# MATHEMATICS:
# DISCOVERED OR CONSTRUCTED?

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The talk was given during the 17th Combinatorial Workshop 2013 in Aussois, France. Since the slides by themselves are not sufficient to convey the content of the presentation, we have added the verbal comments from memory, using a smaller font as done here. We also have revised some slides to clarify points and correct errors, and have included some material that could not be discussed due to the one-hour time limit for the talk.

# Mathematics: Discovered or Constructed?

In 1978, J. Edmonds arranged that we could visit the University of Waterloo for one year. It was a terrific opportunity.

During the day, we would work on mathematical problems at the university. At night when we would be too tired to continue, we started to study the writings of the philosopher L. Wittgenstein. Our uncle F. Hülster had done so years earlier and had written up his understanding of the books *Tractatus Logico-Philosophicus*, 1921, and *Philosophische Untersuchungen*, 1953, in two enlightening manuscripts.

In 1980, back at our home institution, the University of Texas at Dallas, the topic of this talk came up in an informal discussion with one of our Ph.D. students. He argued that mathematics was discovered, with a reasoning that did not seem convincing. We made up our mind that we should look into the matter and study what others had said about the question.

## Mathematics: Discovered or Constructed?

We use "discovered" in the sense that an existing thing is detected, and "constructed" in the sense that a previously not existing thing is produced. Here are some examples.

Discovered:

- Antarctica
- DNA double helix

Constructed:

- Water pump of Archimedes
- Constitution of the USA

# Mathematics: Discovered or Constructed?

Let us look at some questions concerning discovery and construction.

Specific questions:

1. Are mathematical theorems discovered or constructed?
2. Are the complex numbers discovered or constructed?
3. Are the natural numbers discovered or constructed?

If we want to argue for discovery, then the task becomes progressively easier as we proceed from case 1 to case 3. Indeed, the items claimed to be discovered become simpler as we proceed down the list. On the other hand, proving construction becomes progressively harder.

## Mathematics: Discovered or Constructed?

General question:

> Are at least some parts of mathematics discovered, or is all of mathematics constructed?

Compared with the previous questions, discovery is easiest to argue, and construction is hardest.

No matter what the view, there is general agreement:

> Counting and the natural numbers are the start of arithmetic. Indeed, they can be viewed as the beginning of mathematics.

There is a subtle point, though. Before we can count things such as apples, we must implicitly have the notion of equivalence classes. It is safe to assume that mankind learned that notion by evolution. For example, the people who didn't recognize an approaching lion as dangerous if they had not seen that particular animal before, were simply eaten.

Counting with the fingers and toes also implicitly requires the notion of equivalence classes.

## General agreement: Two examples

R. Dedekind (*Stetigkeit und irrationale Zahlen*, 1872), a representative of the Axiomatic Method: "I view all of Arithmetic to be a necessary or at least natural result of the most elementary act, the counting. And counting is nothing but the successive creation of an infinite sequence of the positive whole numbers, in which each individual is defined from its immediate predecessor."

H. Weyl (*Über die neue Grundlagenkrise der Mathematik*, 1921), a representative of Intuitionism: "Mathematics starts with the series of natural numbers, that is, the law that from nothing creates the number 1 and from each already created number the successor."

The Axiomatic Method and Intuitionism are two opposing philosophies about mathematics. Oversimplifying, one might say that Intuitionism does not allow use of an element of a set based solely on a proof that the set is nonempty. The Axiomatic Method has no compunction about such use.

C. Thiel of the University of Erlangen/Nürnberg pointed out to us that both statements contain some vagueness. Dedekind does not specify who does the counting, and Weyl does not say where the cited law comes from. Thiel says that both are "hedging their bets."

# A wide range of opinions about discovery vs. construction

Over at least 2,400 years, a wide variety of opinions have been offered about discovery versus construction.

Plato (424–348 BC, quoted by Euthydemos 290 BC, translation by C. Thiel): "Geometers, arithmeticians, and astronomers are in a sense seekers since they do not produce their figures and other symbols at will, but only explore what is already there."

