The Hypergraph Assignment Problem

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DFG Research Center MATHEON
Mathematics for key technologies

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Contents

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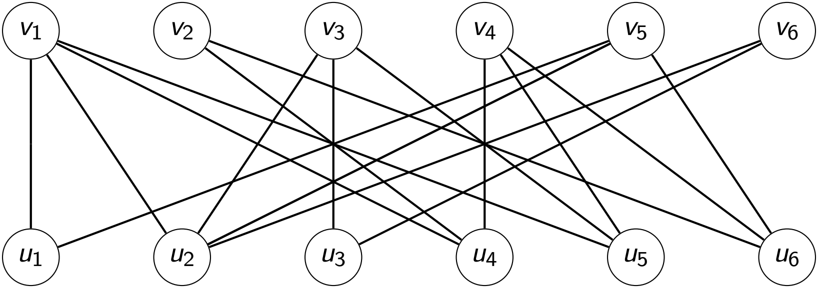
- a set E of edges connecting U and V,

an assignment is a subset H of E such that there is exactly one incident edge in H for each vertex.



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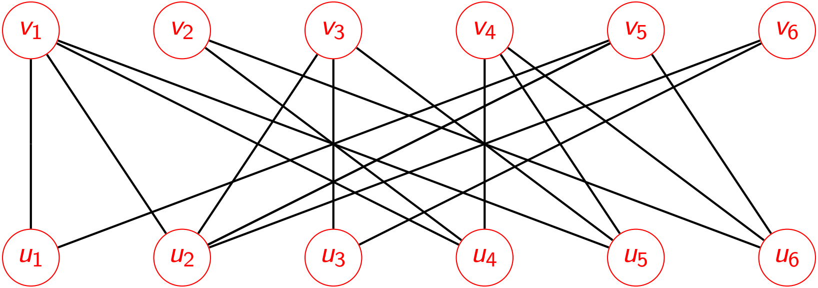

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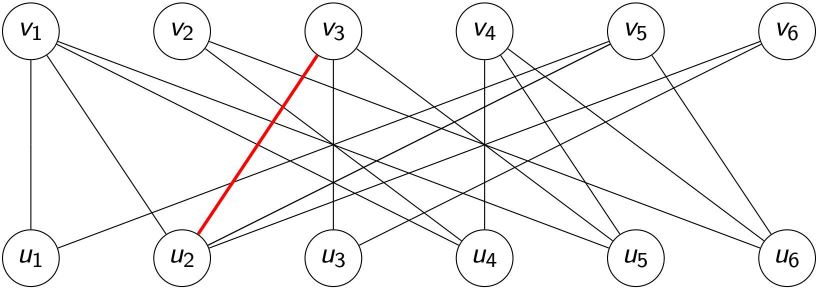

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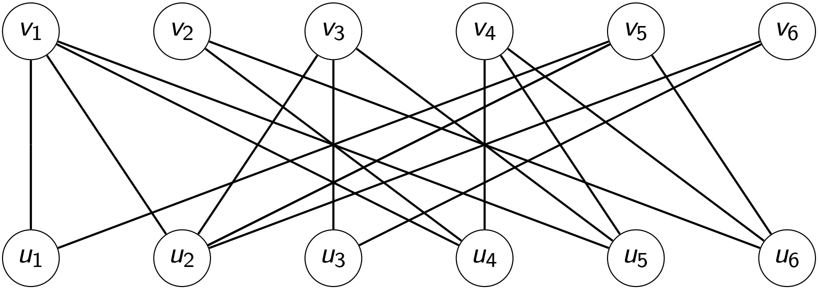

