Extended Formulations of Stable Set Polytopes via Decomposition

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Aussois, January 2013

History

Grötschel, Lovász, Schrijver 1981

A polynomial time algorithm that computes a stable set of maximum weight in a perfect graph based on the ellipsoid method.

Grötschel, Lovász, Schrijver 1986

A compact SDP-extended formulation for the stable set polytope of perfect graphs.

Chudnovsky, Robertson, Seymour, Thomas 2003

The Strong Perfect Graph Theorem.

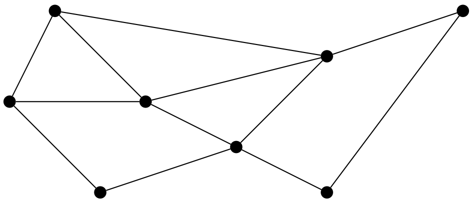
Outline

Stable Set Polytope

Stable Set Polytope

The *stable set polytope* $\mathrm{P}_{\text{stable}}(G) \subseteq \mathbb{R}^E$ of the graph $G = (V, E)$ is defined by

$$\mathrm{P}_{\text{stable}}(G) = \mathrm{conv}(\{\chi(S) : S \text{ is a stable set in } G\}).$$



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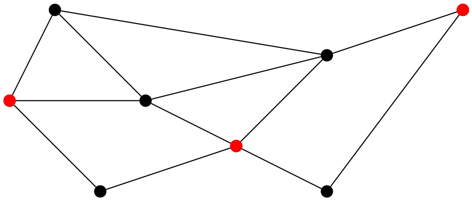


Extended Formulations of Polytopes

Extension

A polyhedron $Q \subseteq \mathbb{R}^d$ and a linear projection $p : \mathbb{R}^d \to \mathbb{R}^m$

- form an *extension* of a polytope $P \subseteq \mathbb{R}^m$ if $P = p(Q)$ holds.
- the *size* of the extension is the number of facets of Q.

Crucial Fact

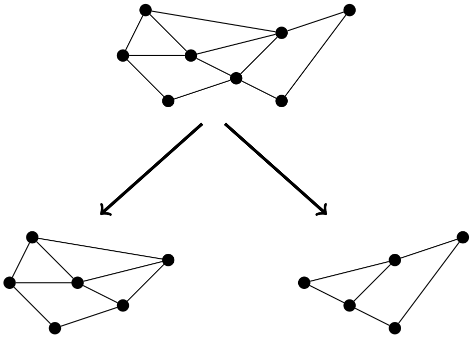
For each $c \in \mathbb{R}^m$, we have

$$\max\{\langle c, x\rangle \ : \ x \in P\} = \max\{\langle T^t c, y\rangle \ : \ y \in Q\}$$

if the linear map $p : \mathbb{R}^d \to \mathbb{R}^m$ is defined as $p(y) = Ty$.

Decomposition

A *decomposition* of an object X is the substitution of X with objects, according to a given decomposition rule R. These objects are the *blocks* of the decomposition of X with R.



Constructing Extended Formulations via Decomposition

Class of Objects

Given a rule R, a class of objects $\mathcal{C}$ and a class of objects $\mathcal{P}$, we say that $\mathcal{C}$ is *decomposable into* $\mathcal{P}$ *with* R if every object in $\mathcal{C}$ can be recursively decomposed with R until all blocks belong to $\mathcal{P}$.

Constructing Extended Formulations via Decomposition

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Extended Formulations via Decomposition

For every object X in $\mathcal{C}$ there exists a compact extended formulation of the polytope $P(X)$ if

- For every object Y in $\mathcal{C}$ which is decomposed by the rule R into objects Y_1, Y_2,..., Y_k with extended formulations for $P(Y_1)$, $P(Y_2)$,..., $P(Y_k)$ of size s_1, s_2,..., s_k there is an extended formulation for the polytope $P(Y)$ of size $s_1 + s_2 + \ldots + s_k$.

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- There is a recursive decomposition of every object in $\mathcal{C}$ by the rule R results into polynomial number of objects in $\mathcal{P}$.

