SIAM J. DISCRETE MATH.
Vol. 27, No. 3, pp. 1610–1624

ON MAXIMAL S-FREE SETS AND THE HELLY NUMBER FOR THE FAMILY OF S-CONVEX SETS*

GENNADIY AVERKOV†

Abstract. We study two combinatorial parameters, which we denote by $f(S)$ and $h(S)$, associated with an arbitrary set $S \subseteq \mathbb{R}^d$, where $d \in \mathbb{N}$. In the nondegenerate situation, $f(S)$ is the largest possible number of facets of a d-dimensional polyhedron L such that the interior of L is disjoint with S and L is inclusion-maximal with respect to this property. The parameter $h(S)$ is the Helly number of the family of all sets that can be given as the intersection of S with a convex subset of $\mathbb{R}^d$. We obtain the inequality $f(S) \le h(S)$ for an arbitrary S, and the equality $f(S) = h(S)$ for every discrete S. Furthermore, motivated by research in integer and mixed-integer optimization, we show that 2^d is the sharp upper bound on $f(S)$ in the case $S = (\mathbb{Z}^d \times \mathbb{R}^n) \cap C$, where $n \ge 0$ and $C \subseteq \mathbb{R}^{d+n}$ is convex. The presented material generalizes and unifies results of various authors, including the result $h(\mathbb{Z}^d) = 2^d$ of Doignon, the related result $f(\mathbb{Z}^d) = 2^d$ of Lovász, and the inequality $f(\mathbb{Z}^d \cap C) \le 2^d$, which has recently been proved for every convex set $C \subseteq \mathbb{R}^d$ by Morán and Dey.

Key words. cutting plane, Doignon's theorem, Helly's theorem, Helly number, intersection cut, lattice-free set, S-convex set, S-free set

AMS subject classifications. Primary, 90C11; Secondary, 52A01, 52C07, 90C10

DOI. 10.1137/110850463

1. Introduction.

For background information from convex geometry, theory of polyhedra, and geometry of numbers we refer the reader to [19, 20, 21, 25, 27, 28]. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. For $m \in \mathbb{N}$ we use the notation $[m] := \{1, \ldots, m\}$. Let $d \in \mathbb{N}$ and $S \subseteq \mathbb{R}^d$. The origin of linear spaces (such as $\mathbb{R}^d$) is denoted by o. In this manuscript we study S-free sets and maximal S-free sets, defined as follows.

DEFINITION 1.1 (S-free set and maximal S-free set). *A set $L \subseteq \mathbb{R}^d$ is said to be* S-free *if L is closed and convex and the interior of L is disjoint with S. An S-free set L in $\mathbb{R}^d$ is said to be* maximal S-free *if there exists no S-free set properly containing L.*

We consider the question of whether for a given S there exists a bound $k \in \mathbb{N}_0$ such that every maximal S-free set is a polyhedron with at most k facets. If such a bound k exists, we are interested in finding an appropriate k explicitly. Our question can be expressed in terms of the following functional.

DEFINITION 1.2 (the functional f—facet complexity of maximal S-free sets). *Let $S \subseteq \mathbb{R}^d$. If there exists $k \in \mathbb{N}_0$ such that every d-dimensional maximal S-free set is a polyhedron with at most k facets, we define $f(S)$ to be the minimal k as above. If there exist no d-dimensional maximal S-free sets (e.g., for $S = \mathbb{R}^d$), we let $f(S) := -\infty$. If there exist maximal S-free sets which are not polyhedra or maximal S-free polyhedra with arbitrarily large number of facets, we define $f(S) := +\infty$.*

Thus, in the qualitative form our question is about conditions which ensure $f(S) < +\infty$. Quantitatively, we are interested in bounds on $f(S)$. More specifically, we are interested in evaluating or estimating $f(S)$ for certain structured sets S that play a

*Received by the editors October 6, 2011; accepted for publication (in revised form) July 5, 2013; published electronically September 26, 2013.
http://www.siam.org/journals/sidma/27-3/85046.html
†Institute for Mathematical Optimization, Faculty of Mathematics, University of Magdeburg, 39106 Magdeburg, Germany (averkov@math.uni-magdeburg.de).

role in optimization. With a view toward applications in the cutting-plane theory from integer and mixed-integer optimization, it is desirable to have upper bounds on $f((\mathbb{Z}^d \times \mathbb{R}^n) \cap C)$, where $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, and $C \subseteq \mathbb{R}^{d+n}$ is convex. See also [1, 4, 6, 12, 13, 16, 31] for information on application of S-free sets for generation of cutting planes. The main topic of this manuscript is the study of the relationship between $f(S)$ and the Helly number $h(S)$ for the family of of S-convex sets. The notions of S-convex set and the Helly number are introduced below.

DEFINITION 1.3 (S-convex set). *Given $S \subseteq \mathbb{R}^d$, a set $A \subseteq \mathbb{R}^d$ is called* S-convex *if $A = S \cap C$ for some convex set $C \subseteq \mathbb{R}^d$.*

In order to avoid possible ambiguities we point out that the literature contains a number of different generalizations of the notion of convexity (see, for example, the monographs [11] and [30]). In [30, section 1.9] the S-convexity as introduced here is called the relative convexity of S. For $S = \mathbb{R}^d$ the S-convexity is reduced to the standard convexity. The notion of S-convexity in the case $S = \mathbb{Z}^d$ was considered by various authors in different contexts (see, for example, [7, 15, 17, 18, 24]). To the best of our knowledge, the study of S-convexity in the case $S = \mathbb{R}^d \times \mathbb{Z}^n$ was initiated in [5].

DEFINITION 1.4 (the functional h—Helly number). *Given a nonempty family $\mathcal{F}$ of sets, the* Helly number $h(\mathcal{F})$ *of $\mathcal{F}$ is defined as follows. For $\mathcal{F} = \{\emptyset\}$ let $h(\mathcal{F}) := 0$. If $\mathcal{F} \neq \{\emptyset\}$ and there exists $k \in \mathbb{N}$ such that*

$$F_1 \cap \cdots \cap F_m = \emptyset \implies \exists i_1, \ldots, i_k \in [m] : F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset \tag{1.1}$$

for all $F_1, \ldots, F_m \in \mathcal{F}$ ($m \in \mathbb{N}$), then we define $h(\mathcal{F})$ to be the minimal k as above. In all other cases we let $h(\mathcal{F}) := +\infty$. For $S \subseteq \mathbb{R}^d$ we use the notation

$$h(S) := h\big(\{S \cap C \,:\, C \subseteq \mathbb{R}^d \text{ is convex}\}\big). \tag{1.2}$$

That is, $h(S)$ is the Helly number of the family of all S-convex sets.

The functional h given by (1.2) has several interpretations in terms of optimization; see [5, Proposition 1.2].

The main results of this manuscript, which we formulate in the next section, can be split into two groups: theorems about $f(S)$ and $h(S)$ for general sets S having no particular (global) structure and, on the other hand, theorems providing bounds on $f(S)$ and $h(S)$ for structured sets S, whose structure is related to integer and mixed-integer programming.

For general sets $S \subseteq \mathbb{R}^d$, we derive the inequality $f(S) \le h(S)$ and the equality $f(S) = h(S)$ in the case that S is discrete (see Theorem 2.1). We also relate $f(S)$ and $h(S)$ to the sequences of values $f(S_t)$ and $h(S_t)$, respectively, in the case of a set sequence $(S_t)_{t\in\mathbb{N}}$ satisfying $S_t \subseteq S_{t+1}$ for all $t \in \mathbb{N}$ and $S = \bigcup_{t\in\mathbb{N}} S_t$ (see Theorem 2.2).

