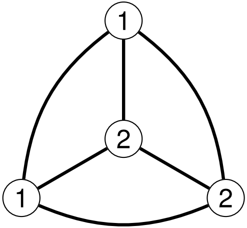
Degree-constrained orientations of embedded graphs

Yann Disser **Jannik Matuschke**

The Combinatorial Optimization Workshop
Aussois, January 9, 2013

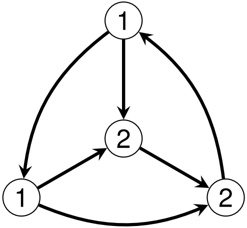
Graph orientation



Problem

Given graph G and $\alpha : V \to \mathbb{N}_0$, is there an orientation s.t. every vertex v has in-degree $\alpha(v)$?

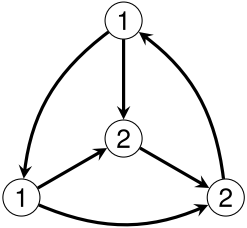
Graph orientation



Problem

Given graph G and $\alpha : V \to \mathbb{N}_0$, is there an orientation s.t. every vertex v has in-degree $\alpha(v)$?

Graph orientation

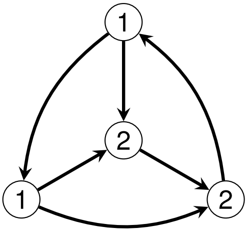


Problem

Given graph G and $\alpha : V \to \mathbb{N}_0$, is there an orientation s.t. every vertex v has in-degree $\alpha(v)$?

- applications in graph drawing, evacuation, data structures, theoretical insights ...

Graph orientation

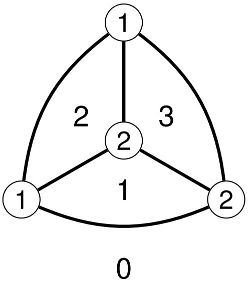


Problem

Given graph G and $\alpha : V \to \mathbb{N}_0$, is there an orientation s.t. every vertex v has in-degree $\alpha(v)$?

- applications in graph drawing, evacuation, data structures, theoretical insights ...
- solvable in poly-time, even for general upper and lower bounds [Hakimi 1965, Frank & Gyárfás 1976]

Graph orientation



Problem

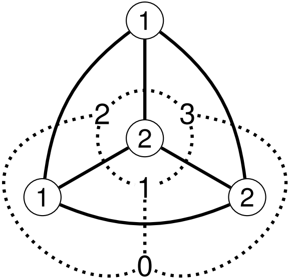
Given graph G and $\alpha : V \to \mathbb{N}_0$, is there an orientation s.t. every vertex v has in-degree $\alpha(v)$?

- applications in graph drawing, evacuation, data structures, theoretical insights ...
- solvable in poly-time, even for general upper and lower bounds [Hakimi 1965, Frank & Gyárfás 1976]

Question

What if we have degree-constraints in primal and dual graph?

Graph orientation



Problem

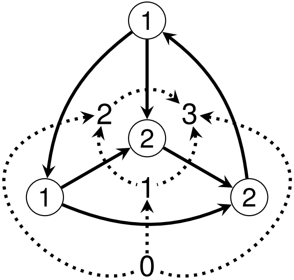
Given graph G and $\alpha : V \to \mathbb{N}_0$, is there an orientation s.t. every vertex v has in-degree $\alpha(v)$?

- applications in graph drawing, evacuation, data structures, theoretical insights ...
- solvable in poly-time, even for general upper and lower bounds
  [Hakimi 1965, Frank & Gyárfás 1976]

Question

What if we have degree-constraints in primal and dual graph?

Graph orientation



Problem

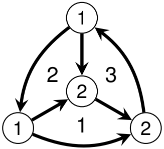
Given graph G and $\alpha : V \to \mathbb{N}_0$, is there an orientation s.t. every vertex v has in-degree $\alpha(v)$?

- applications in graph drawing, evacuation, data structures, theoretical insights ...
- solvable in poly-time, even for general upper and lower bounds
  [Hakimi 1965, Frank & Gyárfás 1976]

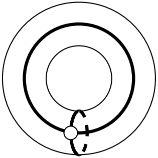
Question

What if we have degree-constraints in primal and dual graph?