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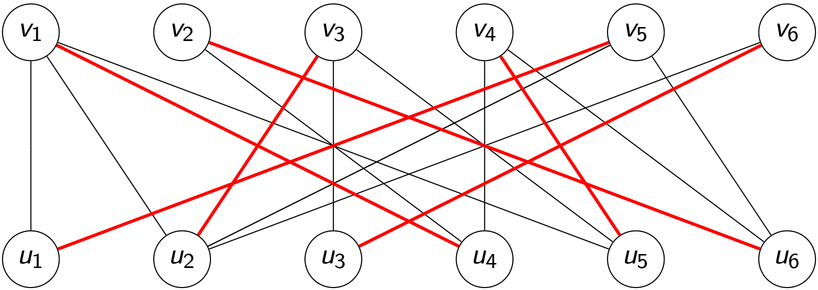

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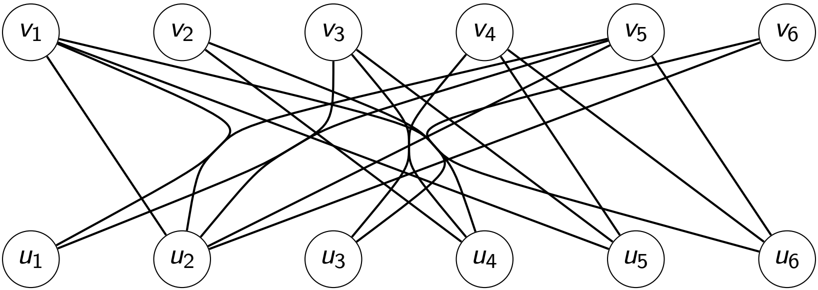

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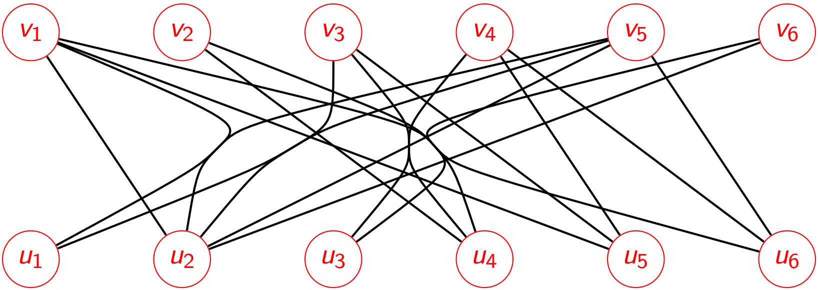

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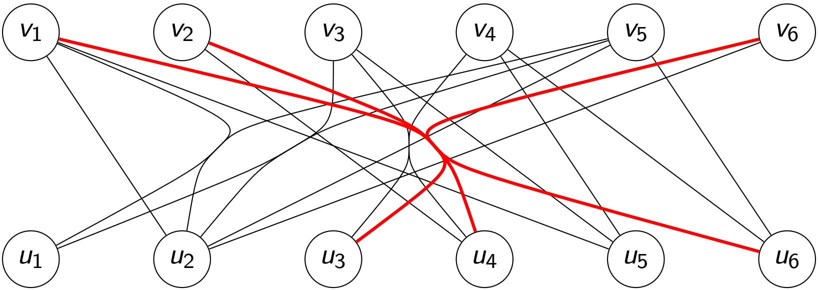

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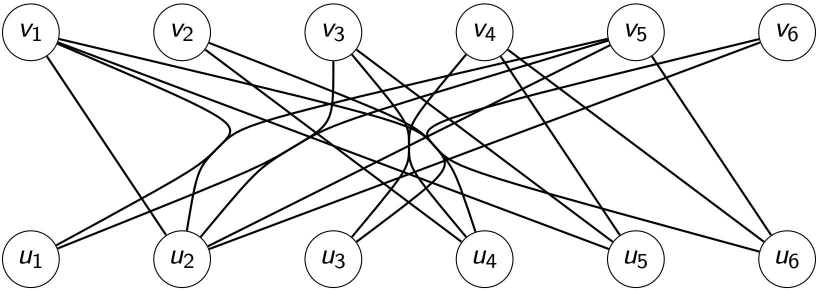