Clearly Plato is in favor of discovery.

The ideas of Plato dominated philosophical thought for centuries. Instead of examining examples so influenced, we jump right up to the 19th century.

C.F. Gauss (letter to Bessel, 1830): "...number is purely a product of our minds ..."

Now here is a contrast. Since Gauss declares numbers to be constructed, he evidently believes that all of mathematics is obtained that way. Gauss was a deeply religious person, so his viewpoint about mathematics may seem surprising. But he felt that he could not draw any conclusion about the mysteries of his belief, and carefully separated facts and arguments about mathematics and the world from his religious beliefs.

## A wide range of opinions about discovery vs. construction (cont'd)

G. Cantor (*Über unendliche, lineare Punktmannigfaltigkeiten*, 1883): "...
the essence of mathematics just consists of its freedom."

Cantor caused an upheaval in mathematics by his investigation into infinite sets and the intro-
duction of transfinite numbers. His friend J. P. G. L. Dirichlet reviewed Cantor's papers prior
to submission to a journal and advised him that publication would be detrimental to his career.
Cantor went ahead anyway. Dirichlet's prediction turned out to be correct, since Cantor never
attained a position at a major university, something he surely deserved.

L. Kronecker (quoted by Weber, 1893): "God made the integers; all else is
the work of man."

Kronecker was a finitist, that is, he only accepted an existence proof if it involved a finite construc-
tion starting with the natural numbers. For consistency, he could not advocate the construction of
the entire set of integers by humans. Instead, he invoked a religious entity for the task.

## A wide range of opinions about discovery vs. construction (cont'd)

R. Dedekind (*Stetigkeit und irrationale Zahlen*, 1872): "... the negative and rational numbers are created by man ..."

We have already seen another citation of the paper, where counting creates the natural numbers and it is not stated who does the counting. In the present case, creation by man is clearly asserted.

The section of the paper that defines the (Dedekind) cut is titled "Creation of the irrational numbers." Dedekind emphasizes that this creation process is an axiom.

The (Dedekind) cut involves splitting the ordered rational numbers into two disjoint sets where all numbers in one set are smaller than the numbers in the other one. Dedekind then postulates that a number lies between the two sets. Carrying out this operation in all possible ways, he creates the real numbers.

This process is unacceptable to an intuitionist since the mere fact that the two sets are disjoint is used to claim existence of a number between them. Worse yet, this is done in one fell swoop for all ways of splitting up the ordered rationals, with no prescription how this is to be accomplished.

## A wide range of opinions about discovery vs. construction (cont'd)

Here is a statement by B. Russell we read many years ago. We have not been able to find the reference, despite Google. If anybody has that information, please send it to us.

B. Russell: "I cannot imagine a world where $1+1$ is not equal to 2."

Russell's statement baffled us for many years. It seemed impossible to find a contradiction to the implied claim that the first counting step is part of every world. We will return to the statement later.

Next are two books. The first one implicitly votes for construction of the complex numbers, and the second one for discovery not just of the natural numbers, but also of the zero. The latter number is essential for implementation of the idea of place value, a powerful idea.

A. Hodges (book *One to Nine*, 2007): "If complex numbers are so intrinsic to reality, why are we unaware of them?"

C. Reid (book *From Zero to Infinity*, 1956): The Zero is the "first of the numbers, was the last to be discovered."

## A wide range of opinions about discovery vs. construction (cont'd)

There are also attempts to start mathematics by some notation or mechanism in nature. Here is one.

P. Lorenzen (*Wie ist Philosophie der Mathematik möglich?*, 1975):
Construction as start of mathematics. Numbers are created by repeated "/"s. The number $n$ is then defined by $n$ "/"s.

Suppose a bird taking off from a beach hops three times before being in the air, thus leaving three scratches in the sand. Do these scratches represent the number three? No! But if somebody comes and counts them, "One, two, three," then indeed that person can view the scratches as a representation of the number 3.