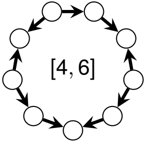
Outline



1. Uniqueness for planar embeddings



2. Bound for general embeddings



3. Hardness for interval version

Problem definition

Primal-dual orientation problem

Input: embedded graph $G = (V, E)$,
$\alpha : V \to \mathbb{N}_0$, $\alpha^* : V^* \to \mathbb{N}_0$

Task: Is there orientation D, s.t.
$|\delta^-_D(v)| = \alpha(v)$ for all $v \in V$ and
$|\delta^-_D(f)| = \alpha^*(f)$ for all $f \in V^*$?

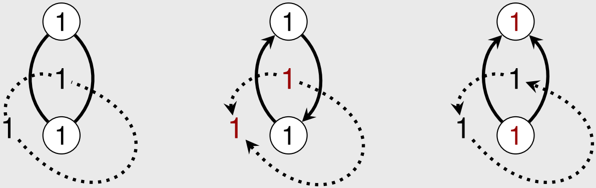
Problem definition

Primal-dual orientation problem

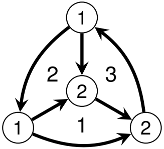
Input: embedded graph $G = (V, E)$,
$\alpha : V \to \mathbb{N}_0$, $\alpha^* : V^* \to \mathbb{N}_0$

Task: Is there orientation D, s.t.
$|\delta_D^-(v)| = \alpha(v)$ for all $v \in V$ and
$|\delta_D^-(f)| = \alpha^*(f)$ for all $f \in V^*$?

Existence of primal and dual solution not sufficient



Outline

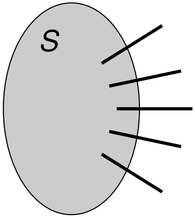


1. Uniqueness for planar embeddings

Directed cuts and rigid edges

Observation

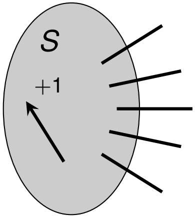
Let $S \subseteq V$. If $\sum_{v \in S} \alpha(v) = |E[S]|$, then all edges in $\delta(S)$ must be oriented away from S.



Directed cuts and rigid edges

Observation

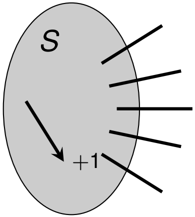
Let $S \subseteq V$. If $\sum_{v \in S} \alpha(v) = |E[S]|$, then all edges in $\delta(S)$ must be oriented away from S.



Directed cuts and rigid edges

Observation

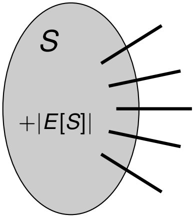
Let $S \subseteq V$. If $\sum_{v \in S} \alpha(v) = |E[S]|$, then all edges in $\delta(S)$ must be oriented away from S.



Directed cuts and rigid edges

Observation

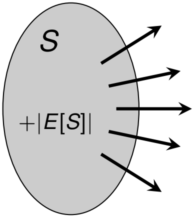
Let $S \subseteq V$. If $\sum_{v \in S} \alpha(v) = |E[S]|$, then all edges in $\delta(S)$ must be oriented away from S.



Directed cuts and rigid edges

Observation

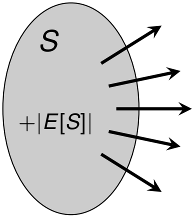
Let $S \subseteq V$. If $\sum_{v \in S} \alpha(v) = |E[S]|$, then all edges in $\delta(S)$ must be oriented away from S.



Directed cuts and rigid edges

Observation

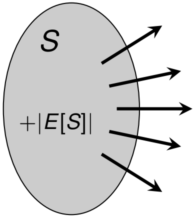
Let $S \subseteq V$. If $\sum_{v \in S} \alpha(v) = |E[S]|$, then all edges in $\delta(S)$ must be oriented away from S.



Definiton

An edge is called **rigid** if $e \in \delta(S)$ for some $S \subseteq V$ with $\sum_{v \in S} \alpha(v) = |E[S]|$. $R := \{e \in E : e \text{ is rigid}\}$.

Directed cuts and rigid edges



Observation

Let $S \subseteq V$. If $\sum_{v \in S} \alpha(v) = |E[S]|$, then all edges in $\delta(S)$ must be oriented away from S.

Definiton

An edge is called **rigid** if $e \in \delta(S)$ for some $S \subseteq V$ with $\sum_{v \in S} \alpha(v) = |E[S]|$. $R := \{e \in E : e \text{ is rigid}\}$.