Constructing Extended Formulations via Decomposition

Class of Objects

Given a rule R, a class of objects $\mathcal{C}$ and a class of objects $\mathcal{P}$, we say that $\mathcal{C}$ is *decomposable into $\mathcal{P}$ with R* if every object in $\mathcal{C}$ can be recursively decomposed with R until all blocks belong to $\mathcal{P}$.

Extended Formulations via Decomposition

For every object X in $\mathcal{C}$ there exists a compact extended formulation of the polytope $P(X)$ if

- For every object Y in $\mathcal{C}$ which is decomposed by the rule R into objects Y_1, $Y_2, \dots$, Y_k with extended formulations for $P(Y_1)$, $P(Y_2), \dots$, $P(Y_k)$ of size s_1, $s_2, \dots$, s_k there is an extended formulation for the polytope $P(Y)$ of size $s_1 + s_2 + \dots + s_k$.
- There is a recursive decomposition of every object in $\mathcal{C}$ by the rule R results into polynomial number of objects in $\mathcal{P}$.
- For every object Y in $\mathcal{P}$ there exists a compact extended formulation of the polytope $P(Y)$.

Outline

Clique Cutset

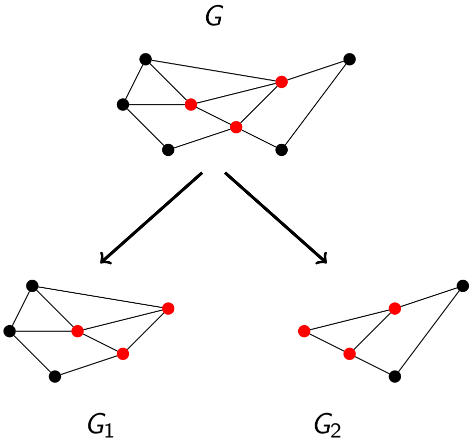
Cutset

A clique $K \subseteq V$ of $G = (V, E)$ is a *clique cutset* if $V \setminus K$ can be partitioned into two nonempty sets V_1 and V_2 such that no node of V_1 is adjacent to V_2.

Clique Cutset Decomposition

The *blocks* of the clique cutset decomposition are the subgraphs G_1 and G_2 of G induced by $V_1 \cup K$ and $V_2 \cup K$, respectively.

Clique Cutset Decomposition



Stable Set Polytope and Clique Cutset

Chvátal 1975

A point lies in $\mathrm{P}_{\text{stable}}(G)$ if only if its restriction to $V_1 \cup K$ lies in $\mathrm{P}_{\text{stable}}(G_1)$ and its restriction to $V_2 \cup K$ lies in $\mathrm{P}_{\text{stable}}(G_2)$.

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Proof

Let x be a point such that its restriction x^1 to $V_1 \cup K$ lies in $\mathrm{P}_{\text{stable}}(G_1)$ and its restriction x^2 to $V_2 \cup K$ lies in $\mathrm{P}_{\text{stable}}(G_2)$. Then,

$$x^i = \sum_{S \in \mathcal{S}(G_i)} \lambda_S^i \chi(S)$$

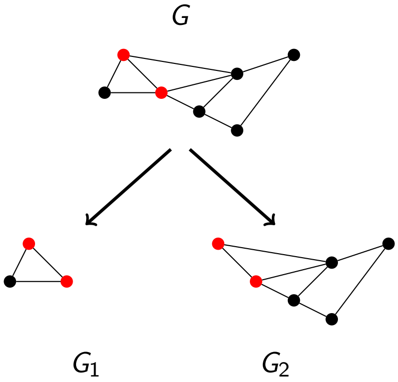
where $\lambda^i \geq 0$, $\sum_{S \in \mathcal{S}(G_i)} \lambda_S^i = 1$. Thus, for every $v \in K$

$$\sum_{\substack{S \in \mathcal{S}(G_1) \\ v \in S}} \lambda_S^1 = \sum_{\substack{S \in \mathcal{S}(G_2) \\ v \in S}} \lambda_S^2 .$$

Clique Cutset Decompositions

Clique Cutset Decomposition

If G has a clique cutset then there exists a clique cutset decomposition of G such that one of the blocks does not have a clique cutset.