Problem: We must understand the number $n$ before we can create $n$ "/"s.

Thus, the definition of the number $n$ by $n$ "/"s requires prior definition of the number $n$, a case of circular definition.

# A wide range of opinions about discovery vs. construction (cont'd)

The next opinion by the eminent physicist and mathematician R. Penrose brings us right back to Plato.

R. Penrose (book *The Road to Reality*, 2004): Objective existence of Platonic world of mathematical forms. For example, Mandelbrot set exists in that world, and existed prior to Mandelbrot displaying it. On the other hand, Penrose is not sure whether the Axiom of Choice exists in that world. A drawing shows the relationships connecting the Physical World, Mental World (= world of perceptions), and Platonic Mathematical World.

P. Erdös (lecture, 1985): "You don't have to believe in God, but you should believe in 'The Book'."

Erdös was an atheist, so the statement is facetious. But lurking in it is the belief that somewhere, somehow, there are best proofs of theorems.

## A wide range of opinions about discovery vs. construction (cont'd)

We digress for the moment to an area outside mathematics. In fact to Physics, where one would think that all is well with respect to philosophical foundation. But there we find the following statement.

S. Hawking and L. Mlodinov (book *The Grand Design*, 2010): "Our brains interpret the input from our sensory organs by making a model of the world." Result is "model-dependent realism."

Model-dependent realism declares that we do not know what is actually happening and that, lacking that knowledge, we simply act as if our model is the correct representation.

Now and then we find a statement about the basis of mathematics that is incomprehensible to us. Here is an example.

O. Becker (*Größe und Grenze der mathematischen Denkweise*, 1959): "We can and must count and calculate, since we are temporary and finite creatures. An eternal infinite creature does not count. It need not count, indeed cannot count. The activity of counting and computing would make no sense for it."

## A wide range of opinions about discovery vs. construction (cont'd)

During a symposium in Berlin in 2012, E. Knobloch of the Technical University Berlin gave several lectures on the history of mathematics. We thought it an opportunity to ask him about the discovery versus construction question.

E. Knobloch (discussion, 2012): "Of course, mathematics is man-made. Nine out of ten mathematicians agree with this answer."

The second part, "Nine out of ten mathematicians agree with this answer," surprised us. In the decades since 1980, when we became interested in the question, we have posed the question a number of times to mathematicians, each time getting the answer that at least some part of mathematics is discovered, if not all of it. We mentioned this to Knobloch, who repeated his statement with conviction, adding "Just read the references."

## Some Arguments for Discovery of Mathematics

**A.** Mathematics is used lots of times in the world.

This is so for humans, but also extends to a number of animals.

Two examples:

> R. Rugani et al. (*Arithmetic in newborn chicks*, 2009): Very young chicks of a few days old have rudimentary counting skills. In a series of experiments, they were made to imprint on plastic balls and could figure out which of two groups of balls hidden behind screens had the most balls.

> M. Tenneson (Scientific American, *More Animals Seem to Have Some Ability to Count*, 2009): Bees, robins, parrots, monkeys can count, some up to about 12. Also some can add and subtract.

**B.** Counting is a natural, unavoidable process.

**C.** A theorem exists regardless of whether somebody states it or proves it.

Suppose we have just proved a theorem. Then the statement will be a theorem from now on. But since it was proved using eternally valid logic, the statement must have been a theorem before we established that fact, indeed since time immemorial. This conclusion seems irrefutable.

**Counterarguments**

Discovery Claim A: Use of mathematics happens lots of times in nature.

Claim A is correct.

Discovery Claim B: Counting is a natural, unavoidable process.

Will show that Claim B is not correct.

For counterarguments to Claim B, we review research covering the Parahã tribe of Brazil's Amazon region.

D. Everett (books *Language - The Cultural Tool*, 2012, and *Don't Sleep, There Are Snakes*, 2008). See also J. Colapinto (The New Yorker, *The Interpreter: Has a remote Amazonian tribe upended our understanding of language?*, 2007). Most statements below are direct citations.