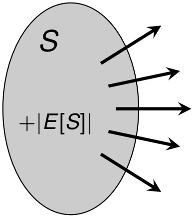
Lemma

If D feasible orientation, then $e \in R$ iff e on directed cut w.r.t. D.

Directed cuts and rigid edges

Observation

Let $S \subseteq V$. If $\sum_{v \in S} \alpha(v) = |E[S]|$, then all edges in $\delta(S)$ must be oriented away from S.



Definiton

An edge is called **rigid** if $e \in \delta(S)$ for some $S \subseteq V$ with $\sum_{v \in S} \alpha(v) = |E[S]|$. $R := \{e \in E : e \text{ is rigid}\}$.

Lemma

If D feasible orientation, then $e \in R$ iff e on directed cut w.r.t. D.

- Same argumentation in dual graph gives set R^*.

Uniqueness of solution in planar embeddings

Theorem

If G is a plane graph and there is a globally feasible orientation D, then $E = R \,\dot{\cup}\, R^*$. Thus, D is the unique solution.

Uniqueness of solution in planar embeddings

Theorem

If G is a plane graph and there is a globally feasible orientation D, then $E = R \dot{\cup} R^*$. Thus, D is the unique solution.

Proof.

- e either on directed cycle or directed cut of G_D

Uniqueness of solution in planar embeddings

Theorem

If G is a plane graph and there is a globally feasible orientation D, then $E = R \dot{\cup} R^*$. Thus, D is the unique solution.

Proof.

- e either on directed cycle or directed cut of G_D
- e on di-cut of $G_D \;\Leftrightarrow\; e \in R$

Uniqueness of solution in planar embeddings

Theorem

If G is a plane graph and there is a globally feasible orientation D, then $E = R \dot{\cup} R^*$. Thus, D is the unique solution.

Proof.

- e either on directed cycle or directed cut of G_D
- e on di-cut of $G_D \Leftrightarrow e \in R$
- e on di-cycle of $G_D \Leftrightarrow e$ on di-cut of $G_D^* \Leftrightarrow e \in R^*$ □

Uniqueness of solution in planar embeddings

Theorem

If G is a plane graph and there is a globally feasible orientation D, then $E = R \,\dot{\cup}\, R^*$. Thus, D is the unique solution.

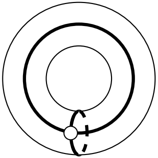
Proof.

- e either on directed cycle or directed cut of G_D
- e on di-cut of $G_D \;\Leftrightarrow\; e \in R$
- e on di-cycle of $G_D \;\Leftrightarrow\; e$ on di-cut of $G_D^* \;\Leftrightarrow\; e \in R^*$ □

Corollary

We can find D in time $\mathcal{O}(|E|^{3/2})$ by computing a feasible orientation in G and G^* and combining their rigid parts.

Outline



2 Bound for general embeddings

Linear algebra for general embeddings

Linear algebra formulation

D: arbitrary orientation $\quad x(e) \in \{0, 1\}$: reverse edge e?

$$\begin{aligned}
\sum_{e \in \delta_D^+(v)} x(e) - \sum_{e \in \delta_D^-(v)} x(e) + |\delta_D^-(v)| &= \alpha(v) \quad \forall v \in V \\
\sum_{e \in \delta_D^+(f)} x(e) - \sum_{e \in \delta_D^-(f)} x(e) + |\delta_D^-(f)| &= \alpha^*(f) \quad \forall f \in V^*
\end{aligned}$$

Linear algebra for general embeddings

Linear algebra formulation

D: arbitrary orientation $\quad x(e) \in \{0,1\}$: reverse edge e?

$$\begin{aligned}
\sum_{e \in \delta_D^+(v)} x(e) - \sum_{e \in \delta_D^-(v)} x(e) &= \alpha(v) - |\delta_D^-(v)| \quad \forall v \in V \\
\sum_{e \in \delta_D^+(f)} x(e) - \sum_{e \in \delta_D^-(f)} x(e) &= \alpha^*(f) - |\delta_D^-(f)| \quad \forall f \in V^*
\end{aligned}$$

Linear algebra for general embeddings

Linear algebra formulation

D: arbitrary orientation $\quad x(e) \in \{0,1\}$: reverse edge e?

$$\begin{aligned}
\sum_{e \in \delta_D^+(v)} x(e) - \sum_{e \in \delta_D^-(v)} x(e) &= \alpha(v) - |\delta_D^-(v)| \quad \forall v \in V \\
\sum_{e \in \delta_D^+(f)} x(e) - \sum_{e \in \delta_D^-(f)} x(e) &= \alpha^*(f) - |\delta_D^-(f)| \quad \forall f \in V^*
\end{aligned}$$

