A 3-index Formulation for the Optimum Communication Spanning Tree Problem

(work in progress)

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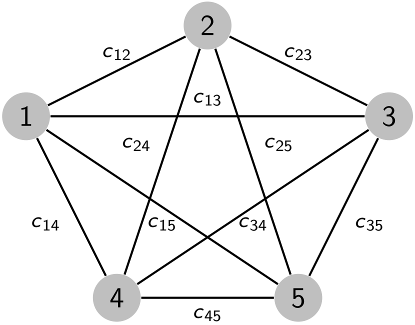
OCSTP - Definitions



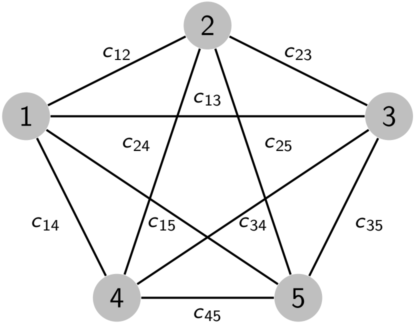
G is an undirected, weighted graph.

Cost Matrix:

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

Where $c_{ij} \geq 0$.

OCSTP - Definitions

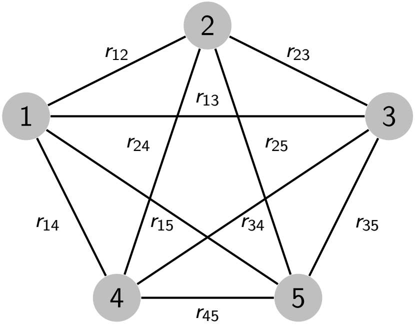


G is an undirected, weighted graph.

Cost Matrix:

- We can consider G complete.
- $c_{ii} = 0$
- $c_{ij} = c_{ji}$
- $c_{ik} \leq c_{ij} + c_{jk}$

OCSTP - Definitions



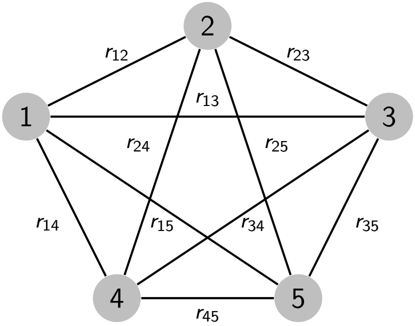
G is an undirected, weighted graph.

Communication Requests Matrix:

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix}$$

Where $r_{ij} \geq 0$.

OCSTP - Definitions



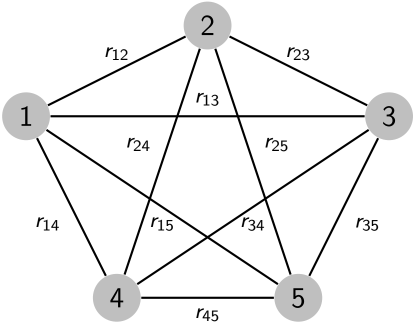
G is an undirected, weighted graph.

Communication Requests Matrix:

$$\begin{pmatrix} 0 & r_{12} + r_{21} & \cdots & r_{1n} + r_{n1} \\ 0 & 0 & \cdots & r_{2n} + r_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

We can consider it upper triangular.

OCSTP - Definitions

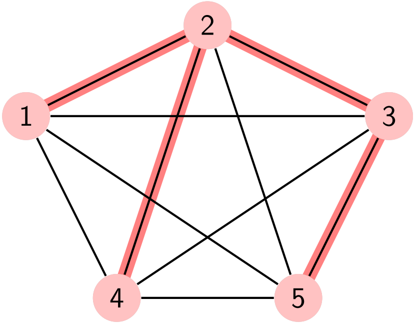


G is an undirected, weighted graph.