We show that the above results yield short and unified proofs of the equality $h(\mathbb{Z}^d) = 2^d$ of Doignon [15] (see also [10, 26, 29]), the equality $f(\mathbb{Z}^d) = 2^d$ of Lovász [2], and the inequality $f(\mathbb{Z}^d \cap C) \le 2^d$ for every convex set $C \subseteq \mathbb{R}^d$ recently proved by Morán and Dey [23]. Note that various special cases of $f(\mathbb{Z}^d \cap C) \le 2^d$ were derived and used as a tool for research in integer optimization in [8, Theorem 1.1], [9, Theorem 2], [14, Proposition 31], and [16, Theorem 3.2]. In [8] the set C is assumed to be an arbitrary affine space, while in [9] the set C is an arbitrary rational polyhedron.

We prove that 2^d is the tight upper bound on $f((\mathbb{Z}^d \times \mathbb{R}^n) \cap C)$ for $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, and an arbitrary convex set $C \subseteq \mathbb{R}^{d+n}$ (see Theorem 2.5). The latter is a generalization of the mentioned result of Morán and Dey to the mixed-integer setting. Observe that the set $\mathbb{Z}^d \times \mathbb{R}^n$ is a Minkowski sum of the lattice $\mathbb{Z}^d \times \{o\}$ of rank d and the linear space $\{o\} \times \mathbb{R}^n$ of dimension n. By this we are motivated to study $f(S)$ and

$h(S)$ for structured sets S, whose structure can be expressed in terms of Minkowski sums and/or lattices. In fact, the upper bound on $f((\mathbb{Z}^d \times \mathbb{R}^n) \cap C)$ will be deduced as a direct consequence of results for such sets S (see Theorems 2.3 and 2.4).

We do not systematically address the interesting question of characterizing maximal S-free sets. Some information on this question can be found in [2, 8, 9, 23, 22].

The manuscript is organized as follows. In section 2 we formulate the main results. Section 3 provides the necessary background material. In section 4 we prove results for general sets S and show how to derive the known relations $h(\mathbb{Z}^d) = f(\mathbb{Z}^d) = 2^d$ and $f(\mathbb{Z}^d \cap C) \leq 2^d$ (for an arbitrary convex $C \subseteq \mathbb{R}^d$) as a consequence. In section 5 we give upper bounds on $f(S)$ and $h(S)$ for structured sets S.

2. Main results.

Results on $f(S)$ and $h(S)$ for general sets S. A set $S \subseteq \mathbb{R}^d$ is said to be *discrete* if every bounded subset of S is finite.

THEOREM 2.1 (relation between f and h). *Let $S \subseteq \mathbb{R}^d$. Then the following statements hold:*

I. $f(S) \leq h(S)$.

II. *If S is discrete, one has $f(S) = h(S)$.*

The following continuity-type result can be used to bound $f(S)$ or $h(S)$ for a "complicated set" S by approximating S with "simple sets" S_t ($t \in \mathbb{N}$).

THEOREM 2.2 (lim inf theorem). *Let $S \subseteq \mathbb{R}^d$. Let $(S_t)_{t=1}^{+\infty}$ be a sequence of sets satisfying $S_t \subseteq S_{t+1} \subseteq \mathbb{R}^d$ for all $t \in \mathbb{N}$ and $S = \bigcup_{t=1}^{+\infty} S_t$. Then*

$$h(S) \leq \liminf_{t \to +\infty} h(S_t), \tag{2.1}$$

$$f(S) \leq \liminf_{t \to +\infty} f(S_t). \tag{2.2}$$

Upper bounds on $f(S)$ and $h(S)$ for structured sets S. For $A, B \subseteq \mathbb{R}^d$ the *Minkowski sum* $A + B$ and *Minkowski difference* $A - B$ of A and B are defined by $A \pm B := \{a \pm b : a \in A, \ b \in B\}$.

THEOREM 2.3 (on adding a convex set). *Let $S \subseteq \mathbb{R}^d$ be closed. Let $C \subseteq \mathbb{R}^d$ be nonempty and convex. Then*

$$h(C + S) \leq (\dim(C) + 1)h(S), \tag{2.3}$$

$$f(C + S) \leq f(S). \tag{2.4}$$

A set $\Lambda \subseteq \mathbb{R}^d$ is said to be a *lattice* if Λ is a discrete subgroup of the additive group $\mathbb{R}^d$. Every lattice $\Lambda \subseteq \mathbb{R}^d$, $\Lambda \neq \{o\}$, can be given by

$$\Lambda = \{t_1 x_1 + \cdots + t_r x_r \, : \, t_1, \ldots, t_r \in \mathbb{Z}\},$$

where $r \in \{1, \ldots, d\}$ and $x_1, \ldots, x_r \in \mathbb{R}^d$ are linearly independent vectors. The value r above is called the rank of the lattice Λ and is denoted by $\operatorname{rank}(\Lambda)$. For $\Lambda = \{o\}$ we define $\operatorname{rank}(\Lambda) = 0$.

THEOREM 2.4. *Let $C, D \subseteq \mathbb{R}^d$ be nonempty convex sets, and let $\Lambda \subseteq \mathbb{R}^d$ be a lattice. Then*

$$h((C + \Lambda) \cap D) \leq (\dim(C) + 1)2^{\operatorname{rank}(\Lambda)}, \tag{2.5}$$

$$f((C + \Lambda) \cap D) \leq 2^{\operatorname{rank}(\Lambda)}. \tag{2.6}$$

THEOREM 2.5. *Let $d \in \mathbb{N}$, $n \in \mathbb{N}_0$ and let $C \subseteq \mathbb{R}^{d+n}$ be convex. Then*

$$f((\mathbb{Z}^d \times \mathbb{R}^n) \cap C) \leq 2^d. \tag{2.7}$$

Furthermore, (2.7) *is attained with equality for $C = \mathbb{R}^{d+n}$.*

If C in (2.6) is a linear space which does not contain nonzero vectors of Λ, inequalities (2.6) and (2.7) are equivalent. In this special case, (2.6) is a "coordinate-free" version of (2.7).

An analogue of Theorem 2.5 for the functional h is provided by [5]. By the main result of [5],

$$h((\mathbb{Z}^d \times \mathbb{R}^n) \cap C) \leq (n+1)2^d \tag{2.8}$$

for every convex set $C \subseteq \mathbb{R}^{d+n}$, with equality attained for $C = \mathbb{R}^{d+n}$.

We emphasize that the assertions for f are the main parts of Theorems 2.2–2.4, while the assertions for h are given as a complement, in order to highlight the analogy between f and h. Inequality (2.1) will be derived as a consequence of basic properties of h, while (2.3) and (2.5) will be shown to follow directly from the results of [5]. The assertions for f in Theorems 2.1 and 2.2 are proved using a compactness argument. The proof of (2.4) relies on basic convex geometry. The proof of (2.6) and (2.7) involves the so-called parity argument, which is presented in the following section.

3. Background material. Throughout the manuscript we use the following notation. The cardinality of a set X is denoted by $|X|$. The standard scalar product of $\mathbb{R}^d$ is denoted by $\langle \cdot\,,\,\cdot \rangle$. By aff, bd, cl, conv, int, and relint we denote the affine hull, boundary, closure, convex hull, interior (of a set), and relative interior (of a convex set), respectively.

The role of the parity argument. Let $d \in \mathbb{N}$. The following implication is usually referred to as the *parity argument*: if $X \subseteq \mathbb{Z}^d$ and $|X| > 2^d$, then there exist $x, y \in X$ with $x \neq y$ and $\frac{1}{2}(x+y) \in \mathbb{Z}^d$. This implication is obtained by comparing the elements of X modulo $2\mathbb{Z}^d$. Note that the parity argument is the key step in the existing proofs of $f(\mathbb{Z}^d) = 2^d$ and $h(\mathbb{Z}^d) = 2^d$.