Observation

- rank of the system is $|V| - 1 + |V^*| - 1$

Linear algebra for general embeddings

Linear algebra formulation

D: arbitrary orientation $\quad x(e) \in \{0, 1\}$: reverse edge e?

$$\begin{aligned}
\sum_{e \in \delta_D^+(v)} x(e) - \sum_{e \in \delta_D^-(v)} x(e) &= \alpha(v) - |\delta_D^-(v)| \quad \forall v \in V \\
\sum_{e \in \delta_D^+(f)} x(e) - \sum_{e \in \delta_D^-(f)} x(e) &= \alpha^*(f) - |\delta_D^-(f)| \quad \forall f \in V^*
\end{aligned}$$

Observation

- rank of the system is $|V| - 1 + |V^*| - 1$
- all solutions in space of dimension $|E| - |V| - |V^*| - 2 = 2g$ (Euler's formula)

Bound on the number of solutions

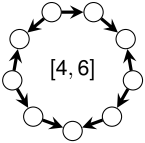
Theorem

- There are at most 2^{2g} feasible orientations.
- All orientations can be found in time $\mathcal{O}(2^{2g}|E|^2 + |E|^3)$.

Remark

The bound on the number of orientations is tight.

Outline



3. Hardness for interval version

Orientations with upper and lower bounds

Bounded primal-dual orientation problem

Input: embedded graph $G = (V, E)$,
$\alpha, \beta : V \to \mathbb{N}_0$, $\alpha^*, \beta^* : V^* \to \mathbb{N}_0$

Task: Is there orientation D, s.t.
$\alpha(v) \leq |\delta_D^-(v)| \leq \beta(v)$ for all $v \in V$ and
$\alpha^*(f) \leq |\delta_D^-(f)| \leq \beta^*(f)$ for all $f \in V^*$?

Orientations with upper and lower bounds

Bounded primal-dual orientation problem

Input: embedded graph $G = (V, E)$,
$\alpha, \beta : V \to \mathbb{N}_0$, $\alpha^*, \beta^* : V^* \to \mathbb{N}_0$

Task: Is there orientation D, s.t.
$\alpha(v) \leq |\delta_D^-(v)| \leq \beta(v)$ for all $v \in V$ and
$\alpha^*(f) \leq |\delta_{\bar{D}}^-(f)| \leq \beta^*(f)$ for all $f \in V^*$?

Theorem

This problem is NP-hard, even for planar embeddings.

Reduction from PLANAR 3-SAT

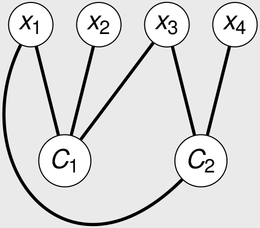
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



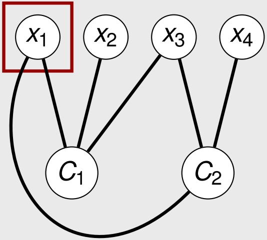
Reduction from PLANAR 3-SAT

PLANAR 3-SAT

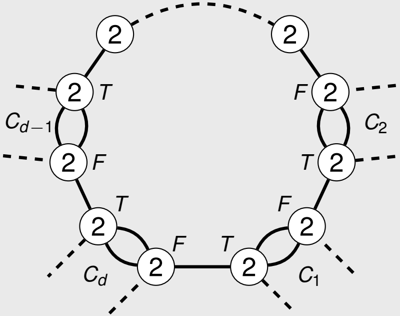
Instance of 3-SAT s.t. the induced bipartite graph is planar.

Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$
$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Variable gadget



Reduction from PLANAR 3-SAT

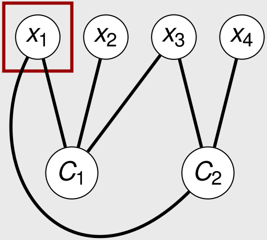
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

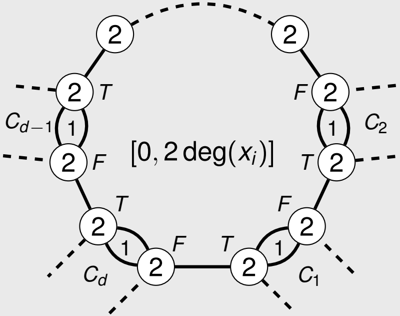
Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Variable gadget