Communication Requests Matrix:

$$R = \begin{pmatrix} 0 & r_{12} & r_{13} & r_{14} & r_{15} \\ 0 & 0 & r_{23} & r_{24} & r_{25} \\ 0 & 0 & 0 & r_{34} & r_{35} \\ 0 & 0 & 0 & 0 & r_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

OCSTP - Definitions



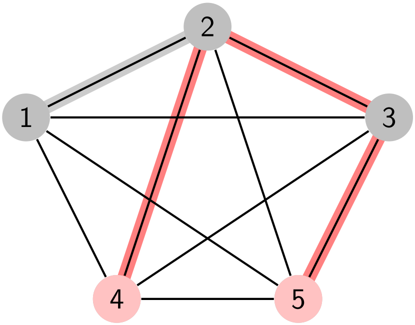
Given an spanning tree T, the Communication Cost of (i, j) is:

$$r_{ij} d_{ij}$$

Where:

$d_{ij} =$ length of the (unique) path between i and j in T.

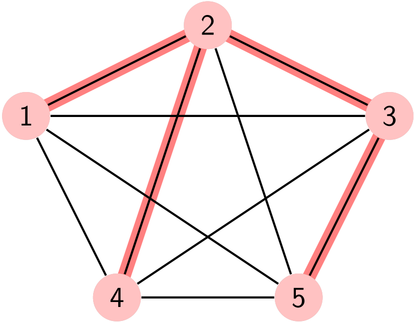
OCSTP - Definitions



Given an spanning tree T, the Communication Cost of $(4, 5)$ is:

$$r_{45} d_{45} = r_{45}(c_{42} + c_{23} + c_{35})$$

OCSTP - Definitions



Given an spanning tree T, the Communication Cost of (i, j) is:

$$r_{ij} d_{ij}$$

the Total Communication Cost is:

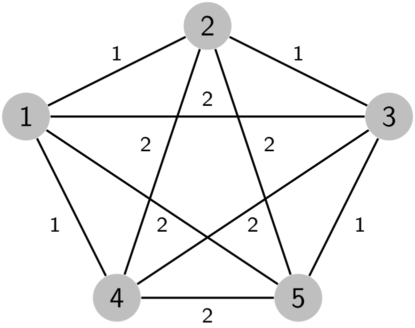
$$\sum_{i \neq j} r_{ij} d_{ij}$$

OCSTP - Definitions

The Optimum Communication Spanning Tree Problem:

Given C and R find the spanning tree T
that **minimizes** the **Total Communication Cost**.

OCSTP vs MST

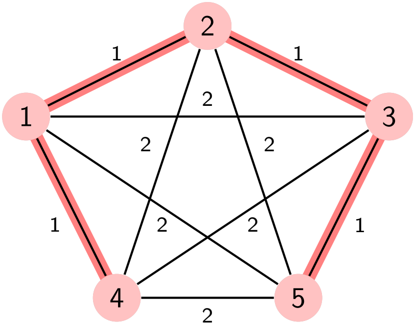


The OCST and the MST can be very different in general:

Communication Requests Matrix:

$$R = \begin{pmatrix} 0 & 1 & 1 & 1 & 20 \\ 0 & 0 & 1 & 20 & 20 \\ 0 & 0 & 0 & 20 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

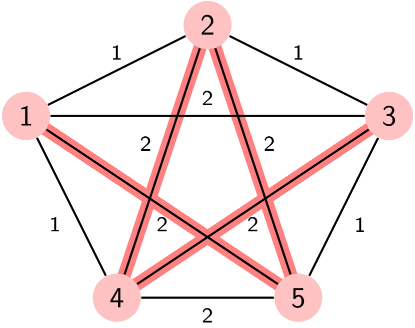
OCSTP vs MST



The OCST and the MST can be very different in general:

- MST weight: 4
- MST com. cost: 210

OCSTP vs MST



The OCST and the MST can be very different in general:

- OCST weight: 8
- OCST com. cost: 192

They don't share any edge!