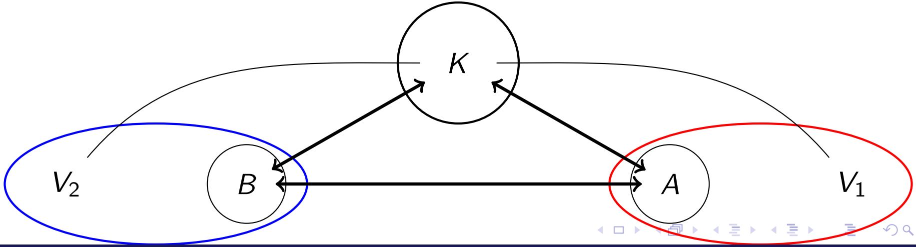
Outline

Amalgam

Amalgam

Triple (A, K, B) is an *amalgam* of a graph $G = (V, E)$ if V can be partitioned into V_1, V_2 and K such that $|V_1| \geq 2$ and $|V_2| \geq 2$ and K is a (possibly empty) clique. V_1 and V_2 contain nonempty subsets A and B such that:

- K is universal to A and B
- A and B are universal
- $V_1 \backslash A$ and $V_2 \backslash B$ are nonadjacent.

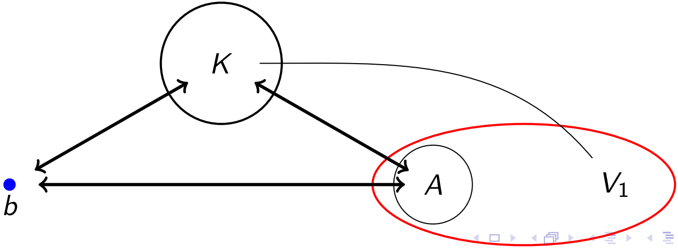


Amalgam Decomposition

Blocks of Amalgam Decomposition

The *blocks* of the amalgam decomposition of G with (A, K, B) are

- the graph obtained by adding a new node b to the subgraph of G induced by $V_1 \cup K$ and adding edges from b to each of the nodes in $K \cup A$
- the graph obtained by adding a new node a to the subgraph of G induced by $V_2 \cup K$ and adding edges from a to each of the nodes in $K \cup B$.



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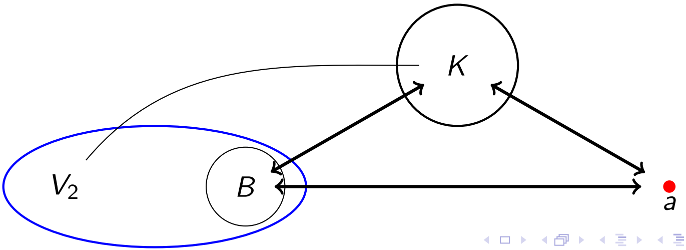
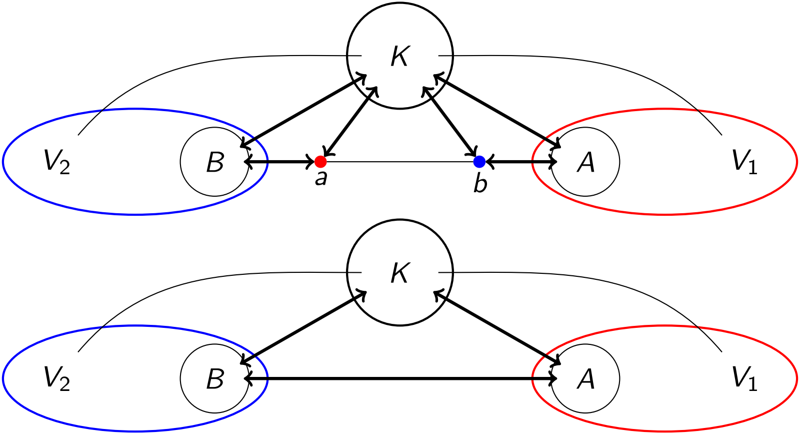