Reduction from PLANAR 3-SAT

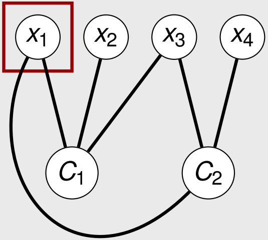
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

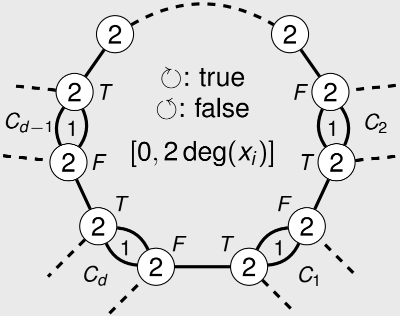
Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Variable gadget



Reduction from PLANAR 3-SAT

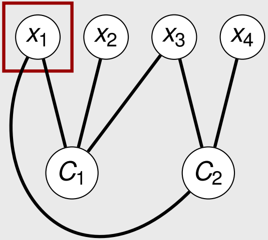
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

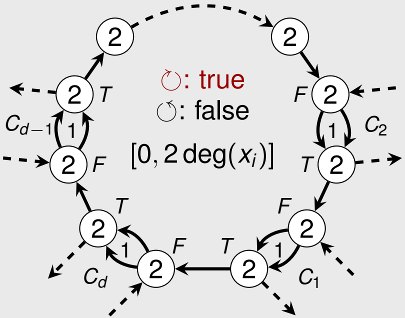
Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Variable gadget



Reduction from PLANAR 3-SAT

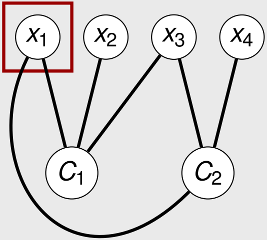
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

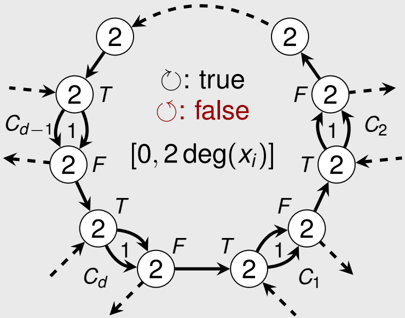
Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Variable gadget



Reduction from PLANAR 3-SAT

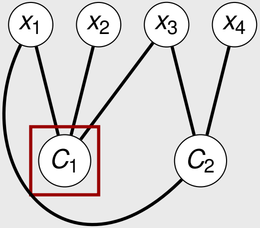
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

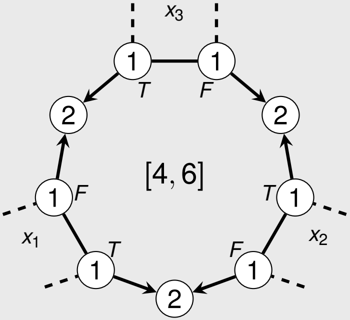
Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Clause gadget



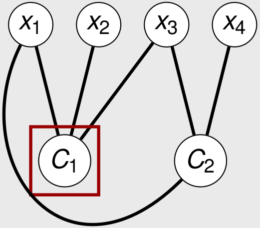
Reduction from PLANAR 3-SAT

PLANAR 3-SAT

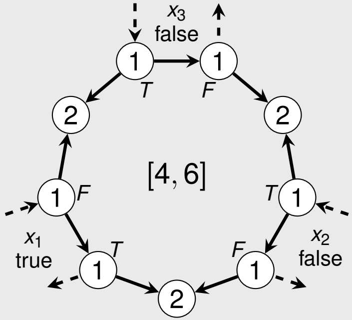
Instance of 3-SAT s.t. the induced bipartite graph is planar.

Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$
$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Clause gadget



Reduction from Planar 3-Sat

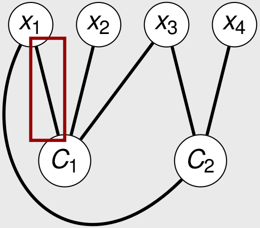
Planar 3-Sat

Instance of 3-Sat s.t. the induced bipartite graph is planar.