OCSTP - History

Originally introduced in:

1974 – T. C. Hu

Optimum Communication Spanning Trees

Proven NP-Hard in:

1978 – Johnson, Lenstra and Rinnooy Kan

The complexity of the network design problem

OCSTP - History

Exact method:

1987 – Ahuja and Murty

Exact and Heuristic Algorithms for the OCSTP

Heuristic methods:

1974 – Now

Several attempts, mainly, **genetic algorithms** and **local searches**

Lagrangean bounds:

2010 – Contreras, Fernández and Marín

Lagrangean bounds for the OCSTP

OCSTP - Particular cases

Optimum Requirement Spanning Tree:

$$c_{ij} = 1 \quad \forall (i,j) \in E$$

The Gomory-Hu tree of mincuts (w.r.t. R) **is an OCST.**

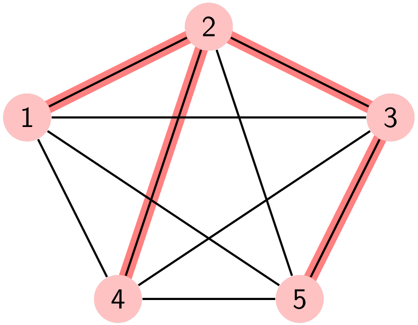
Optimum Distance Spanning Tree:

$$r_{ij} = 1 \quad \forall (i,j) \in E \quad + \quad \textit{additional condition}$$

There is an OCST which is a star.

MIP formulations for the OCSTP

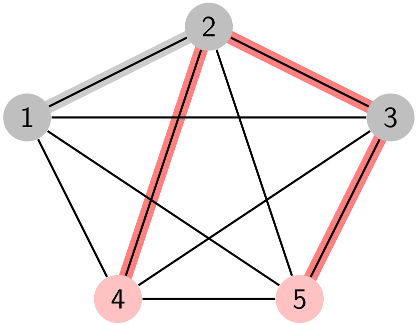
MIP formulations for the OCSTP



Alternative Objective Functions

Communication cost of T:

MIP formulations for the OCSTP

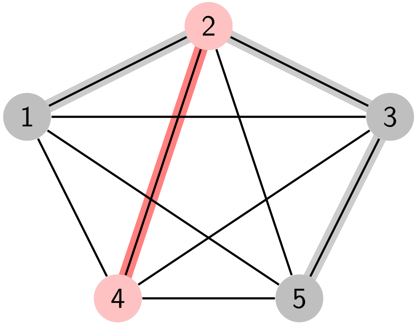


Alternative Objective Functions

Communication cost of T:

$$\min \sum_{r_{od}>0} r_{od}\, d_{od}$$

MIP formulations for the OCSTP



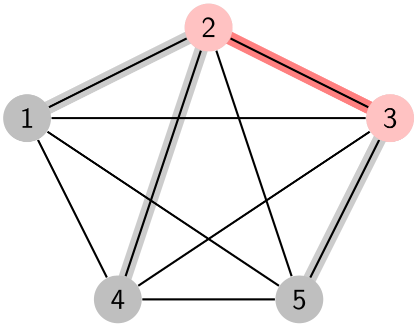
Alternative Objective Functions

Communication cost of T:

$$\min \sum_{r_{od}>0} r_{od}\, d_{od}$$

$$\min \sum_{r_{od}>0} r_{od} \sum_{i \neq j} c_{ij} x_{ij}^{od}$$

MIP formulations for the OCSTP



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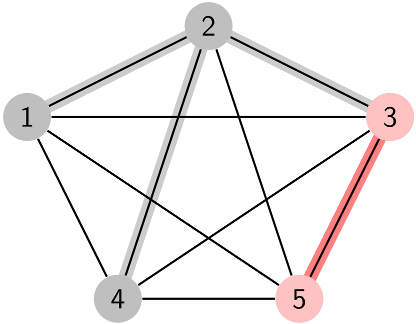

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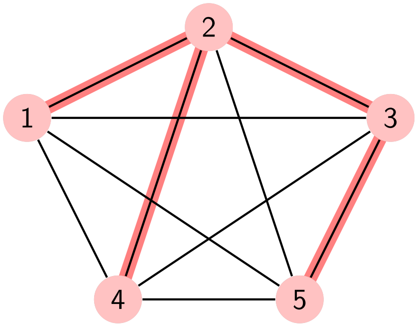