## Counterarguments (cont'd)

Pirahã tribe of Amazon region, size 400–600.

D. Everett learned the Pirahã language over a period of 30 years, and likely is the only outsider who has fully mastered the language.

The Pirahã language uses 8 consonants and 3 vowels, one of the simplest sound systems known. There is no plural.

Since there are so few consonants and vowels, words have to be either long or differentiated by further means. The Pirahã language relies on the second option.

Differentiation by tones, stresses, and syllable length. The complexity of these modifications is such that speakers can dispense with vowels and consonants altogether and sing, hum or whistle conversations.

## Counterarguments (cont'd)

Indeed, there are five types of speech:

1. Regular speech: Uses consonants and vowels. Like a few other Amazonian tongues, there are male and female versions of spoken language: the women use one fewer consonant than the men do.

2. Hum speech: for privacy, like whispering in English. Can disguise one's identity. Mothers use it with children, or when mouth is full.

3. Yell speech: used during loud rain and thunder. Vowel "a" is used almost exclusively, and one or two consonants. Used to communicate over long distances, for example, across a wide river.

4. Musical speech: exaggerates changes of pitch and changing of rhythm of words and phrases.

5. Whistle speech: only used by males; for example, while hunting.

For hunting, the whistling of language is an extraordinary advantage.

A thought: Could it be that the songs of whales and the clicking sequences of dolphins have far richer meaning than we give these animals credit for?

## Counterarguments (cont'd)

The Pirahã consider all forms of human discourse other than their own to be laughably inferior, and they are unique among Amazonian peoples in remaining monolingual.

Pirahã terms:

"Straight head" = Pirahã language

"Crooked head" = any other language, derogatory expression

The Pirahã have no numbers, no fixed color terms, no perfect tense, no deep memory, no tradition of art or drawing, and no words for "all," "each," "every," "most," or "few"—terms of quantification believed by some linguists to be among the common building blocks of human cognition.

# Digression: Language and recursion

We digress for a moment to discuss aspects of linguistics. N. Chomsky is a famous linguist. Starting in the late 1950s, he created a theory that involves a universal grammar and recursion.

N. Chomsky: Recursion is the cornerstone of all languages. Indeed, a language not using recursion cannot produce complex utterances of infinitely varied meaning.

An example illustrates recursion. We begin with the sentence "The man sat on the chair." Then we expand this by recursion to "The man, who wore a hat, sat on the chair." Then "The man, who wore a hat, sat on the red chair."

D. Everett: The Pirahã language displays no evidence of recursion and thus is a counterexample.

Among the evidence cited by Everett is a Pirahã story concerning a panther. The sentences are very short statements involving no recursion. Yet, they convey the story. In a moment we will see how this is possible.

There has been a heated discussion between Chomsky and researchers supporting Chomsky's conclusions on one side and Everett on the other. How will this ultimately be resolved? Who knows. But viewing the controversy from a different vantage point, there are indications that recursion may not be mandatory for rich human communication. Let's look at some arguments supporting that heretical thought.

## Language and recursion (cont'd)

L. Wittgenstein (book *Philosophische Untersuchungen*, 1953): Language Games.

Wittgenstein invented language games to investigate the role of words and their meaning in a laboratory-like environment. Here, we cannot do justice to this concept and the profound way Wittgenstein used it, and only state that particular sentence structure is not part of Wittgenstein's arguments.

Let's look at an important paragraph.

"One can easily imagine a language that consists only of commands and reports in a battle. Or a language that consists only of questions and expressions for acceptance and denial. And countless others.—And to imagine a language means to imagine a form of life."

The last sentence is the key insight: A language points to a form of life. If we take "form of life" to be one of the possible meanings of the word "culture," then we can say instead that the form of a language gives us a view into the culture of a people.

In his books, Everett puts forth the stronger claim that culture shapes language. He uses the Pirahã language to bolster the claim. The language is perfectly suited for survival of a people in a very hostile environment, with limited tools and weapons. The language gives the Pirahã people a means to communicate effectively across distances and, during hunting, undetected by animals.

