in \mathcal{S}_u \setminus \text{Span}(\mathcal{S}_v)$, determine in polynomial time whether D' is \mathcal{M}' -connected using the submodularity of $\rho_{D'}(X) + r_{\mathcal{M}'}(\mathcal{S}'_X)$, the oracle for the rank function $r_{\mathcal{M}'}$ (that is easily computed from $r_{\mathcal{M}}$) and Theorem 4.1. Either all arcs are good or we find a bad arc uv and $s \in \mathcal{S}_u \setminus \text{Span}(\mathcal{S}_v)$ satisfying Claim 2.4. In the first case, by Claim 2.3, the required packing is found. In the second case, it leads to the computation of an \mathcal{M}' -basic packing in the digraph with roots (D', \mathcal{S}', π') which contains less arcs than D . ■

By the submodularity of $x(\varrho_D(X)) + r_{\mathcal{M}}(\mathbf{S}_X)$ and Theorem 4.1, $P_{\mathcal{M},D}$ can be separated in polynomial time. Thus, using the ellipsoid method, by Grötschel, Lovász and Schrijver [6], and by Theorem 4.2, we have the following result.

Theorem 4.3. *Let (D, \mathbf{S}, π) be a digraph with roots, \mathcal{M} a matroid on \mathbf{S} and c a cost function on the set of arcs of D . If there exists an \mathcal{M} -basic packing of arborescences in (D, \mathbf{S}, π) then one of minimum cost can be found in polynomial time.*

5 Final remarks

We finish the paper with a related problem. Given a digraph with roots (D, \mathbf{S}, π) , a matroid \mathcal{M} on \mathbf{S} with rank function $r_{\mathcal{M}}$ and a bound $b : V \rightarrow \mathbb{Z}$, an (\mathcal{M}, b) -packing of arborescences is a set $\{T_1, \dots, T_t\}$ of pairwise arc-disjoint arborescences such that T_i is rooted at $\mathbf{s}_i \in \mathbf{S}$ for $i = 1, \dots, t$ and $r_{\mathcal{M}}(\{\mathbf{s}_i \in \mathbf{S} : v \in V(T_i)\}) \geq b(v)$ for all $v \in V$. When b is constant, using Theorem 1.6 and matroid truncation, one can derive a characterization of digraphs with roots admitting an (\mathcal{M}, b) -packing of arborescences. On the other hand, for general b , the problem turns out to be NP-complete since it contains the disjoint Steiner arborescences problem that is to find 2 arc-disjoint arborescences both rooted at the same vertex and both covering a specified subset of vertices.

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