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$$\min \sum_{r_{od}>0} \sum_{i \neq j} r_{od} c_{ij} x_{ij}^{od}$$

MIP formulations for the OCSTP

4-index variables

$$x_{ij}^{od} = \begin{cases} 1 & \text{if the directed arc } (i,j) \text{ is on} \\ & \text{the path from } o \text{ to } d. \\ 0 & \text{otherwise.} \end{cases}$$

$$y_{ij} = \begin{cases} 1 & \text{if the edge } \{i,j\} \text{ is in the tree.} \\ 0 & \text{otherwise.} \end{cases}$$

MIP formulations for the OCSTP

$$\min \sum_{r_{od}>0} \sum_{i \neq j} r_{od}\, c_{ij}\, x_{ij}^{od}$$

$$\text{s.t.} \quad \sum x_{oj}^{od} = 1 \qquad \forall\, r_{od} > 0 \qquad (1)$$

$$\sum x_{ij}^{od} - \sum x_{jk}^{od} = 0 \qquad \forall\, r_{od} > 0 \quad \forall j \neq o, d \qquad (2)$$

$$x_{ij}^{od} + x_{ji}^{od} \leq y_{ij} \qquad \forall\, i \neq j \qquad (3)$$

$$\sum y_{ij} = |V| - 1 \qquad (4)$$

$$x_{ij}^{od} \geq 0 \qquad \forall\, i \neq j \quad \forall\, r_{od} > 0 \qquad (5)$$

$$y_{ij} \in \{0, 1\} \qquad \forall\, i \neq j \qquad (6)$$

MIP formulations for the OCSTP

Advantages of the 4-index formulation

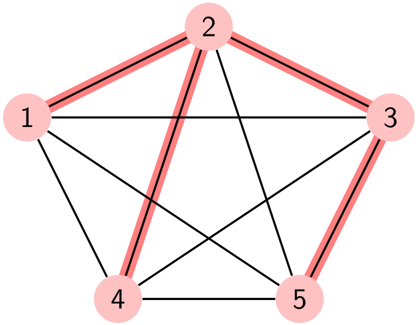
- Very good lower bounds.
- Takes advantage of instances with few $r_{ij} > 0$.

Drawbacks of the 4-index formulation

- Unable to load in memory problems of just 40 nodes.

A 3-index formulation for the OCSTP

A 3-index MIP formulation

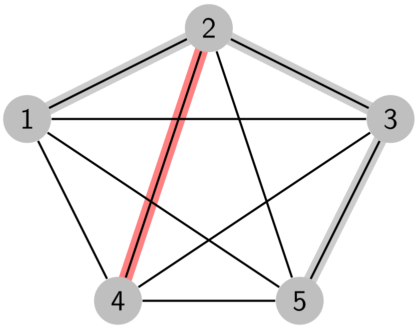


Alternative Objective Functions

Communication cost of T:

$$\min \sum_{r_{od}>0} \sum_{i \neq j} r_{od} c_{ij} x_{ij}^{od}$$

A 3-index MIP formulation



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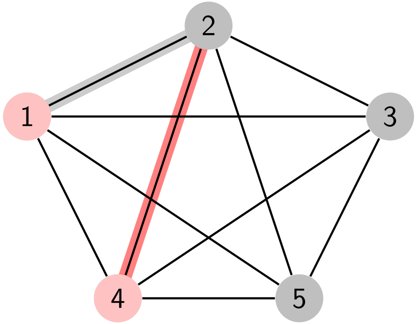
A 3-index MIP formulation



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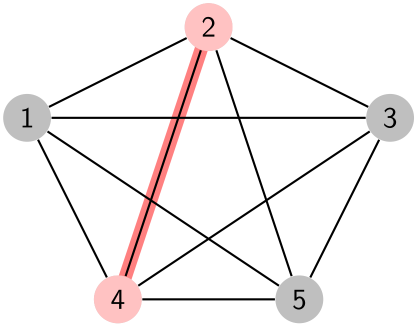