## Language and recursion (cont'd)

Let's look at some example statements that use no recursion.

E. Hemingway made a bet that he could tell a story using just six words.

He won the bet with the following six words.

"For sale: Baby shoes. Never worn."

The shortest telegram ever was sent by Oscar Wilde to his publisher to inquire how his newest book was doing in the market.

Oscar Wilde telegram: "?"

The publisher's response was equally short.

Publisher's response: "!"

How is it possible that we construct a story from Hemingway's six words, assembled in three sentences with two words each, none of which involves recursion? Or that we understand the exchange between Oscar Wilde and his publisher?

## Language and recursion (cont'd)

A convincing answer is provided by results in Brain Science.

B. K. Berger (book *Louder Than Words: The New Science of How the Mind Makes Meaning*, 2012): Hypothesis of Embodied Simulation: The mind produces meaning by simulating the experience of what the language describes.

Pardon my irreverence, but the definition of Embodied Simulation seems a strong competitor for complex mathematical definitions. To be fair, the definition is not indicative of the rest of Berger's book, which is as lucid and clear as one would ever want in a text written by an expert for nonspecialists.

Instead of working through the definition, let's look at an example. Suppose we grab a key, walk to a car, and unlock the door on the driver's side. Certain cells of the brain control the muscles of hands and legs to accomplish these actions.

In an entirely different setting, suppose somebody asks us, "Please take the key and unlock the car door." Our brain then computes the meaning of this sentence using the very same cells that previously controlled the muscles of hands and legs! Of course, this time, these brain cells do not activate the muscles of hands or legs.

An astonishing fact, isn't it. The same results hold for other parts of our body, such as the eyes and ears. So when we see an egg and interpret the image by certain brain cells, then these very brain cells later interpret the word "egg." It even holds for abstract concepts or symbols processed by a certain part of our brain, such as a question mark or an exclamation mark.

# Language and recursion (cont'd)

We have seen that Embodied Simulation, by its very mechanism, can provide rich meanings of words, groups of words, even of entire sentences, based on prior experiences. For this reason, one could call Embodied Simulation "Reality Replay." Note that recursion plays no role in these results.

We are finished with our excursion into linguistics and return to the task at hand, which is refutation of Discovery Claim B. It states that counting is a natural, unavoidable process.

## Back to: Counterarguments (cont'd)

The Pirahã do not count and do not understand the concept of counting. Everett tried for 8 months to teach rudimentary mathematics. After that time: Not one Pirahã learned to count to 10. None learned to add $3 + 1$ or even $1 + 1$. Only occasionally would some get the right answer.

An astonishing result.

D. Everett: It is hard for westerners to realize that math is detachable from human existence.

## Digression: Axiomatic Method

Before we can state counterarguments concerning Discovery Claim C, which effectively states that theorems are discovered and not constructed, we need to review the Axiomatic Method.

1. Postulate axioms.

2. Create concepts, structures, rules of relationships.

3. Create interesting statements and attempt to establish them as theorems.

The process is recursive in the sense described by N. Bourbaki (*L'Architecture des Mathématiques*, 1961): "Mathematics is similar to a large city where suburbs grow into the surrounding land and where the center is periodically rebuilt, each time according to a clearly defined plan and according to a new, more impressive order."

"N. Bourbaki" is the pseudonym of a group of mathematicians attempting a comprehensive coverage of mathematics. Professor Bourbaki works at the University of Nancago, which, one would guess from the name, is situated halfway between Nancy, France, and Chicago, USA, and thus in the middle of the Atlantic Ocean. Maybe it is the Atlantis of lore?

The cited paper is well written. And yet, we think it a bit quaint. One reason is the subsequent proof of Fermat's Last Theorem by A. J. Wiles who, in the terminology of the Bourbaki paper, builds an extraordinary bridge connecting two distant suburbs of the city of mathematics.

# Axiomatic Method (cont'd)

Knowledgeable as we are about mathematics, we tend to overlook that things seeming trivial to us actually were major insights in times past.