We illustrate how to use the parity argument by sketching a proof of the inequality $f(\mathbb{Z}^d \cap C) \leq 2^d$ in the case that C is a *bounded* convex subset of $\mathbb{R}^d$. Let $S := \mathbb{Z}^d \cap C$. If $S = \emptyset$, there is nothing to prove. Thus, we assume $S \neq \emptyset$. Let L be an arbitrary d-dimensional maximal S-free set. Using separation theorems for L and each of the (finitely many) points of S, we see that L is a polyhedron. Let m be the number of facets of L. Taking into account the finiteness of the set S and using basic facts from the theory of polyhedra, one can show that the relative interior of every facet of L contains a point of S (this is proved rigorously in section 4; see Lemma 4.4.II). Consider a sequence of points $x_1, \ldots, x_m \in S$ constructed by picking one point of S from the relative interior of each facet of L. If $m > 2^d$, then the parity argument yields the existence of indices $i, j \in [m]$ with $i \neq j$ and $\frac{1}{2}(x_i + x_j) \in \mathbb{Z}^d$. Clearly, $\frac{1}{2}(x_i + x_j) \in \operatorname{int}(L) \cap S$, which is a contradiction to the fact that L is S-free. Thus, $m \leq 2^d$.

It should be mentioned that the above proof idea cannot be directly extended to the case of an arbitrary convex set C for the following reason. For certain choices of C there exist d-dimensional maximal S-free polyhedra L such that the relative interior of some facets of L does not contain points of S. Take, for example, $C = \{x \in \mathbb{R}^2 : \langle x\,,\,u \rangle \leq 1\}$ and $L = \{x \in \mathbb{R}^2 : \langle x\,,\,u \rangle \geq 1\}$ with $u := \binom{2}{\sqrt{2}}$. Thus, in some cases, using the parity argument, one can give a relatively simple proof of

$f(\mathbb{Z}^d \cap C) \le 2^d$, but for a proof in the case of a general convex set C additional work is needed. Our proofs of the inequality $f(\mathbb{Z}^d \cap C) \le 2^d$ and its generalizations (2.6) and (2.7) will combine the parity argument with elementary tools from analysis (such as limits and the compactness argument).

Basic facts about *f*, *h*, and maximal *S*-free sets. Let us list some simple properties of the functional h. The definition of the Helly number yields the equality

$$h(\mathcal{F}) = h(\{F_1 \cap \cdots \cap F_t \,:\, t \in \mathbb{N},\ F_1, \ldots, F_t \in \mathcal{F}\}) \tag{3.1}$$

for every nonempty family $\mathcal{F}$.

In the following proposition we present several relations for $h(S)$, where $S \subseteq \mathbb{R}^d$.

PROPOSITION 3.1. *Let $S \subseteq \mathbb{R}^d$, let $C \subseteq \mathbb{R}^d$ be convex, and let ϕ be an affine mapping on $\mathbb{R}^d$. Then*

$$h(S) = \lim_{t \to +\infty} h(S \cap [-t,t]^d), \tag{3.2}$$

$$h(S) \le |S|, \tag{3.3}$$

$$h(S \cap C) \le h(S), \tag{3.4}$$

$$h(\phi(S)) \le h(S). \tag{3.5}$$

Proof. Let us prove (3.2). The sequence of values $h(S \cap [-t,t]^d)$ with $t \in \mathbb{N}$ is monotonically nondecreasing and thus convergent, with the limit belonging to $\mathbb{N}_0 \cup \{+\infty\}$. If the limit of this sequence is $+\infty$, one can see that $h(S) = +\infty$. Otherwise there exists $k \in \mathbb{N}_0$ such that $h(S \cap [-t,t]^d) = k$ for all sufficiently large $t \in \mathbb{N}$. Consider arbitrary convex sets $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ with $m \in \mathbb{N}$ and $C_1 \cap \cdots \cap C_m \cap S = \emptyset$. By the choice of k, for every sufficiently large $t \in \mathbb{N}$ one can choose $i_1(t), \ldots, i_k(t) \in [m]$ such that $C_{i_1(t)} \cap \cdots \cap C_{i_k(t)} \cap S \cap [-t,t]^d = \emptyset$. By the finiteness of $[m]$ there exist indices $i_1, \ldots, i_k \in [m]$ and an infinite set $N \subseteq \mathbb{N}$ such that $h(S \cap [-t,t]^d) = k$ and $i_1 = i_1(t), \ldots, i_k = i_k(t)$ for every $t \in N$. The latter yields $C_{i_1} \cap \cdots \cap C_{i_k} \cap S = \emptyset$ and shows $h(S) \le k$.

Inequality (3.3) is trivial if S is empty or infinite. Assume that S is nonempty and finite, say $S = \{s_1, \ldots, s_k\}$, where $k := |S| \in \mathbb{N}$. Consider arbitrary convex sets $C_1, \ldots, C_m \subseteq \mathbb{R}^d$ with $m \in \mathbb{N}$ and $C_1 \cap \cdots \cap C_m \cap S = \emptyset$. For every $j \in [k]$ one can choose C_{i_j} with $s_j \notin C_{i_j}$. This yields $C_{i_1} \cap \cdots \cap C_{i_k} \cap S = \emptyset$ and shows $h(S) \le |S|$.

Inequality (3.4) follows from the fact that every $(S \cap C)$-convex set is also S-convex. For the proof of (3.5) we can assume $k := h(S) \in \mathbb{N}$, since otherwise the inequality is trivial. Consider arbitrary convex sets $C_1, \ldots, C_m \subseteq \phi(\mathbb{R}^d)$ with $C_1 \cap \cdots \cap C_m \cap \phi(S) = \emptyset$. The latter implies $\phi^{-1}(C_1) \cap \cdots \cap \phi^{-1}(C_m) \cap S = \emptyset$. Since the sets $\phi^{-1}(C_1), \ldots, \phi^{-1}(C_m)$ are convex, there exist $i_1, \ldots, i_k \in [m]$ with $\phi^{-1}(C_{i_1}) \cap \cdots \cap \phi^{-1}(C_{i_k}) \cap S = \emptyset$. Thus $C_{i_1} \cap \cdots \cap C_{i_k} \cap \phi(S) = \emptyset$, which shows $h(\phi(S)) \le k$. ∎

In the rest of this section we collect well-known facts about maximal S-free sets and the functional f. If L is a d-dimensional S-free polyhedron in $\mathbb{R}^d$ such that the relative interior of each facet of L contains a point of S, then L is maximal S-free. Every d-dimensional S-free set can be extended to a maximal S-free set. That is, if K is an S-free set, then there exists a maximal S-free set L with $K \subseteq L$. This follows directly from Zorn's lemma. One can also give a different proof by adapting the approach from [3, Proposition 3.1]. It is not hard to see that $f(S) = -\infty$ if and only if $\mathrm{cl}(S) = \mathbb{R}^d$. Furthermore, one can verify that

$$f(S \times \mathbb{R}^n) = f(S) \tag{3.6}$$

for every $n \in \mathbb{N}$. Equality (3.6) follows from the fact that every $(d+n)$-dimensional $(S \times \mathbb{R}^n)$-free set L is a subset of the $(S \times \mathbb{R}^n)$-free set $L' := L + \{o\} \times \mathbb{R}^n$, where L' can be represented as a Cartesian product of a d-dimensional S-free set and $\mathbb{R}^n$.

4. Proofs of results on $f(S)$ and $h(S)$ for general sets S.

The inequality $f(S) \le h(S)$ (Theorem 2.1.I) and its consequences. In this section Theorem 2.1.I is derived as a consequence of the following lemma.

