Graph Orientation and Network Flows Over Time

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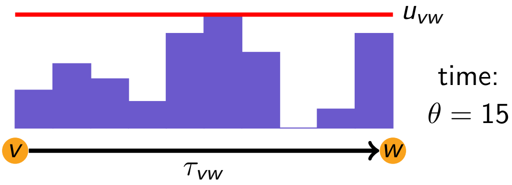
Flows Over Time

Given:

- digraph $D = (V, A)$,
- capacities u_e,
- transit times τ_e
- time horizon T.



Definition.

A flow over time with time horizon T is a family of functions

$$f_e : \{1, \ldots, T\} \to \mathbb{R}_{\geq 0}, \qquad \text{for } e \in A,$$

such that

- $f_e(\theta) \leq u_e$ for all e, θ (capacity constraints),
- flow conservation constraints hold.

Maximum s-t-Flows and Transshipments Over Time

Algorithm (Ford & Fulkerson 1958).

1. compute static min-cost s-t-flow x in D;
2. decompose x into flows x_P on s-t-paths P;
3. send flow at constant rate x_P into paths P over time;

Theorem (Ford & Fulkerson 1958).

This temporally repeated flow is a maximum s-t-flow over time.

Transshipment problem: Several sources/sinks with supplies/demands.

Theorem (Hoppe & Tardos 2000).

The transshipment over time problem can be solved in polynomial time.

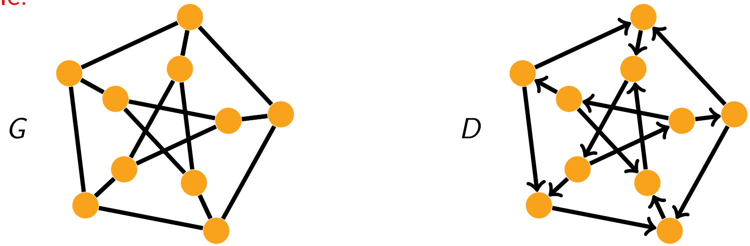
Orientations of Undirected Graphs

$G = (V, E)$ undirected graph, $D = (V, A)$ digraph.

Definition.

D is an orientation of G if G is the underlying undirected graph of D.

Example:



Theorem (Robbins 1939).

G has a strongly connected orientation $\iff$ G is 2-edge connected.

Nash-Williams' Orientation Theorem

Corollary.

G has k-arc-connected orientation $\iff$ G is $2k$-edge connected.

Notation: For $s, t \in V$

$$\lambda_G(s,t) := \# \text{ edge-disjoint } s\text{-}t\text{-paths in } G,$$
$$\lambda_D(s,t) := \# \text{ arc-disjoint } s\text{-}t\text{-paths in } D.$$

Theorem (Nash-Williams 1960).

G has an orientation D with

$$\lambda_D(s,t) \geq \lfloor \tfrac{1}{2}\lambda_G(s,t) \rfloor \qquad \text{for all } s,t \in V.$$

Bottom line: Orienting costs factor $\frac{1}{2}$ in connectivity (up to rounding).

Static Network Flows in Undirected Graphs

Let $b \in \mathbb{R}^V$ with $\sum_{v \in V} b_v = 0$,

$$S^+ := \{v \in V \mid b_v > 0\} \qquad \text{(sources)},$$

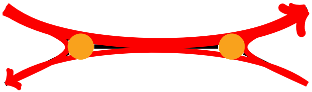
$$S^- := \{v \in V \mid b_v < 0\} \qquad \text{(sinks)}.$$

Definition.

$x : E \to \mathbb{R}$ is a *b-transshipment in G* if there is an orientation D of G such that x is a b-transshipment in D.

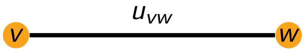
Remarks.

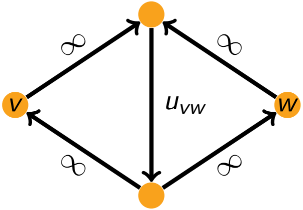
- By definition, each edge is only used in one direction.
- Using edges in both directions is unnecessary (flow cancelation).



Reduction to Digraphs

Reduce b-transshipment problem in G to a directed graph:



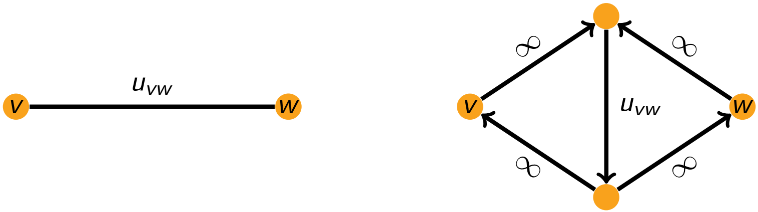


Remarks.

- Directed and undirected case "equivalent" for single-commodity flows.
- But: Undirected case often easier for multiple commodities, e. g.,
  - fixed number of edge/arc-disjoint paths,
  - Hu's 2-commodity flow theorem.

Network Flows Over Time in Undirected Graphs

Define by reduction to directed graph:



Notice: Edge might be used in different directions over time!

Observation.

There is a maximum s-t-flow over time such that each edge is only used in one direction.

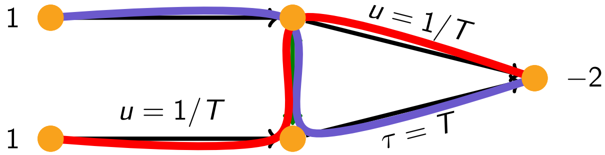
Proof: Temporally repeated flow derived from static min-cost flow. □

Transshipment Over Time

Theorem (Kim & Shekhar 2005; Rebennack et al. 2010).