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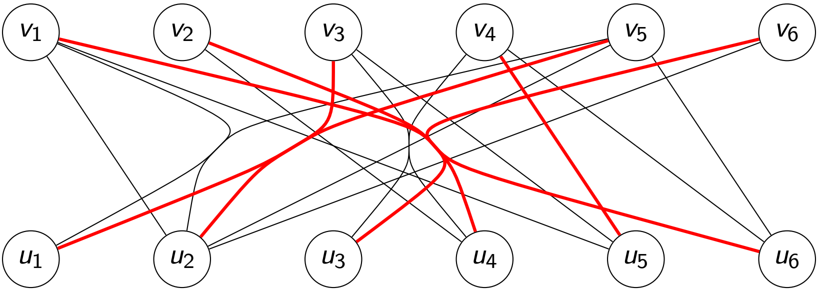


Definition

A *bipartite hypergraph* $G = (U, V, E)$ is a triple of two disjoint vertex sets U, V and a set of hyperedges $E \subseteq 2^{U \dot\cup V}$. We assume that the vertex sets have the same size $|U| = |V|$, and that every hyperedge $e \in E$ has the same number $|e \cap U| = |e \cap V| > 0$ of vertices in U and V. We denote by $|e|$ the *size* of the hyperedge $e \in E$, and call a hyperedge of size 2 an *edge*.

Definition

For a vertex subset $W \subseteq U \cup V$ we define the *incident hyperedges* $\delta(W) := \{e \in E : e \cap W \neq \emptyset, e \setminus W \neq \emptyset\}$ to be the set of all hyperedges having at least one vertex in both U and $(U \cup V) \setminus W$. We also write $\delta(v) = \delta(\{v\})$ if v is a vertex.

Hypergraph Assignment Problem (HAP)

Definition

Let $G = (U, V, E)$ be a bipartite hypergraph. A *hyperassignment* in G is a subset $H \subseteq E$ of hyperedges such that every $v \in U \cup V$ is contained in exactly one hyperedge $e \in H$.

Hypergraph Assignment Problem

Input: A pair (G, c_E) consisting of a bipartite hypergraph $G = (U, V, E)$ and a cost function $c_E : E \to \mathbb{R}$.
Output: A minimum cost hyperassignment in G w. r. t. c_E, i. e., a hyperassignment H^* in G such that

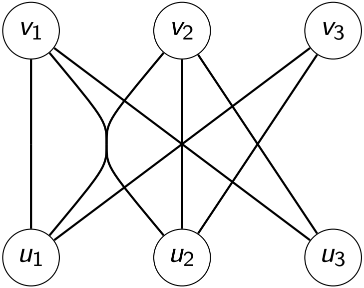
$$c_E(H^*) = \min\{c_E(H) : H \text{ is a hyperassignment in } G\},$$

or the information that no hyperassignment exists.

Theorem (B., He. [2011])

1. *The hypergraph assignment problem (HAP) is NP-hard.*
2. *The HAP is APX-hard.*
3. *The LP/IP gap of HAP can be arbitrarily large.*
4. *The determinants of basis matrices of HAP can be arbitrarily large.*

$$\begin{aligned} \min_{x \in \mathbb{R}^E} \quad & \sum_{e \in E} c_E(e) x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e = 1 \qquad \forall v \in U \cup V \\ & x \geq 0 \\ & x \in \mathbb{Z}^E \end{aligned}$$



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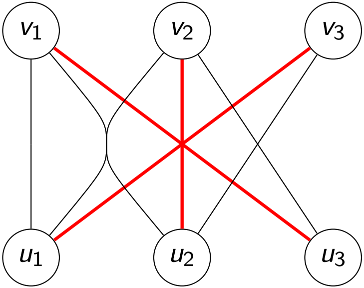