LEMMA 4.1. *Let $S \subseteq \mathbb{R}^d$ and $k \in \mathbb{N}$. Assume that every d-dimensional S-free polyhedron P is contained in an S-free polyhedron Q with at most k facets. Then every d-dimensional maximal S-free set is a polyhedron with at most k facets.*

Proof. Let L be an arbitrary d-dimensional S-free set. It suffices to show that L is contained in an S-free polyhedron with at most k facets. We consider a sequence $(P_t)_{t=1}^{+\infty}$ of d-dimensional polytopes such that

$$P_t \subseteq P_{t+1} \quad \forall t \in \mathbb{N} \tag{4.1}$$

and

$$\operatorname{int}(L) = \bigcup_{t=1}^{+\infty} P_t. \tag{4.2}$$

Such a sequence can be constructed as follows. Let $(z_t)_{t=1}^{+\infty}$ be any ordering of the rational points of $\operatorname{int}(L)$. Then, for every $t \in \mathbb{N}$, we define $P_t := \operatorname{conv}(\{z_1, \ldots, z_{t+d}\})$. With an appropriate choice of $z_1, \ldots, z_{d+1}$ the polytope P_1 and, by this, also every other polytope P_t is d-dimensional. By the assumption, each P_t is contained in an S-free polyhedron Q_t having at most k facets. Every Q_t can be represented by

$$Q_t = \big\{x \in \mathbb{R}^d : \langle u_{1,t}, x\rangle \le \beta_{1,t}, \ldots, \langle u_{k,t}, x\rangle \le \beta_{k,t}\big\},$$

where $u_{1,t}, \ldots, u_{k,t} \in \mathbb{R}^d$, $\beta_{1,t}, \ldots, \beta_{k,t} \in \mathbb{R}$. In the degenerate situation $Q_t = \mathbb{R}^d$ we let $u_{i,t} := o$ and $\beta_{i,t} := 1$ for every $i \in [k]$. Otherwise we can assume $u_{i,t} \ne o$ for every $i \in [k]$. After an appropriate renormalization we assume that $\binom{u_{i,t}}{\beta_{i,t}} \in \mathbb{R}^{d+1}$ is a vector of unit Euclidean length, for every $i \in [k]$ and $t \in \mathbb{N}$. By compactness of the unit sphere in $\mathbb{R}^{d+1}$, there exists an infinite set $N \subseteq \mathbb{N}$ such that, for every $i \in [k]$, the vector $u_{i,t}$ converges to some vector $u_i \in \mathbb{R}^d$ and $\beta_{i,t}$ converges to some $\beta_i \in \mathbb{R}$, as t goes to infinity over points of N. Clearly, $\binom{u_i}{\beta_i} \in \mathbb{R}^{d+1}$ is a vector of unit length for every $i \in [k]$. We define the polyhedron

$$Q := \big\{x \in \mathbb{R}^d : \langle u_1, x\rangle \le \beta_1, \ldots, \langle u_k, x\rangle \le \beta_k\big\}.$$

Let us show $\operatorname{int}(L) \subseteq Q$. Consider an arbitrary $x' \in \operatorname{int}(L)$. One has $x' \in P_t \subseteq Q_t$ for all sufficiently large $t \in N$. Thus, for each $i \in [k]$, the inequality $\langle u_{i,t}, x'\rangle \le \beta_{i,t}$ holds if $t \in N$ is sufficiently large. Passing to the limit, we get $\langle u_i, x'\rangle \le \beta_i$ for every $i \in [k]$. Hence $x' \in Q$. The inclusion $\operatorname{int}(L) \subseteq Q$ implies $L \subseteq Q$. It remains to show that Q is S-free. We assume that Q is not S-free. Then there exists $x' \in S \cap \operatorname{int}(Q)$. Taking into account that $u_i \ne o$ or $\beta_i \ne 0$ for every $i \in [k]$,

$$\operatorname{int}(Q) = \big\{x \in \mathbb{R}^d : \langle u_1, x\rangle < \beta_1, \ldots, \langle u_k, x\rangle < \beta_k\big\}.$$

Consequently $\langle u_i, x\rangle < \beta_i$ for all $i \in [k]$, and thus $\langle u_{i,t}, x'\rangle < \beta_{i,t}$ for all $i \in [k]$ if $t \in N$ is sufficiently large. Hence $x' \in S \cap \operatorname{int}(Q_t)$ for all sufficiently large $t \in N$, contradicting the fact that Q_t is S-free. ∎

Proof of Theorem 2.1.I. Without loss of generality let $S \neq \emptyset$ and $k := h(S) < +\infty$, since otherwise the assertion is trivial. The condition $S \neq \emptyset$ implies $k > 0$. Let us verify the assumption of Lemma 4.1. Let $P \subseteq \mathbb{R}^d$ be an arbitrary d-dimensional S-free polyhedron. We have $S \neq \emptyset$ and thus $P \neq \mathbb{R}^d$. We represent P by $P = H_1 \cap \cdots \cap H_m$, where $m \in \mathbb{N}$ and $H_1, \ldots, H_m \subseteq \mathbb{R}^d$ are closed halfspaces. Then $\operatorname{int}(H_1) \cap S, \ldots, \operatorname{int}(H_m) \cap S$ are S-convex sets whose intersection is empty. By the definition of $h(S)$, there exist indices $i_1, \ldots, i_k \in [m]$ such that $\operatorname{int}(H_{i_1}) \cap \cdots \cap \operatorname{int}(H_{i_k}) \cap S = \emptyset$. It follows that $P \subseteq Q := H_{i_1} \cap \cdots \cap H_{i_k}$, where Q is an S-free polyhedron with at most k facets. We conclude by Lemma 4.1 that every d-dimensional S-free set is contained in a polyhedron with at most $h(S)$ facets. Hence $f(S) \leq h(S)$. ∎

Remark 4.2 (deriving the result of Morán and Dey from Doignon's theorem). Let C be a convex subset of $\mathbb{R}^d$. Having obtained Theorem 2.1.I, the inequality $f(C \cap \mathbb{Z}^d) \leq 2^d$ of Morán and Dey follows directly from Doignon's theorem, $h(\mathbb{Z}^d) = 2^d$. By Theorem 2.1.I, $f(C \cap \mathbb{Z}^d) \leq h(C \cap \mathbb{Z}^d)$. By the convexity of C, we have $h(C \cap \mathbb{Z}^d) \leq h(\mathbb{Z}^d)$. Taking into account Doignon's theorem, we arrive at $f(C \cap \mathbb{Z}^d) \leq h(\mathbb{Z}^d) = 2^d$.

Remark 4.3 (a weak version of Theorem 2.5). Let $d \in \mathbb{N}$, $n \in \mathbb{N}_0$, and let $C \subseteq \mathbb{R}^{d+n}$ be convex. We can use the equality $h(\mathbb{Z}^d \times \mathbb{R}^n) = (n+1)2^d$, which was proved in [5], to find an upper bound on $f((\mathbb{Z}^d \times \mathbb{R}^n) \cap C)$. Analogously to Remark 4.2, we get $f((\mathbb{Z}^d \times \mathbb{R}^n) \cap C) \leq h((\mathbb{Z}^d \times \mathbb{R}^n) \cap C) \leq h(\mathbb{Z}^d \times \mathbb{R}^n) = (n+1)2^d$. The upper bound $(n+1)2^d$ derived in this way is weaker than the bound 2^d provided by Theorem 2.5.

The equality $f(S) = h(S)$ for discrete sets (Theorem 2.1.II). Theorem 2.1.II is proved by reducing the case of a general discrete set S to the case of a finite S. The following lemma presents properties of S-free sets in the case of finite S.