Example taken from L. Wittgenstein (book *Remarks on the Foundations of Mathematics*, 1953):

Given natural numbers $a$ and $n$. Define: $a^n = a \cdot a \cdot \ldots \cdot a$ ($n$ times)

Is this just a different notation?

No:  $a \cdot a \cdot \ldots \cdot a$ ($n$ times) $\cdot a \cdot a \cdot \ldots \cdot a$ ($m$ times) $= a^{m+n}$

Thus, multiplication is effectively turned into addition. This notion leads to the core idea for the use of logarithms.

We are ready to examine Discovery Claim C.

## Counterarguments (cont'd)

Discovery Claim C: A theorem exists regardless of whether somebody states it or proves it.

Claim C implicitly not only assumes that the axioms invoked by the theorem exist independently of the person defining them, but that the axioms of logic used to prove the theorem exist independently as well.

The assumption of prior axioms is implicit in the previously seen statement of B. Russell: "I cannot imagine a world where $1 + 1$ is not equal to 2."

For many years, we focused on the number 2 in the statement. But the resolution lies in the 1+1 part, where the number 1 occurs as axiom.

The world of the Pirahã tribe is exactly one of the worlds Russell cannot imagine, due to absence of the number "1".

## The Discovery claim has additional difficulties

We discuss two examples.

- Axiom of Choice (AC): Given an index set $S$ and a collection of nonempty sets $A_j, j \in S$, there exists a collection $a_j \in A_j, j \in S$.

The axiom has consequences that violate our sense of reality. (Banach-Tarski paradox, *Sur la décomposition des ensembles des points en parties respectivement congruentes*, 1924).

Specifically, the paper shows that we can start with a solid sphere of any given diameter $d$, then, using the Axiom of Choice, take it apart, and reassemble from the pieces two solid spheres with same diameter $d$. An astonishing feat. It implies that we should never invoke the Axiom of Choice when modeling practical optimization problems of logistics.

**The Discovery claim has additional difficulties (cont'd)**

- Continuum Hypothesis (CH) of G. Cantor, 1878: There is no set whose cardinality is strictly between that of the integers and that of the real numbers.

Suppose Cantor discovered the CH. The theorems of K. Gödel, 1940, and P. Cohen, 1963, say that the standard Zermelo-Fraenkel set theory remains consistent if we add CH or declare that CH does not hold.

Specifically, Gödel showed that CH cannot be disproved from the Zermelo-Fraenkel set theory, and Cohen showed that CH cannot be proved from that theory.

Thus, CH is independent from Zermelo-Fraenkel set theory and must be considered an axiom when it or its negation is invoked. If we declare CH or its negation to be a discovery, then almost anything developed by the human brain would have to be considered a discovery.

# Analogous cases: Discovery vs. Construction

Let's take a fun detour into other areas of creative human activity. We begin with the composition of music.

1. J. S. Bach *Toccata and Fugue in D minor*, BWV 538: Discovered or constructed?

### Nature: Waves

Nature provides the fundamental means for music, in the form of waves.

### Selection: Acoustical Waves

We select waves that our ears can perceive.

### Definition: notes $C$, $D$, $E$, ..., $C\flat$, $C\sharp$, ...

Next we select particular wave frequencies, or do we discover them?

### Construct: scales, rules of harmony.

We organize notes in various ways, or do we discover the rules?

### Construct: Toccata and Fugue using scales, rules of harmony.

Finally J. S. Bach composes the Toccata and Fugue, or does he discover the piece?

Our guess is that any sensible person grants Bach that his composition is constructed and not discovered.

## Analogous cases: Discovery vs. Construction (cont'd)

Now look at the entire sequence: We definitely start with a step involving discovery and end with a step involving construction. Thus, going down the list of statements, there is a first step of construction. For the arguments to follow, it does not matter which step has this feature. So feel free to make a choice most satisfactory to you.