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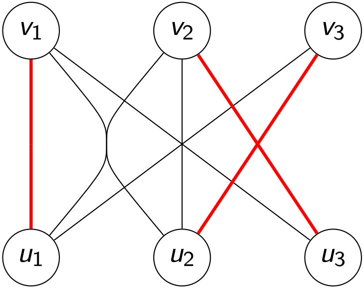

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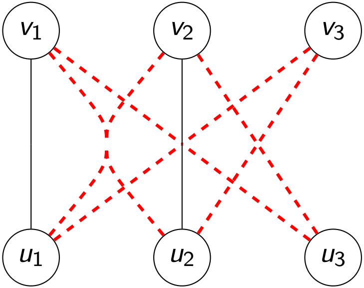
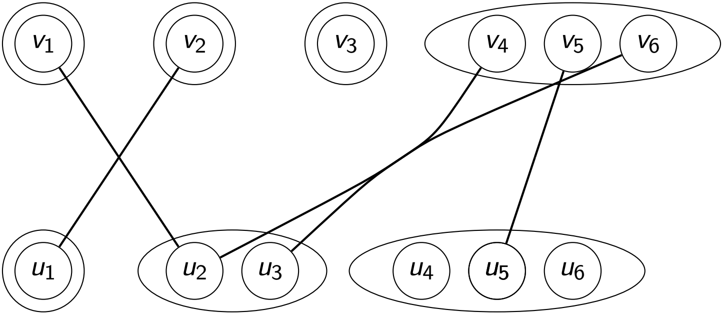


Definition

$G = (U, V, E)$ is called a *partitioned* bipartite hypergraph with *maximum part size* $d \in \mathbb{N}$ if additionally there exist pairwise disjoint $\leq d$-element sets $U_1, \dots, U_p$ and $V_1, \dots, V_q$ called the parts of H such that $\dot\bigcup_{i=1}^{p} U_i = U$, $\dot\bigcup_{i=1}^{q} V_i = V$, and $E \subseteq \bigcup_{i=1}^{p} \bigcup_{j=1}^{q} 2^{U_i \cup V_j}$, i. e., every hyperedge intersects only one part in U and one part in V.



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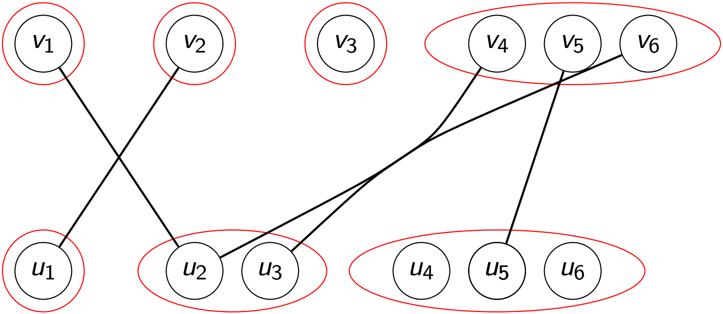

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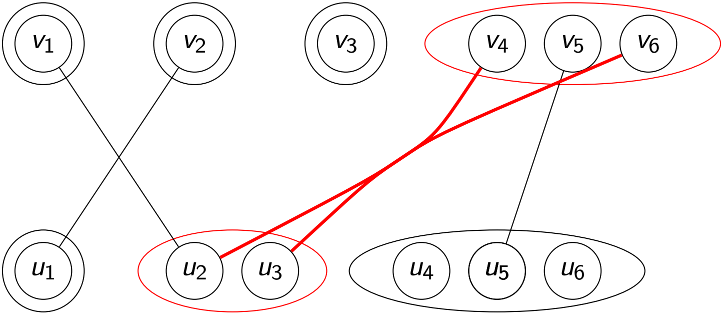