LEMMA 4.4. *Let S be a finite subset of $\mathbb{R}^d$. Then the following statements hold:*

I. $f(S) = h(S)$.

II. *Every maximal S-free set L is a d-dimensional polyhedron such that the relative interior of each facet of L contains a point of S.*

Proof. It suffices to consider the nontrivial case $S \neq \emptyset$. By Theorem 2.1.I, $f(S) \leq h(S)$. Thus, for showing $h(S) = f(S)$ one needs to verify $h(S) \leq f(S)$. Let A be an arbitrary S-convex set. Applying separation theorems for $\operatorname{conv}(A)$ and every point of $S \setminus A$, we see that A can be represented as intersection of S with finitely many open halfspaces. Taking into account (3.1), the latter implies the equality $h(S) = h(\mathcal{F})$ for the family $\mathcal{F}$ consisting of sets $\{x \in S : \langle u, x \rangle < \beta\}$, where $u \in \mathbb{R}^d \setminus \{o\}$ and $\beta \in \mathbb{R}$. We will verify $h(\mathcal{F}) \leq f(S)$. Consider an arbitrary system of strict inequalities

$$\langle u_i, x \rangle < \beta_i \qquad \forall i \in [m], \tag{4.3}$$

with $m \in \mathbb{N}$, $u_1, \ldots, u_m \in \mathbb{R}^d \setminus \{o\}$, $\beta_1, \ldots, \beta_m \in \mathbb{R}$ such that (4.3) has no solution in S, that is,

$$\{x \in S : \langle u_i, x \rangle < \beta_i \ \forall i \in [m]\} = \emptyset. \tag{4.4}$$

For proving $h(\mathcal{F}) \leq f(S)$ it suffices to show that (4.3) has a subsystem with at most $f(S)$ inequalities which has no solution in S. For $j \in [m]$, we say that the jth constraint of (4.3) is redundant if, after the removal of this constraint, the new system still has no solution in S, that is,

$$\{x \in S : \langle u_i, x \rangle < \beta_i \ \forall i \in [m] \setminus \{j\}\} = \emptyset.$$

We say that the jth constraint of the system (4.3) is blocked if

$$\{x \in S \,:\, \langle u_i\,,\, x\rangle < \beta_i \ \forall i \in [m] \setminus \{j\},\ \langle u_j\,,\, x\rangle = \beta_j\} \neq \emptyset.$$

For every $j \in [m]$, the jth constraint is either redundant or blocked by S or otherwise $\beta_j < \beta_j'$, where $\beta_j' \in \mathbb{R}$ is given by

$$\beta_j' := \min\{x \in S \,:\, \langle u_i\,,\, x\rangle < \beta_i \ \forall i \in [m] \setminus \{j\}\}.$$

If the jth constraint is nonredundant and $\beta_j < \beta_j'$, then this constraint becomes blocked if we replace β_j by β_j'. Note that the above operation preserves (4.4) and, furthermore, every constraint which was previously blocked remains blocked. Removing all redundant constraints and consecutively making all the nonredundant constraints blocked, we can modify every system (4.3) to a system

$$\langle u_i\,,\, x\rangle < \gamma_i \qquad \forall i \in I, \tag{4.5}$$

with $I \subseteq [m]$ and $\gamma_i \geq \beta_i$ for all $i \in I$, such that (4.5) has no solution in S and every constraint of (4.5) is blocked. The index set I is nonempty since $S \neq \emptyset$. Every constraint of (4.5) is blocked, and so there exist points $s_i \in S$ $(i \in I)$ such that $\langle u_i\,,\, s_i\rangle = \gamma_i$ for all $i \in I$ and $\langle u_i\,,\, s_j\rangle < \gamma_i$ for all $i, j \in I$ with $i \neq j$. Consider the polyhedron $L = \left\{x \in \mathbb{R}^d \,:\, \langle u_i\,,\, x\rangle \leq \gamma_i \ \forall i \in I\right\}$. By construction, L is S-free. Furthermore, L is d-dimensional since for any $j \in I$ one has

$$s_j - \varepsilon u_j \in \left\{x \in \mathbb{R}^d \,:\, \langle u_i\,,\, x\rangle < \gamma_i \ \forall i \in I\right\} = \operatorname{int}(L)$$

if $\varepsilon > 0$ is sufficiently small. From the properties of the points s_i $(i \in I)$ we conclude that L has precisely $|I|$ facets and each facet of L contains a point of S. Thus, L is a d-dimensional maximal S-free set with $|I|$ facets. Hence $|I| \leq f(S)$. It follows that the subsystem of (4.3) consisting of the inequalities indexed by $i \in I$ has no solution in S. This shows $h(\mathcal{F}) \leq f(S)$ and concludes the proof of $f(S) = h(S)$.

It remains to prove Part II. Consider an arbitrary S-free polyhedron P (not necessarily nonempty or d-dimensional). Then P can be given by

$$P = \left\{x \in \mathbb{R}^d \,:\, \langle u_i\,,\, x\rangle \leq \beta_i \ \forall i \in [m]\right\}$$

for some system (4.3) satisfying (4.4). Applying the arguments used in the proof of $f(S) = h(S)$, we see that P is a subset of a d-dimensional maximal S-free polyhedron L such that the relative interior of each facet of L contains a point of S. This yields the second part of the assertion. ∎

Proof of Theorem 2.1.II. Let $S \subseteq \mathbb{R}^d$ be discrete. In view of Lemma 4.4, we restrict ourselves to the case of infinite S. The inequality $f(S) \leq h(S)$ is provided by Theorem 2.1.I. It remains to show $h(S) \leq f(S)$. Without loss of generality we assume $f(S) < +\infty$, since otherwise the assertion is trivial. Note that $h(S) \geq 1$ since S is nonempty.

We have $h(S) = \lim_{t \to +\infty} h(S_t)$ for $S_t := S \cap [-t, t]^d$. Thus, it suffices to show $h(S_t) \leq f(S)$ for every $t \in \mathbb{N}$. Let us fix an arbitrary $t \in \mathbb{N}$ and let $k := h(S_t)$. By Lemma 4.4.I, $k = h(S_t) = f(S_t)$. Thus, there exists a d-dimensional maximal S_t-free polyhedron L with k facets. We observe that L is not S-free in general. Let $F_1, \ldots, F_k$ be all facets of L.

In the rest of the proof we proceed as follows. Using the fact that S is discrete, one can deduce the existence of an S-free polyhedron P contained in L such that, for every $i \in [k]$, $P \cap F_i$ is $(d-1)$-dimensional and the relative interior of $P \cap F_i$ contains a point of S. It will be shown that every maximal S-free polytope containing P has at least k facets.

By Lemma 4.4.II,

$$\mathrm{relint}(F_i) \cap S_t \neq \emptyset \qquad \forall i \in [k]. \tag{4.6}$$

For each $i \in [k]$ we choose a point $x_i \in \mathrm{relint}(F_i) \cap S_t$. With this choice one obtains $\mathrm{conv}(\{x_1, \dots, x_k\}) \subseteq \mathrm{int}(L) \cup \{x_1, \dots, x_k\}$. Because L is S_t-free, we get

$$[-t, t]^d \cap S \cap \mathrm{conv}(\{x_1, \dots, x_k\}) = S_t \cap \mathrm{conv}(\{x_1, \dots, x_k\}) = \{x_1, \dots, x_k\}.$$

Since S is discrete, we have

$$[-t-\varepsilon, t+\varepsilon]^d \cap S = [-t, t]^d \cap S$$

for a sufficiently small $\varepsilon > 0$. It follows that the polytope

$$P := L \cap [-t-\varepsilon, t+\varepsilon]^d$$

is S-free. By construction, for every $i \in [k]$, $F_i' := F_i \cap [-t-\varepsilon, t+\varepsilon]^d$ is a facet of P and $x_i \in \mathrm{relint}(F_i')$. There exists a maximal S-free set Q with $P \subseteq Q$. Since we assume $f(S) < +\infty$, Q is a polyhedron. For each $i \in [k]$, the $(d-1)$-dimensional polyhedron F_i' is a subset of a facet of Q, since otherwise the point $x_i \in \mathrm{relint}(F_i')$ would be in the interior of Q. Let $i, j \in [k]$ and $i \neq j$. Since F_i and F_j are distinct facets of L, it follows that the sets F_i' and F_j' are subsets of distinct facets of Q. Hence Q has at least k facets. This yields $h(S_t) \leq f(S)$ for every $t \in \mathbb{N}$ and implies $h(S) \leq f(S)$. ∎

The lim inf theorem (Theorem 2.2) and its consequences.