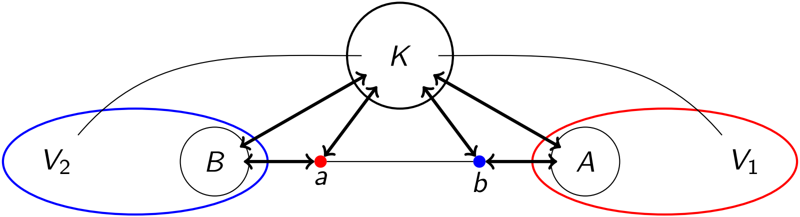
Extended Formulation via Amalgam Decomposition



Extended Formulation via Amalgam Decomposition

Conforti, Gerards, P. 2012

A point lies in $P_{\text{stable}}(G)$ if and only if it can be extended by x_a, x_b such that its restriction to $V_1 \cup K \cup \{a\}$ lies in $P_{\text{stable}}(G_1)$ and its restriction to $V_2 \cup K \cup \{b\}$ lies in $P_{\text{stable}}(G_2)$ and $x_a + x_b + \sum_{v \in K} x_v = 1$.



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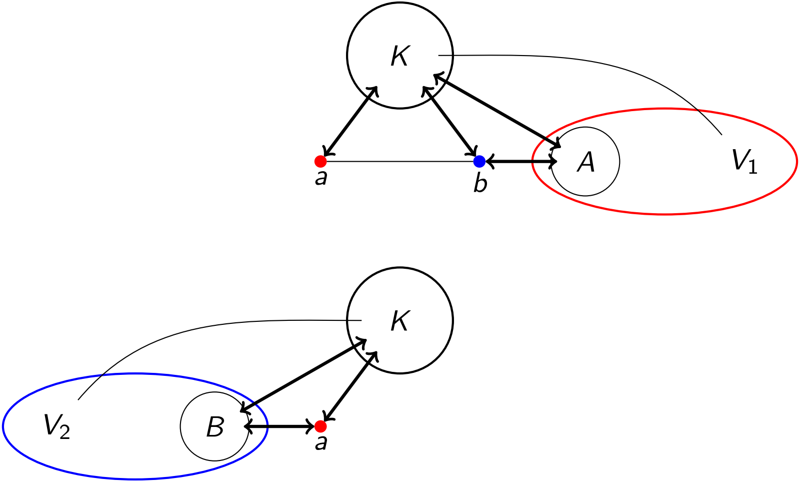

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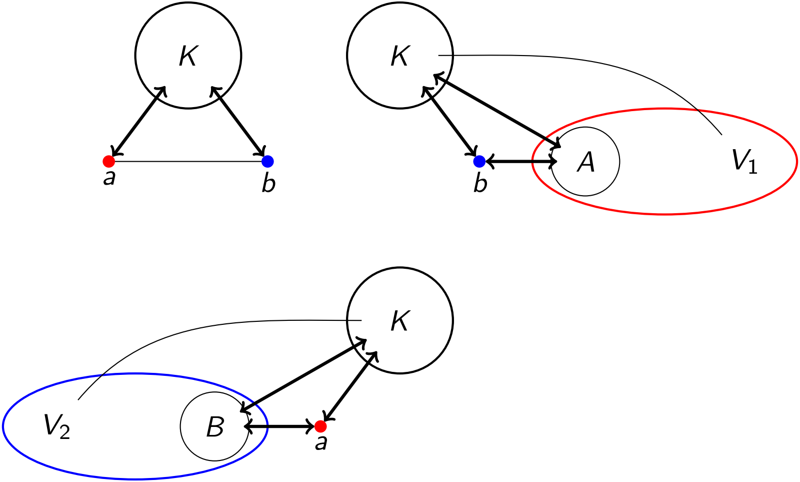