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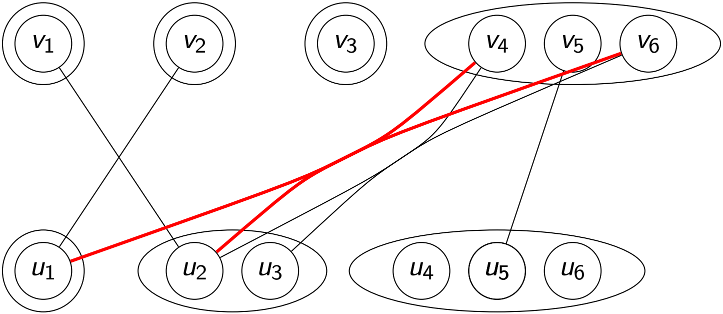


Results for Partitioned Hypergraphs

Theorem (B., He. [2012])

Every HAP can be polynomially transformed into a HAP on a partitioned hypergraph.

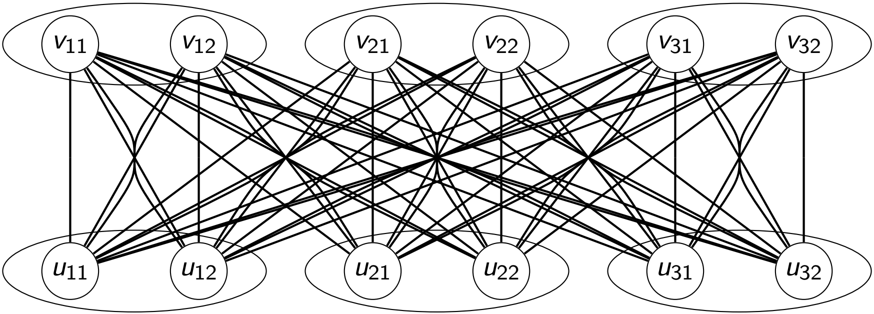
A *clique* $Q \subseteq E$ is a set of hyperedges such that every pair of hyperedges in Q has a nonempty intersection.

Theorem (B., He. [2011])

Every clique in a partitioned hypergraph is a subset of the incident hyperedges $\delta(P)$ of some part P.
There exists an extended formulation with $O(|U|^{d+1})$ variables that implies all clique inequalities.

Let $G_{2,3} = (U, V, E)$ be the complete partitioned bipartite hypergraph with

- parts $\{u_{11}, u_{12}\}$, $\{u_{21}, u_{22}\}$, $\{u_{31}, u_{32}\}$ in U and
- parts $\{v_{11}, v_{12}\}$, $\{v_{21}, v_{22}\}$, $\{v_{31}, v_{32}\}$ in V.



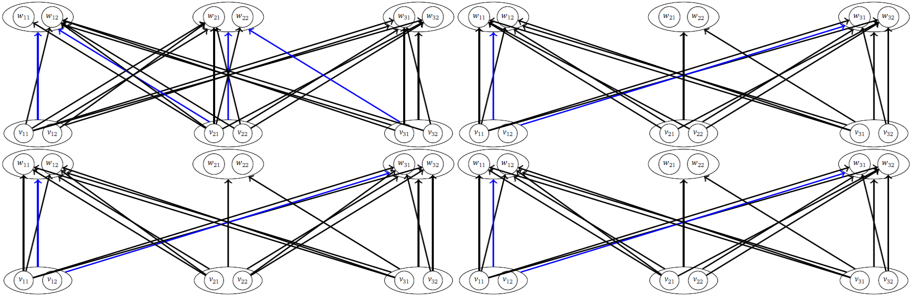
Let $P(G_{2,3})$ be the HAP polytope associated with $G_{2,3}$.

$P(G_{2,3})$ is completely described by 14049 facets.

- Every facet of $P(G_{2,3})$ can be described by many different inequalities (polytope description includes 11 equations).
- All facets can be described in the form

$$\sum_{e \in E_1} x_e - \sum_{e \in E_2} x_e \leq 1.$$

- So far we have no normal form.



Polyhedral Results cont. (B., He.)