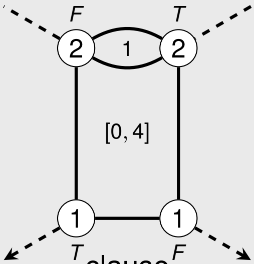
Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Edge gadget



Reduction from PLANAR 3-SAT

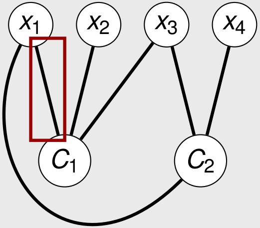
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

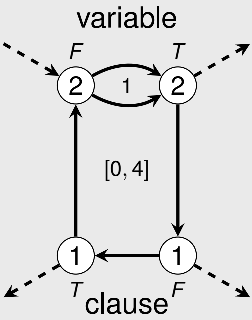
Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Edge gadget



Reduction from PLANAR 3-SAT

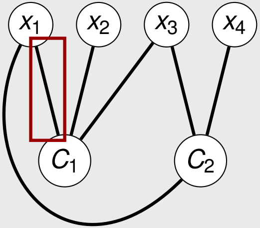
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

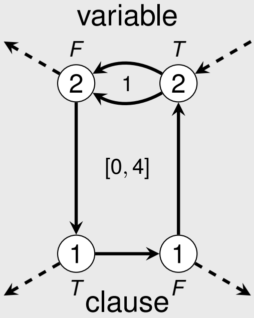
Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Edge gadget



Reduction from PLANAR 3-SAT

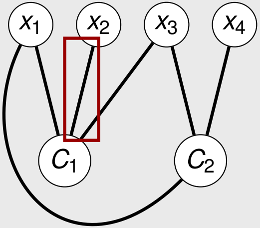
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

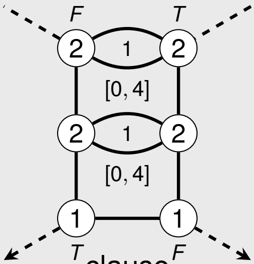
Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$

$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Edge gadget (negated)



Reduction from PLANAR 3-SAT

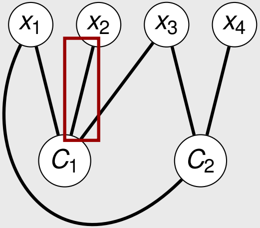
PLANAR 3-SAT

Instance of 3-SAT s.t. the induced bipartite graph is planar.

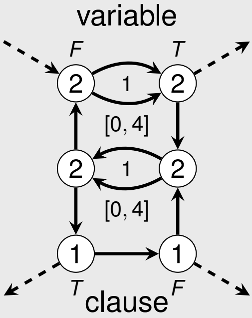
Example

$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$

$C_2 = x_2 \vee x_3 \vee \neg x_4$



Edge gadget (negated)



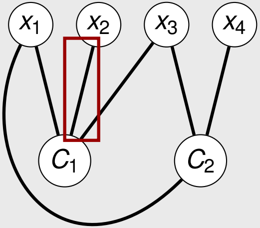
Reduction from PLANAR 3-SAT

PLANAR 3-SAT

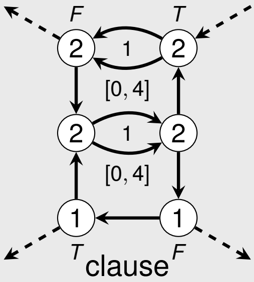
Instance of 3-SAT s.t. the induced bipartite graph is planar.

Example

$$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$$
$$C_2 = x_2 \vee x_3 \vee \neg x_4$$



Edge gadget (negated)



Summary

The primal-dual orientation problem ...

- has at most 2^{2g} solutions if degrees are fixed numbers (enumeration in $\mathcal{O}(2^{2g}|E|^2 + |E|^3)$).
- is NP-hard if only upper and lower bounds are given, even for planar embeddings.

Summary

The primal-dual orientation problem ...

- has at most 2^{2g} solutions if degrees are fixed numbers (enumeration in $\mathcal{O}(2^{2g}|E|^2 + |E|^3)$).
- is NP-hard if only upper and lower bounds are given, even for planar embeddings.

Open question

- Can we find a feasible orientation in time poly$(g, |E|)$?

Summary

The primal-dual orientation problem ...

- has at most 2^{2g} solutions if degrees are fixed numbers (enumeration in $\mathcal{O}(2^{2g}|E|^2 + |E|^3)$).
- is NP-hard if only upper and lower bounds are given, even for planar embeddings.

Open question

- Can we find a feasible orientation in time poly$(g, |E|)$?

Thank you!