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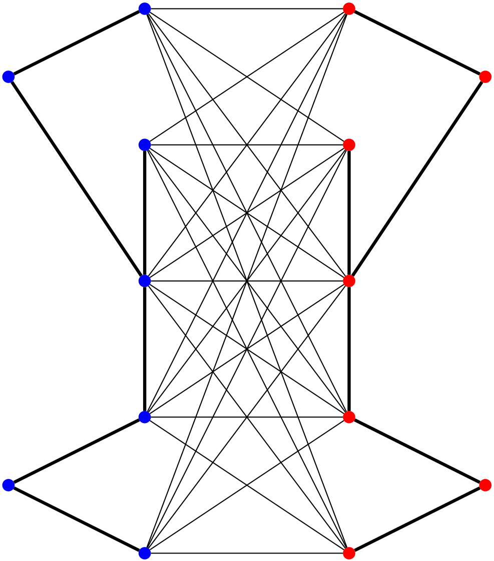

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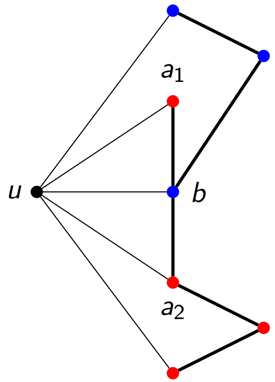


Conforti, Gerards, P. 2012

Every recursive amalgam decomposition results into polynomial number of graphs without an amalgam and polynomial number of cliques (no clique is decomposed during the recursion).

Outline

Cutset Decomposition

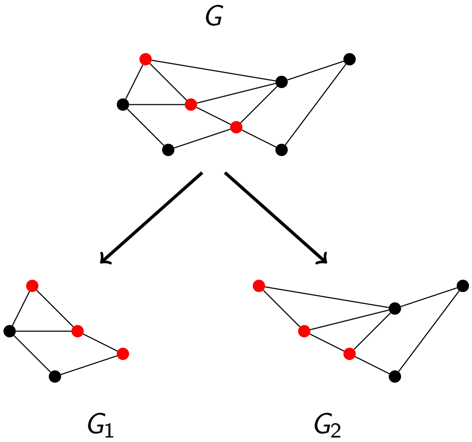
Cutset

A vertex set $K \subseteq V$ of $G = (V, E)$ is a *cutset* if $V \setminus K$ can be partitioned into two nonempty sets V_1 and V_2 such that no node of V_1 is adjacent to V_2.

Cutset Decomposition

The *blocks* of the cutset decomposition are the subgraphs G_1 and G_2 of G induced by $V_1 \cup K$ and $V_2 \cup K$, respectively.

Stable Set Polytope and Cutset



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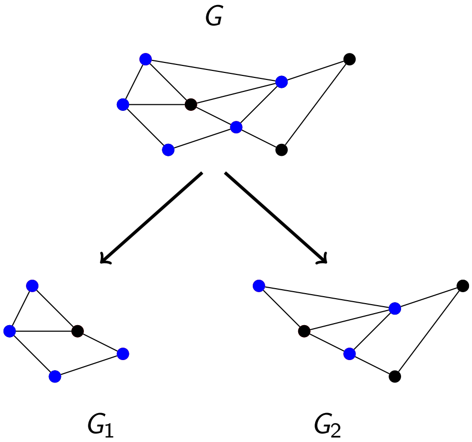


Template

Template

A *template* is a pair $(G, \mathcal{X})$ where $G = (V, E)$ is a graph and $\mathcal{X} = \{X_1, \ldots, X_k\}$ is a collection of subsets of V.