Proof of Theorem 2.2. We first show (2.1). Let $k := \liminf_{t \to +\infty} h(S_t)$. It suffices to consider the case $0 < k < +\infty$. There exists an infinite set $N \subseteq \mathbb{N}$ such that $h(S_t) = k$ for every $t \in N$.

Consider arbitrary convex sets $C_1, \dots, C_m$ in $\mathbb{R}^d$ with $m \in \mathbb{N}$ and assume that $C_1 \cap \dots \cap C_m \cap S = \emptyset$. We show the existence of indices $i_1, \dots, i_k \in [m]$ satisfying $C_{i_1} \cap \dots \cap C_{i_k} \cap S = \emptyset$. By the definition of the Helly number for every $t \in N$ there exist $i_1(t), \dots, i_k(t) \in [m]$ with $C_{i_1(t)} \cap \dots \cap C_{i_k(t)} \cap S_t = \emptyset$. By the finiteness of $[m]$, there exist $i_1, \dots, i_k \in [m]$ such that $i_1 = i_1(t), \dots, i_k = i_k(t)$ for all $t \in N'$, where N' is an infinite subset of N. The monotonicity property $S_t \subseteq S_{t+1}$ for all $t \in \mathbb{N}$ implies $S = \bigcup_{t \in N'} S_t$. Hence, $C_{i_1} \cap \dots \cap C_{i_k} \cap S = \emptyset$.

We now show (2.2). Let $k := \liminf_{t \to +\infty} f(S_t)$. We restrict ourselves to the nontrivial case $k < +\infty$. There exists an infinite set $M \subseteq \mathbb{N}$ such that $f(S_t) = k$ for every $t \in M$. In the degenerate cases one can have $k = -\infty$ or $k = 0$. If $f(S_t) = -\infty$, then $\mathrm{cl}(S_t) = \mathbb{R}^d$ and, consequently, $\mathrm{cl}(S) = \mathbb{R}^d$. This yields $f(S) = -\infty$. In the case $k = 0$ the assertion is trivial since $S_t = \emptyset$ for all $t \in M$ and thus $S = \emptyset$. Let $k \in \mathbb{N}$. We consider an arbitrary d-dimensional S-free set $L \subseteq \mathbb{R}^d$ and show that L is a subset of an S-free polyhedron with at most k facets. For every $t \in M$ there exists a maximal S_t-free set Q_t with $L \subseteq Q_t$. Since $f(S_t) = k$, Q_t is a polyhedron with at most k facets. We proceed similarly to the proof of Lemma 4.1. For every $t \in M$, Q_t can be given by

$$Q_t = \{x \in \mathbb{R}^d : \langle u_{1,t}, x \rangle \leq \beta_{1,t}, \dots, \langle u_{k,t}, x \rangle \leq \beta_{k,t}\},$$

where $u_{1,t}, \dots, u_{k,t} \in \mathbb{R}^d$, $\beta_{1,t}, \dots, \beta_{k,t} \in \mathbb{R}$, and, for every $i \in [k]$ and $t \in M$, $\binom{u_{i,t}}{\beta_{i,t}} \in \mathbb{R}^{d+1}$ is a vector of unit Euclidean length. By compactness, there exists an infinite $N \subseteq M$ such that, for every $i \in [k]$, $u_{i,t}$ converges to some $u_i \in \mathbb{R}^d$ and $\beta_{i,t}$ converges to some $\beta_i \in \mathbb{R}$ as t goes to infinity over points of N. Consider the polyhedron

$$Q = \{x \in \mathbb{R}^d : \langle u_1 , x\rangle \le \beta_1, \dots, \langle u_k , x\rangle \le \beta_k\}.$$

Let us verify the inclusion $L \subseteq Q$. The inclusions $L \subseteq Q_t$ for all $t \in \mathbb{N}$ imply $\langle x , u_{i,t}\rangle \le \beta_{i,t}$ for all $x \in L, i \in [k], t \in \mathbb{N}$. Passing to the limit, as t goes to infinity over points of N, we obtain $\langle x , u_i\rangle \le \beta_i$ for every $i \in [k]$. Hence $L \subseteq Q$. Let us show that Q is S-free. For this it suffices to show that, for every $t \in \mathbb{N}$, Q is S_t-free. Assume that Q is not S_t-free. Then there exists a point $x' \in S_t$ lying in $\operatorname{int}(Q)$, that is, satisfying $\langle u_i , x'\rangle < \beta_i$ for every $i \in [k]$. This points satisfies $\langle u_{i,t'} , x'\rangle < \beta_{i,t'}$ for every $i \in [k]$, where $t' \in N$, $t' \ge t$, is sufficiently large. We have $S_t \subseteq S_{t'}$, and so $x' \in S_{t'} \cap \operatorname{int}(Q_{t'})$, which contradicts the fact that $Q_{t'}$ is $S_{t'}$-free. ∎

Remark 4.5 (alternative proofs of results of Doignon and Lovász). By Theorem 2.1.II, $h(\mathbb{Z}^d) = f(\mathbb{Z}^d)$. This shows that the result $h(\mathbb{Z}^d) = 2^d$ of Doignon and the result $f(\mathbb{Z}^d) = 2^d$ of Lovász are equivalent. Let us give self-contained proofs of these two results. Let $k = f(\mathbb{Z}^d) = h(\mathbb{Z}^d)$. The family of the $\mathbb{Z}^d$-convex sets $\{0,1\}^d \setminus \{z\}$ with $z \in \{0,1\}^d$ contains 2^d elements and has empty intersection. Every proper subfamily of this family has nonempty intersection. Hence $k = h(\mathbb{Z}^d) \ge 2^d$. For deriving $k = f(\mathbb{Z}^d) \le 2^d$ we first apply (2.2), which yields $k \le \liminf_{t\to+\infty} f(S_t)$, where $S_t := \mathbb{Z}^d \cap [-t,t]^d$ for all $t \in \mathbb{N}$. Since $[-t,t]^d$ is bounded and convex, we get $f(S_t) \le 2^d$ by the parity argument, as explained in section 3. Thus, $k \le 2^d$.

Note that, in contrast to the above approach, the existing proofs of $f(\mathbb{Z}^d) = 2^d$ use tools from the geometry of numbers. (See the sources [2] and [8], which use Minkowski's first fundamental theorem and Diophantine approximation, respectively.)

5. Proofs of upper bounds on $f(S)$ and $h(S)$ for structured sets S.

Proof of the theorem on adding a convex set (Theorem 2.3).