Suppose all of mathematics is discovered and not constructed. Alongside the above statements for music, we write corresponding statements of mathematics. We begin with the existence of the real numbers in the first step, select integers in the second step, define features of the integers such as the concept of prime number in the fourth step, and finally discover the theorem that there is no largest prime number in the fourth step.

We now compare the music step where construction is claimed for the first time, with the corresponding mathematics step. For the music step, we have construction, and for the mathematics step discovery. Which steps are involved depends on the choice you made earlier. Regardless of the case, it will be enlightening to compare the discovery arguments for mathematics with the construction arguments for music.

You may object to the comparison of composing music with working on mathematical problems. There seem to be many more choices open to the composer than to the mathematician, since mathematical arguments seem rigidly constrained by rules of logic. But those rules admit huge collections of theorems, indeed gazillions of them, that we discard and declare trivial or uninteresting, or replace by a few axioms.

For example, consider the theorem that for any two integers $a$ and $b$ there is an integer $c$ so that $a + b = c$. In *Principia Mathematica*, it takes B. Russell and A. N. Whitehead several hundred pages to prove $1 + 1 = 2$. Similarly, we could spend our lifetime to prove other addition theorems for the integers.

We pass over those theorems as uninteresting and instead invent, say, the axioms of the additive group of integers. In one step, all those addition theorems disappear, or rather, no longer need to be proved. We do this lots of times, replacing a huge collection of theorems by a few simple axioms. So there are many, many theorems in mathematics, but we ignore almost all of them and focus on comparatively few we feel are interesting, just as a composer considers only note sequences that he finds interesting.

## Analogous cases: Discovery vs. Construction (cont'd)

We continue our fun detour.

**2.** Shakespeare's *Romeo and Juliet*: Discovered or constructed?

Try to construct steps for the drama Romeo and Juliet as done a moment ago for the Toccata and Fugue. Maybe you want to introduce the alphabet in the first step, and then go on.

**3.** Michelangelo's *David*: Discovered or constructed?

Again try to construct steps and make a comparison.

**4.** A. Renoir's *Luncheon of the Boating Party*: Discovered or constructed?

It maybe hard to see how an Impressionist painting can be viewed in steps analogous to the situations above. But that argument overlooks that the Impressionists such as Renoir made major inventions. Or were they discoveries? An example is the use of the color blue to create the effect of vivid shadows in a bright sunlit scene. Another example is the almost mathematical treatment of color by the pointillist G. P. Seurat. Is his famous painting *A Sunday on La Grande Jatte* discovered or constructed?

We hasten to say that we do not claim that working in mathematics is essentially the same as composing music, writing dramas, or creating works of art. But the comparisons may convince you that creativity plays the most significant role in mathematics.

## Analogous cases: Discovery vs. Construction (cont'd)

It is interesting to examine how over centuries the notion of construction has changed for music, literature, sculptures, paintings. The defining rules were gradually relaxed and, during the 20th century, largely abandoned.

Correspondingly, mathematics has broadened its definitions and axioms, at times over major objections, for example, based on the ideas of Finitism or Intuitionism.

Since the 1930s, the Axiomatic Method has become the most widely accepted approach.

## Previously controversial axioms or definitions, now accepted

Living in the world of the Axiomatic Method, we tend to forget how controversial some concepts were when first introduced.

- Imaginary numbers

The strange name "imaginary number" expresses exactly the response these numbers evoked when first introduced. They were not really numbers, indeed the term "imaginary" was meant to be derogatory until the work of L. Euler in the 18th century.

- Axiom:

  For any set $A$, there exists the power set
  $P(A) = \{B \mid B \text{ is a subset of } A\}$.

Here is an operation that is extraordinarily powerful, if you pardon the pun. For example, if we take the set of integers as $A$, then the power set $P(A)$ is readily interpreted as the set of reals of the interval [0,1], as is well known. Indeed, from a subset $S$ of the integers, a fractional binary representation of the corresponding real number is defined by taking the $i$th digit to be 1 if $i$ is in $S$, and to be 0 otherwise. No wonder the intuitionists were upset by this use of the power set, just as they were by the use of the Dedekind cut.