Template Decomposition

A node set K *decomposes the template* $(G, \mathcal{X})$ if $V \backslash K$ can be partitioned into nonempty subsets V_1, V_2 such that

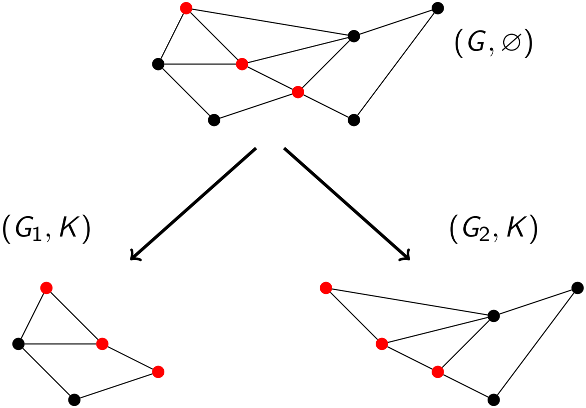
- no edge of G connects V_1 and V_2
- every set in $\mathcal{X}$ is a subset of $V_1 \cup K$ or of $V_2 \cup K$.

Blocks of Template Decomposition

The *blocks* of decompostion are $(G_1, \mathcal{X}_1)$ and $(G_2, \mathcal{X}_2)$ where

- $\mathcal{X}_i$ consists of K together with the members of $\mathcal{X}$ that are contained in $V_i \cup K$
- G_i is the subgraph of G induced by $V_i \cup K$.

Templates



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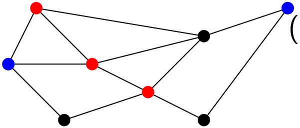
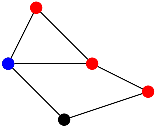
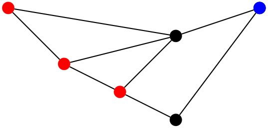


$(G, \{X\})$





Decomposition of Templates: Polytopes

Template Polytope

The polytope $P^*_{\text{stable}}(G, \mathcal{X})$ of the template $(G, \mathcal{X})$ is defined by

$$P^*_{\text{stable}}(G, \mathcal{X}) = \text{conv}(\{\chi^*(S) : S \text{ is a stable set in } G\}).$$

where $\chi^*(S)$ is a zero-one vector which has a coordinate for each pair of a set X in the collection $\mathcal{X} \cup V$ and a nonempty stable set $Z \subseteq X$ of G; and this coordinate equals one if and only if $S \cap X$ equals Z.

Conforti, Gerards, P. 2012

Let a template $(G, \mathcal{X})$ be decomposed by K into templates $(G_1, \mathcal{X}_1)$ and $(G_2, \mathcal{X}_2)$.

Then a point lies in $P^*_{\text{stable}}(G, \mathcal{X} \cup \{K\})$ if and only if its restriction to the variables of $P^*_{\text{stable}}(G_i, \mathcal{X}_i)$ lies in $P^*_{\text{stable}}(G_i, \mathcal{X}_i)$.

Outline

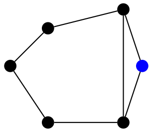
Cap-Free Odd-Signable Graphs

Odd-Signable

A graph is *odd-signable* if it contains a subset of the edges that meets every chordless cycle (i.e. every triangle and hole) an odd number of times.

Cap

A *cap* is a hole together with a node adjacent to exactly two adjacent nodes on the hole.



Conforti, Gerards, P. 2012

For every cap-free odd-signable graph the stable set polytope admits a compact extended formulation.

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Conforti, Cornuéjols, Kapoor, Vušković 1999

For every cap-free graph one of the following holds:

- The graph contains an amalgam.
- The graph is triangulated.
- The graph is biconnected triangle-free with at most one node which is universal to all other nodes.

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Dirac 1961, Hajnal, Suryani 1958

For every triangulated graph one of the following holds:

- The graph contains a clique cutset.
- The graph is a clique.

Conforti, Cornuéjols, Kapoor, Vušković 1996

For every triangle-free odd-signable graph containing a cube one of the following is true:

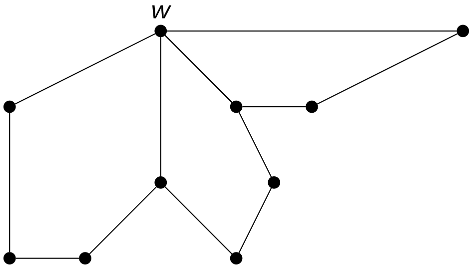
- The graph contains a clique cutset.
- The graph is a cube.