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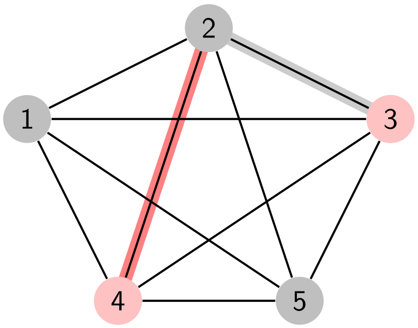

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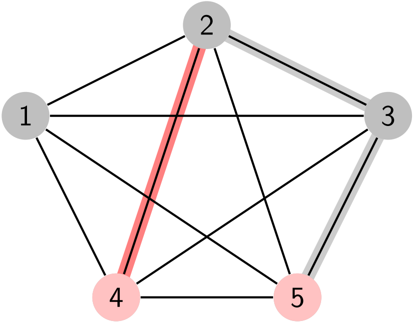
A 3-index MIP formulation



Alternative Objective Functions

Communication cost of T:

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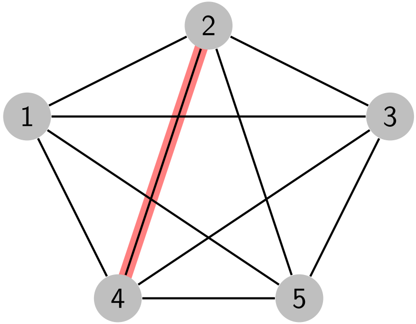
A 3-index MIP formulation



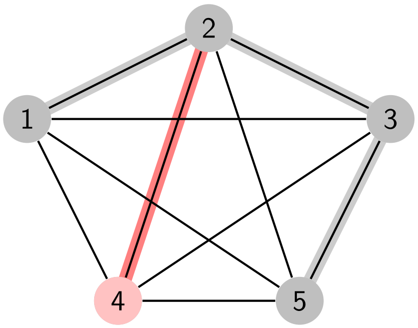
Alternative Objective Functions

Communication cost of T:

$$\min \sum_{r_{od}>0} \sum_{i \neq j} r_{od} c_{ij} x_{ij}^{od}$$

$$\min \sum_{i \neq j} c_{ij} \sum_{o} \sum_{d} r_{od} x_{ij}^{od}$$

A 3-index MIP formulation



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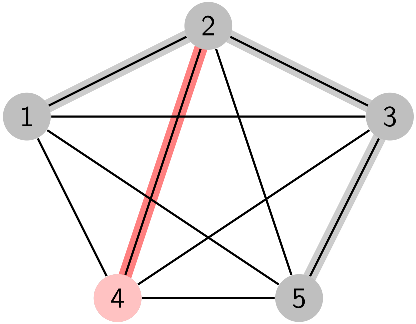
A 3-index MIP formulation



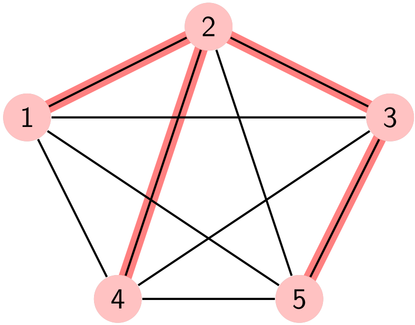
Alternative Objective Functions

Communication cost of T:

$$\min \sum_{r_{od}>0} \sum_{i \neq j} r_{od} c_{ij} x_{ij}^{od}$$

$$\min \sum_{i \neq j} c_{ij} \sum_{o} f_{ij}^{o}$$

A 3-index MIP formulation



Alternative Objective Functions

Communication cost of T:

$$\min \sum_{r_{od}>0} \sum_{i \neq j} r_{od} c_{ij} x_{ij}^{od}$$

$$\min \sum_{i \neq j} \sum_{o} c_{ij} f_{ij}^{o}$$

A 3-index MIP formulation

Variables

f_{ij}^{o} = Flow, with origin at o, that traverses the arc (i,j).

$$y_{ij} = \begin{cases} 1 & \text{if the edge } \{i,j\} \text{ is in the tree.} \\ 0 & \text{otherwise.} \end{cases}$$