- Axiom of Choice

We have seen that mathematicians should not use the Axiom of Choice when practical problems are modeled. But in topology it is an essential tool.

## How is it possible that there are so many opinions?

We are coming to the last section of this talk. How can it be that so many intelligent people come to conclusions that are clearly incompatible? For an answer, we first turn to Wittgenstein, and then to recent results of Brain Science.

When Wittgenstein realized that his first book, *Tractatus Logico-Philosophicus* published in 1921, contained, in his words, "grave errors," he set out to develop a method of investigation that prevented such pitfalls.

Almost all of his subsequent work was dedicated to that goal, with astonishing results. Here we just touch on the main ideas and discuss an example in a moment.

L. Wittgenstein (most books, in particular *Philosophische Untersuchungen*, 1953): Most problems of philosophy arise because of

- treatment as if they were problems of science;

- conceptual confusion surrounding language;

- the fact that people do not look at enough examples.

The fly in the bottle: A seemingly inescapable but wrong conclusion. The fly is totally convinced that it cannot escape, though straight up is a way out.

## How is it possible that there are so many opinions? (cont'd)

The concept of Embodied Simulation provides an explanation why looking at various examples helps us avoid pitfalls of philosophy.

Take the verb "measure." It can mean evaluation of time intervals, lengths, audio signals, or vision acuity, to name a few uses. Each of these concepts triggers thinking by different brain cells.

Here are some examples: for "measuring time", maybe the brain cells that evaluate a calendar; for " length", maybe the brain cells that control the hand muscles operating a caliper; for "audio signal", maybe the brain cells that analyze sound; for "vision acuity", maybe the brain cells that process the image as we look into the apparatus at an optometrist's office.

As a result, a rich panorama of thoughts and ideas surfaces in various parts of the brain when we go through examples involving the word "measure."

**How is it possible that there are so many opinions? (cont'd)**

Example where Wittgenstein applies his method:

J. W. von Goethe (book *Zur Farbenlehre*, 1810): covers the nature of colors and their perception by humans. He believed that the theory was his most profound intellectual achievement.

L. Wittgenstein (book *Remarks on Colour*, 1950): proves Goethe's theory to be wrong. Indeed, he shows that there cannot be a theory of color in the sense proposed by Goethe.

**How is it possible that there are so many opinions? (cont'd)**

Revolution in Brain Science, since 1990s, based on MRI.

New notion: Plasticity of the brain.

Example: Davidson and Begley, book *The Emotional Life of Your Brain*, 2012: Meditating restructures regions of the brain and creates stronger neural connections.

MRI proves conjecture by Gauss: "It may be true that men who are mere mathematicians have certain specific shortcomings; but that is not the fault of mathematics, for it is equally true of every other exclusive occupation."

## How is it possible that there are so many opinions? (cont'd)

Kahneman, book *Thinking, Fast and Slow*, 2011: Hypothesis of two systems, called System 1 and System 2, and of biased analysis. Statements below are citations.

- System 1: automatic operation that uses associate memory to construct a coherent interpretation of the world.

- System 2: controlled operations.

System 1 effortlessly originates impressions and feelings that are the main sources of the explicit beliefs and deliberate choices of System 2.

## How is it possible that there are so many opinions? (cont'd)

- What You See Is All There Is (WYSIATI):

Excessive confidence in what we believe we know, and our apparent inability to acknowledge the full extent of our ignorance and the uncertainty of the world we live in.

L. Wittgenstein's fly-in-the-bottle is System 1 and WYSIATI at work.

How can we avoid the problems produced by System 1 and WYSIATI?

Recall that Wittgenstein advocates looking at lots of examples, and that Embodied Simulation explains why this is a good thing: Various brain regions are activated by the different examples.

As a result, when many examples are considered, then System 1 produces a rich collection of concepts and ideas, and WYSIATI does not have its detrimental effect.

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Whatever errors remain are solely mine.