The polytope is highly symmetric. The symmetries are generated by:

- $u_{ij} \mapsto v_{ij}$, $v_{ij} \mapsto u_{ij}$ for all i, j
- $u_{ij} \mapsto u_{ij}$, $v_{ij} \mapsto v_{\sigma(i)j}$ for some $\sigma \in S_3$
- $u_{11} \mapsto u_{11}$, $u_{12} \mapsto u_{11}$, $u_{ij} \mapsto u_{ij}$ for $i \neq 1$, $v_{ij} \mapsto v_{ij}$

This results in 4608 vertex permutations, which imply permutations of the hyperedge variables.

The 14049 facets of $P(G_{2,3})$ fall into 30 symmetry classes.

We have understood 16 facet classes:

- trivial facets: hyperedge ≥ 0
- cliques
- odd clique set inequalities (see next slides)

14 facet classes are still to be understood.

Classification of Facets Without Normal Form

Given:

- permutation of variables
- vertices of the polytope
- facet inequalities of the polytope

How to classify the facets into symmetry classes?

- identify every facet with the incident vertices of the polytope
- permutation of variables implies permutation of vertices
- permutation of vertices implies permutation of facets
- implemented in general ("HUHFA")

Odd set cuts for the matching polytope of a graph $G = (N, E)$, $N' \subseteq N$, $|N'| = 2k + 1$ odd (Edmonds [1965]):

$$\sum_{e \in E : e \subseteq N'} x_e \leq k$$

or

$$\sum_{e \in E} \left\lfloor \frac{|\{v \in N' : e \in \delta(v)\}|}{2} \right\rfloor x_e \leq k.$$

Generalization for a hypergraph $G = (U, V, E)$, $N' \subseteq U \cup V$, $|N'| = 2k + 1$ odd:

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Generalization of Odd Set Inequalities

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Replace N' by a set of cliques: $\mathcal{Q} \subseteq 2^E$, $|\mathcal{Q}| = 2k+1$ odd number of cliques in G. Odd clique set cut:

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Odd Clique Set Inequalities (cont.)

- Not all odd clique set inequalities are facets for the HAP polytope.
- Separation?
- Different generalization of odd set cuts: "Generalized clique family inequalities for claw-free graphs" (Pêcher, Wagler [2006])

$$\begin{aligned} p &\leq |\mathcal{Q}| \\ 0 &\leq r \leq R = |\mathcal{Q}| \mod p \\ 0 &\leq J \leq p - r \\ E_p &:= \{e \in E : |\{Q \in \mathcal{Q} : e \in Q\}| \geq p\} \\ E_{p-j} &:= \{e \in E : |\{Q \in \mathcal{Q} : e \in Q\}| = p - j\} \end{aligned}$$

$$\sum_{0 \leq j \leq J} (p - r - j) \sum_{e \in E_{p-j}} x_e \leq b$$

do not lead to facets of $P(G_{2,3})$

Contents

Overview of Heuristic Approaches

- Constructive heuristics:
  - greedy with coefficients per vertex
  - Hungarian method with vertex groups
- Local search:
  - Hungarian method with vertex groups
  - 2-opt
  - dynamic k-opt
- Perturbation heuristics:
  - greedy insert with randomization

Results of first tests:

instance name	bipartite hypergraph	arcs	2-hyperedges	optimal value	heuristic result	gap	run time (sec.)
Random10	$G_{2,10}$	400	100	88	88	0 %	52.9
Random20	$G_{2,20}$	1600	400	84	85	1.2 %	53.7
Random35	$G_{2,35}$	4900	1225	92	129	40.2 %	57.8
Random50	$G_{2,50}$	10000	2500	112	144	28.6 %	54.4
Random75	$G_{2,75}$	22500	5625	95	140	47.4 %	105.8
Random100	$G_{2,100}$	40000	10000	93	155	66.7 %	223.5

- costs of hyperedges i. i. d. from $\{0, \ldots, 100\}$
- some variability in results and run times due to randomization
- many parameter changes possible

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