A 3-index MIP formulation

$$\min \sum_{i \neq j} c_{ij}\, f_{ij}^o$$

$$\text{s.t.} \quad \sum f_{oj}^o = \sum r_{od} \qquad \forall\, o \qquad (1)$$

$$\sum f_{ij}^o - \sum f_{jk}^o = r_{oj} \qquad \forall j \neq o \qquad (2)$$

$$f_{ij}^o + f_{ji}^o \leq M y_{ij} \qquad \forall\, i \neq j \qquad (3)$$

$$\sum y_{ij} = |V| - 1 \qquad (4)$$

$$f_{ij}^o \geq 0 \qquad \forall\, i \neq j \quad \forall\, o$$

$$y_{ij} \in \{0, 1\} \qquad \forall\, i < j$$

3-index MIP Formulation

Drawbaks of the 3-index formulation

- Weak lower bounds (66% – 78%).
- Too slow to solve problems of just 20 nodes.

Advantages of the 3-index formulation

- Can load bigger problems in memory.
- Adding new valid inequalities seems easy.

Branch & Cut for the 3-index Formulation

Branch & Cut for the 3-index Formulation

Valid inequalities

$$\sum f_{io}^{o} = 0 \qquad \forall\, o \qquad (5)$$

$$y\,(\delta(S)) \geq 1 \qquad \forall\, S \subset V \qquad (6)$$

$$m_{ij} y_{ij} \leq \sum f_{ij}^{o} + \sum f_{ji}^{o} \qquad \forall\, i < j \qquad (7)$$

Where m_{ij} is the value of the min-cut separating i and j.

Branch & Cut for the 3-index Formulation

Extended formulation

f_{ij}^{o} = Flow, with origin at o, that traverses the arc (i,j).

$$x_{ij}^{o} = \begin{cases} 1 & \text{if the directed arc } (i,j) \text{ is used} \\ & \text{to deliver flow from } o. \\ 0 & \text{otherwise.} \end{cases}$$

$$y_{ij} = \begin{cases} 1 & \text{if the edge } \{i,j\} \text{ is in the tree.} \\ 0 & \text{otherwise.} \end{cases}$$

Branch & Cut for the 3-index Formulation

Extended valid inequalities

$$\sum x_{ij}^{o} = |V| - 1 \qquad \forall\, o \qquad (8)$$

$$\sum x_{oj}^{o} \geq 1 \qquad \forall\, o \qquad (9)$$

$$\sum x_{ij}^{o} = 1 \qquad \forall\, o \neq j \qquad (10)$$

$$x_{ij}^{o} + x_{ji}^{o} \leq y_{ij} \qquad \forall\, o \quad \forall\, i < j \qquad (11)$$

$$f_{ij}^{o} \leq M x_{ij}^{o} \qquad \forall\, o \quad \forall\, i \neq j \qquad (12)$$

Branch & Cut for the 3-index Formulation

Extended valid inequalities

$$r_{oj}x_{ij}^o + r_{oi}x_{ji}^o \leq f_{ij}^o + f_{ji}^o \qquad \forall\, o \quad \forall\, i < j \quad (13)$$

$$k_{ij}^o x_{ij}^o + k_{ji}^o x_{ji}^o \leq \sum f_{ij}^o + \sum f_{ji}^o \qquad \forall\, o \quad \forall\, i < j \quad (14)$$

Where $k_{ij}^o = \max\{m_{oj}, m_{ij}\}$.

Branch & Cut for the 3-index Formulation

Preliminary Results

# Nodes	LP1	LP2	LP3	4-index LP	Optimum
6	78.2%	78.2%	97.7%	100%	693180
9	73.8%	73.8%	95.6%	99.6%	1857749
10	69.7%	69.7%	94.2%	98.4%	2320671
12	66.2%	66.2%	93.2%	96.4%	3428509
16	75.1%	75.1%	92.2%	94.3%	460091

- **LP1:** Basic formulation
- **LP2:** Extended formulation
- **LP3:** Extended formulation + cuts

Questions? Ideas?

Thank You!