Proof of Theorem 2.3. Let us show (2.3). Let $n := \dim(C)$. Consider an appropriate injective affine mapping $\psi : \mathbb{R}^n \to \mathbb{R}^d$ and an n-dimensional convex set $C' \subseteq \mathbb{R}^n$ such that $\psi(C') = C$. We introduce the mapping $\phi : \mathbb{R}^{d+n} \to \mathbb{R}^d$ by $\phi\binom{x}{y} = x + \psi(y)$ for every $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^n$. By construction, $\phi(S \times C') = S + C$. Applying the inequality

$$h(S \times \mathbb{R}^n) \le (n+1)h(S),$$

proven in [5], together with (3.4) and (3.5), we obtain

$$\begin{aligned} h(S + C) = h(\phi(S \times C')) &\le h(S \times C') = h((S \times \mathbb{R}^n) \cap (\mathbb{R}^d \times C')) \\ &\le h(S \times \mathbb{R}^n) \le (n+1)h(S). \end{aligned}$$

Now we derive (2.4). Assume $k := f(S) < +\infty$, since otherwise the assertion is trivial. Consider an arbitrary d-dimensional $(C+S)$-free set $L \subseteq \mathbb{R}^d$; that is, L is d-dimensional, closed, convex and $\operatorname{int}(L) \cap (C+S) = \emptyset$. It suffices to show that L is contained in a maximal $(C+S)$-free polyhedron with at most k facets. Let us first show the equality

$$\operatorname{int}(L) - C = \operatorname{int}(L - C). \tag{5.1}$$

The left-hand side is an open set, since it can be represented as the union of the open sets $\mathrm{int}(L) - c$ with $c \in C$. Hence $\mathrm{int}(L) - C = \mathrm{int}(\mathrm{int}(L) - C) \subseteq \mathrm{int}(L - C)$. Let us show the converse inclusion $\mathrm{int}(L - C) \subseteq \mathrm{int}(L) - C$. Using the additivity of the operator relint (see [25, section 6]) and the d-dimensionality of L, we obtain $\mathrm{int}(L - C) = \mathrm{int}(L) - \mathrm{relint}(C) \subseteq \mathrm{int}(L) - C$.

In view of (5.1), we get

$$\mathrm{int}(L) \cap (C + S) = \emptyset \iff (\mathrm{int}(L) - C) \cap S = \emptyset \iff \mathrm{int}(L - C) \cap S = \emptyset.$$

Thus, $\mathrm{cl}(L - C)$ is S-free, and by this there exists a d-dimensional S-free polyhedron P with at most k facets and $L - C \subseteq P$. The polyhedron P can be given by

$$P = \left\{x \in \mathbb{R}^d : \langle u_1 , x\rangle \le \beta_1, \ldots, \langle u_k , \beta_k\rangle \le \beta_k\right\},$$

where $u_1, \ldots, u_k \in \mathbb{R}^d \setminus \{o\}$ and $\beta_1, \ldots, \beta_k \in \mathbb{R}$. One has $\langle x - c , u_i\rangle \le \beta_i$ for all $x \in L, c \in C$, $i \in [k]$. Hence

$$L \subseteq P' := \left\{x \in \mathbb{R}^d : \langle u_1 , x\rangle \le \beta'_1, \ldots, \langle u_k , x\rangle \le \beta'_k\right\},$$

where

$$\beta'_i := \beta_i + \inf_{c \in C} \langle u_i , c\rangle \qquad \forall i \in [k].$$

It remains to show that P' is $(C + S)$-free. Assume the contrary. Then one can find $x' \in S$ and $c' \in C$ such that $\langle u_i , x' + c'\rangle < \beta_i + \inf_{c \in C} \langle u_i , c\rangle$ for every $i \in [k]$. It follows that $\langle u_i , x'\rangle < \beta_i$ for every $i \in [k]$, and hence P is not S-free, which is in contradiction to the choice of P. ∎

Proofs of Theorems 2.4 and 2.5. In the proof of Theorem 2.4 we shall need properties of d-dimensional maximal S-free sets in the case that S is a union of finitely many polyhedra. Such maximal S-free sets will be discussed in Lemma 5.2. For the proof of Lemma 5.2 we shall need a result on separation of polyhedra, which we formulate in Lemma 5.1. Lemma 5.3 is another auxiliary result used in the proof of Theorem 2.4.

If $P \subseteq \mathbb{R}^d$ is a polyhedron, then every nonempty face F of P can be given as

$$F = F(P, u) := \{x \in P : \langle u , y\rangle \le \langle u , x\rangle \ \forall y \in P\}$$

for an appropriate $u \in \mathbb{R}^d$. The notation $F(P, u)$ is used in Lemma 5.1.

LEMMA 5.1. *Let $P, Q \subseteq \mathbb{R}^d$ be polyhedra such that* $\dim(P) = d$*, $Q \ne \emptyset$, and* $\mathrm{int}(P) \cap Q = \emptyset$*. Assume that for some facet G of P one has* $\mathrm{relint}(G) \cap Q = \emptyset$*. Then there exists a closed halfspace $H \subseteq \mathbb{R}^d$ such that $P \subseteq H$,* $\mathrm{relint}(G) \subseteq \mathrm{int}(H)$*, and* $Q \cap \mathrm{int}(H) = \emptyset$*.*

Proof. It is known that the set $P - Q$ is a polyhedron.

Case 1: $o \notin P - Q$. We can separate o and $P - Q$ by a hyperplane. That is, there exists $u \in \mathbb{R}^d \setminus \{o\}$ such that $\sup_{x \in P - Q} \langle u , x\rangle < 0$. It follows that $\sup_{x \in P} \langle u , x\rangle - \inf_{y \in Q} \langle u , y\rangle < 0$. We can define $H = \left\{x \in \mathbb{R}^d : \langle u , x\rangle \le \beta\right\}$, where $\beta \in \mathbb{R}$ is any value satisfying $\sup_{x \in P} \langle u , x\rangle < \beta < \inf_{y \in Q} \langle u , y\rangle$. For H as above, one has $P \subseteq \mathrm{int}(H)$ and $H \cap Q = \emptyset$.

Case 2: $o \in P - Q$. The set $P - Q$ is d-dimensional since $\dim(P) = d$ and $Q \ne \emptyset$. Observe that $o \in \mathrm{bd}(P - Q)$. In fact, otherwise we would have

$o \in \operatorname{int}(P-Q) = \operatorname{int}(P) - \operatorname{relint}(Q)$, which implies $\operatorname{int}(P) \cap Q \neq \emptyset$, yielding a contradiction. Since $o \in \operatorname{bd}(P-Q)$, there exists $u \in \mathbb{R}^d \setminus \{o\}$ such that $F(P-Q,u)$ is a face of $P-Q$ with $o \in \operatorname{relint}(F(P-Q,u))$. It is known that $F(P-Q,u) = F(P,u) - F(Q,u)$. Taking into account the additivity of the relint operator, we obtain $\operatorname{relint}(F(P-Q,u)) = \operatorname{relint}(F(P,u)-F(Q,u)) = \operatorname{relint}(F(P,u)) - \operatorname{relint}(F(Q,u))$. One has $F(P,u) \neq G$, since otherwise $o \in \operatorname{relint}(G) - Q$, and by this $\operatorname{relint}(G) \cap Q \neq \emptyset$, which contradicts the assumptions. Therefore, we have $F(P,u) \cap \operatorname{relint}(G) = \emptyset$. Let $\beta := \sup_{x \in P} \langle u\,,\, x\rangle = \inf_{y \in Q} \langle u\,,\, y\rangle$ and $H := \left\{x \in \mathbb{R}^d : \langle u\,,\, x\rangle \leq \beta\right\}$. The assertion for H can be verified in a straightforward manner. ∎

We also observe that Lemma 5.1 can be proved algebraically using Motzkin's transposition theorem (a generalization of Farkas' lemma to systems involving strict as well as nonstrict linear inequalities; see [28, Corollary 7.1k]).