It is NP-complete to decide if there is a b-transshipment over time where each edge is only used in one direction.

Example: $u_e = \infty$ and $\tau_e = 0$, unless stated otherwise:



The "Price of Orienting"

Nash-Williams' theorem: Orienting costs factor $\frac{1}{2}$ in connectivity.

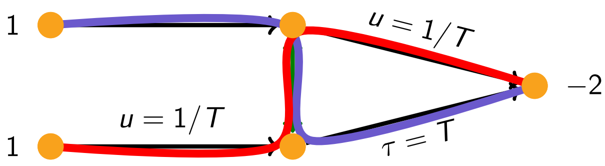
Question: What is the "price of orienting" for transshipments over time?

Definition ("Price of Orienting").

Suppose there is a b-transshipment over time in G.

What fraction of total supply/demand can be satisfied in orientation of G?

Example:



- Price of orienting is no better than factor $\frac{1}{2}$.
- No positive factor for concurrent problem possible.

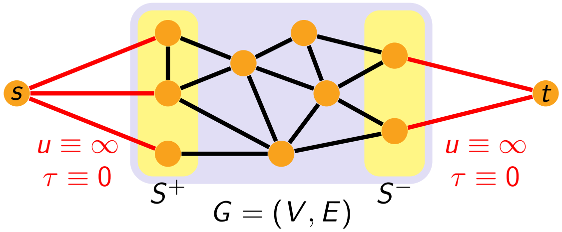
Main Result

Theorem (Arulselvan, Groß, Sk. 2013).

Price of orienting for transshipments over time is no worse than factor $\frac{1}{3}$.

Proof idea:

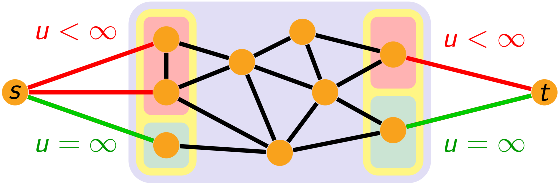
Find temporally repeated flow with Ford-Fulkerson algorithm:



Problem: Flow through nodes in S^+ or S^- can exceed supply/demand.

Solution idea: Introduce finite capacities on red edges.

How to Find Appropriate Capacities



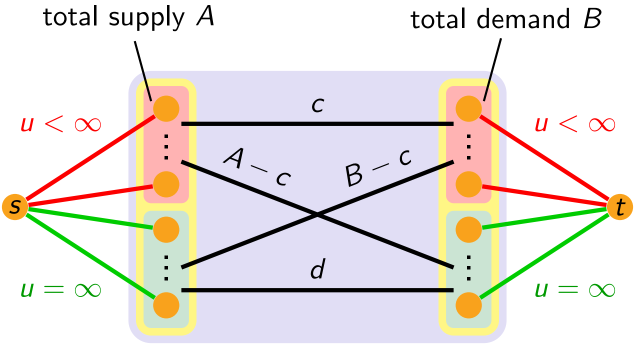
Key Lemma.

There are capacities $u_e \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ for the red edges $e \in \delta(s) \cup \delta(t)$ and a maximum s-t-flow over time f such that

- the amount of flow sent through a node $v \in S^+ \cup S^-$ is at most $|b_v|$;
- if strictly less than $|b_v|$ units of flow go through node $v \in S^+ \cup S^-$, then $u_e = \infty$ for the corresponding edge $e = sv$ or $e = vt$.

Proof idea: Brouwer's fixed-point theorem. □

Proof of Main Theorem



Notice that $c \leq \min\{A, B\} \leq \frac{1}{2}(A + B)$

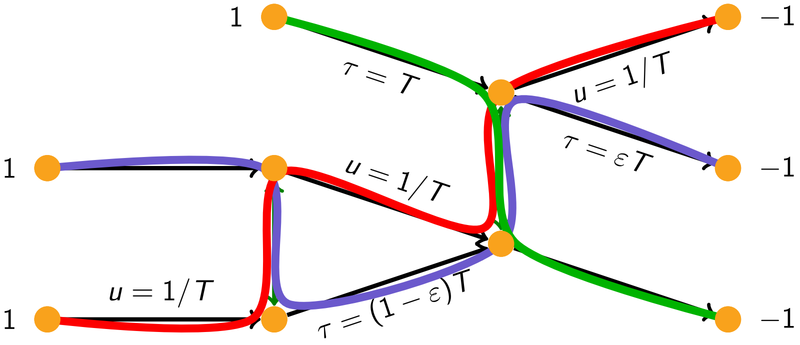
Value of maximum s-t-flow over time: $M = A + B - c + d$

Let S be the sum of all supplies, then: $M \geq S - A - B$

$$\implies \quad 3 \cdot M \;\geq\; 2 \cdot (A + B - c) + S - A - B \;\geq\; S$$

$\square$

Instance with Price of Orienting 1/3



Thus, price of orienting for transshipments over time is exactly $\frac{1}{3}$.

Conclusion, Further Results, and Open Problems

- Price of orienting for transshipments over time is exactly $\frac{1}{3}$.
- Only existence proof – most probably there is an algorithm!
- For single source or sink case the factor can be improved to $\frac{1}{2}$ (tight!).
- NP-complete to decide if factor $\frac{1}{2} + \varepsilon$ is possible for given instance.
- Question: Is it possible to satisfy all supplies/demands in an orientation by increasing the time horizon by some small factor?
- Earliest arrival s-t-flows use edges in both directions.
- Price of orienting is unbounded for earliest arrival s-t-flows.

Bad Instance for Earliest Arrival s-t-Flows