Conforti, Cornuéjols, Kapoor, Vušković 1996

For every triangle-free odd-signable graph G, containing no cube as an induced subgraph, one of the following is true:

- The graph has a clique cutset.
- The template $(G, \varnothing)$ can be recursively decomposed into basic templates $(G_1, \mathcal{X}_1)$, $(G_2, \mathcal{X}_2)$, $\ldots$, $(G_k, \mathcal{X}_k)$.

Basic Templates

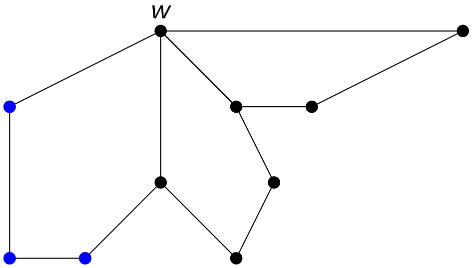


Basic Templates

A template $(G, \mathcal{X})$ is *basic* if it satisfies the conditions below:

- The graph G is a fan.
- Each set in $\mathcal{X}$ is a triple, and all of these triples except (possibly) one consist of a vertex and two its neigbours in one of the sectors of the fan G.

Basic Templates



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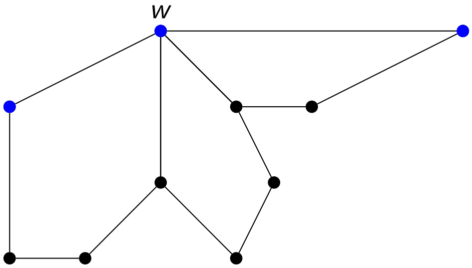

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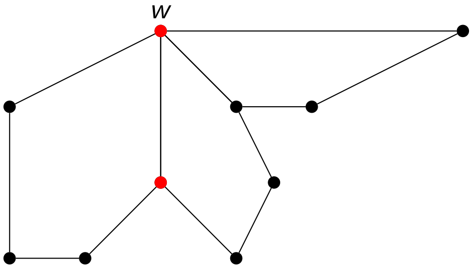

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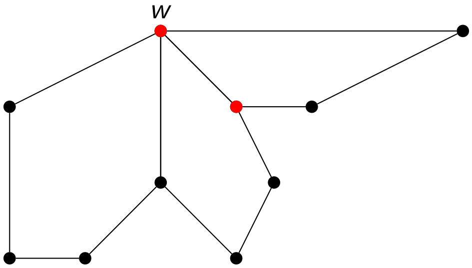

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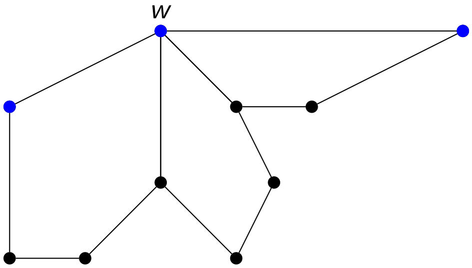

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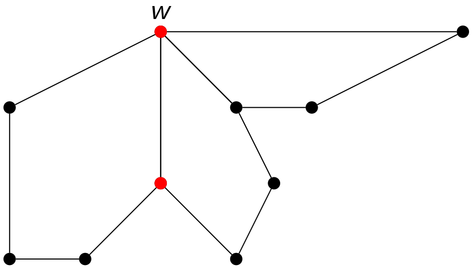
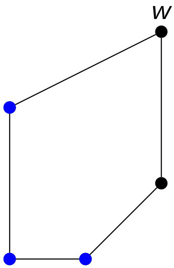


End of the Proof

To finish the proof we have to provide a compact extended formulation of $P^*_{\text{stable}}(G, \mathcal{X})$ where $(G, \mathcal{X})$ is basic and G is a hole.

Basic Templates



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Thank you!