LEMMA 5.2. *Let $d, k \in \mathbb{N}$, and let $S := C_1 \cup \cdots \cup C_k$, where $C_1, \ldots, C_k \subseteq \mathbb{R}^d$ are nonempty polyhedra. Let L be a d-dimensional maximal S-free set in $\mathbb{R}^d$. Then L is a polyhedron, and the relative interior of each facet of L contains a point of S.*

Proof. For each $i \in [k]$, L and C_i can be separated by a hyperplane. In view of the maximality of L, it follows that L is a polyhedron with at most k facets. We consider an arbitrary facet F of L and show that $\operatorname{relint}(F) \cap S \neq \emptyset$, arguing by contradiction. Assume that $\operatorname{relint}(F) \cap S = \emptyset$; that is, $\operatorname{relint}(F) \cap C_i = \emptyset$ for every $i \in [k]$. By Lemma 5.1, for every $i \in [k]$ there exists a closed halfspace H_i in $\mathbb{R}^d$ such that $L \subseteq H_i$, $\operatorname{relint}(F) \subseteq \operatorname{int}(H_i)$, and $\operatorname{int}(H_i) \cap C_i = \emptyset$. We define $P := H_1 \cap \cdots \cap H_k$. By construction, P is S-free. From the construction we also get $L \subseteq P$ and $\operatorname{relint}(F) \subseteq \operatorname{int}(P)$, which yields $L \subsetneq P$. This contradicts the maximality of the S-free set L. ∎

LEMMA 5.3. *Let $(C_t)_{t=1}^{+\infty}$ be a sequence of convex sets in $\mathbb{R}^d$, and let $S = \bigcup_{t=1}^{+\infty} C_t$. Let $S' \subseteq \mathbb{R}^d$ be a set satisfying*

$$\bigcup_{t=1}^{+\infty} \operatorname{relint}(C_t) \subseteq S' \subseteq \bigcup_{t=1}^{+\infty} \operatorname{cl}(C_t).$$

Then $f(S) = f(S')$.

Proof. Let $L \subseteq \mathbb{R}^d$ be a d-dimensional closed convex set. We show that L is S-free if and only if L is S'-free. One has

$$\begin{aligned}
S \cap \operatorname{int}(L) = \emptyset &\Longrightarrow C_t \cap \operatorname{int}(L) = \emptyset && \forall t \in \mathbb{N}\\
&\Longrightarrow C_t \subseteq \mathbb{R}^d \setminus \operatorname{int}(L) && \forall t \in \mathbb{N}\\
&\Longrightarrow \operatorname{cl}(C_t) \subseteq \mathbb{R}^d \setminus \operatorname{int}(L) && \forall t \in \mathbb{N}\\
&\Longrightarrow S' \subseteq \mathbb{R}^d \setminus \operatorname{int}(L)\\
&\Longrightarrow S' \cap \operatorname{int}(L) = \emptyset.
\end{aligned}$$

Conversely,

$$\begin{aligned}
S' \cap \operatorname{int}(L) = \emptyset &\Longrightarrow \operatorname{relint}(C_t) \cap \operatorname{int}(L) = \emptyset && \forall t \in \mathbb{N}\\
&\Longrightarrow \operatorname{relint}(C_t) \subseteq \mathbb{R}^d \setminus \operatorname{int}(L) && \forall t \in \mathbb{N}\\
&\Longrightarrow \operatorname{cl}(\operatorname{relint}(C_t)) \subseteq \mathbb{R}^d \setminus \operatorname{int}(L) && \forall t \in \mathbb{N}\\
&\Longrightarrow \operatorname{cl}(C_t) \subseteq \mathbb{R}^d \setminus \operatorname{int}(L) && \forall t \in \mathbb{N}\\
&\Longrightarrow S \subseteq \mathbb{R}^d \setminus \operatorname{int}(L)\\
&\Longrightarrow S \cap \operatorname{int}(L) = \emptyset.
\end{aligned}$$

Thus, we get the assertion. ∎

Proof of Theorem 2.4. Let us prove (2.5). Since D is convex we obtain $h((C+\Lambda)\cap D)) \le h(C+\Lambda)$. By (2.3) we have $h(C+\Lambda) \le (\dim(C)+1)h(\Lambda)$. Doignon's theorem yields $h(\Lambda) = 2^{\mathrm{rank}(\Lambda)}$. This implies (2.5).

Now let us show (2.6). Let $S := (C+\Lambda)\cap D$, and let $L \subseteq \mathbb{R}^d$ be a d-dimensional maximal S-free set. We prove that L is a polyhedron with at most $2^{\mathrm{rank}(\Lambda)}$ facets. Without loss of generality let $S \ne \emptyset$. Consider the following two cases.

Case 1: *C and D are both polytopes.* In this case S is a union of finitely many polytopes. By Lemma 5.2 the set L is a polyhedron. Let $F_1,\ldots,F_m$ be all facets of L, where $m \in \mathbb{N}_0$. By Lemma 5.2, for every $i \in [m]$, the relative interior of F_i contains a point of the form $x_i + y_i$, where $x_i \in \Lambda$, $y_i \in C$, and $x_i + y_i \in D$. If $m > 2^{\mathrm{rank}(\Lambda)}$, then by the parity argument (see section 3) one has $\frac{1}{2}(x_i + x_j) \in \Lambda$ for some $1 \le i < j \le m$. Using the convexity of C, D, and L, and the fact that $x_i + y_i$ and $x_j + y_j$ lie in the relative interior of two different facets of L, we deduce $\frac{1}{2}((x_i+y_i)+(x_j+y_j)) \in \mathrm{int}(L)\cap S$. Hence L is not S-free, which is a contradiction. It follows that $m \le 2^{\mathrm{rank}(\Lambda)}$.

Case 2: *C or D is not a polytope.* We introduce a sequence $(z_i)_{i=1}^{+\infty}$ such that $\{z_i : i \in \mathbb{N}\} = \{z \in \Lambda : (z+C)\cap D \ne \emptyset\}$. Then

$$S = \bigcup_{i=1}^{\infty} Q_i,$$

where $Q_i := (z_i + C)\cap D \ne \emptyset$. For each Q_i we fix a countable dense subset $\{q_{i,j} : j \in \mathbb{N}\}$ of $\mathrm{relint}(Q_i)$. With this choice one has $\mathrm{relint}(Q_i) = \mathrm{conv}(\{q_{i,j} : j \in \mathbb{N}\})$. For every $t \in \mathbb{N}$ we define

$$S_t := (\Lambda + C_t)\cap D_t,$$

where

$$\begin{aligned} C_t &:= \mathrm{conv}(\{q_{i,j} - z_i : i,j \in [t]\}),\\ D_t &:= \mathrm{conv}(\{q_{i,j} : i,j \in [t]\}). \end{aligned}$$

By construction, one has $C_t \subseteq C_{t+1} \subseteq C$ and $D_t \subseteq D_{t+1} \subseteq D$ for every $t \in \mathbb{N}$. The latter also implies $S_t \subseteq S_{t+1} \subseteq S$ for every $t \in \mathbb{N}$. We have

$$\begin{aligned} \bigcup_{i=1}^{\infty} \mathrm{relint}(Q_i) &= \bigcup_{i=1}^{\infty} \mathrm{conv}(\{q_{i,j} : j \in \mathbb{N}\}) = \bigcup_{i=1}^{\infty}\bigcup_{t=i}^{\infty} \mathrm{conv}(\{q_{i,j} : j \in [t]\})\\ &= \bigcup_{t=1}^{\infty}\bigcup_{i=1}^{t} \mathrm{conv}(\{q_{i,j} : j \in [t]\}) \subseteq \bigcup_{t=1}^{\infty}(\Lambda + C_t)\cap D_t = \bigcup_{t=1}^{\infty} S_t \subseteq S, \end{aligned}$$

where above we use the equality

$$\mathrm{conv}(\{q_{i,j} : j \in \mathbb{N}\}) = \bigcup_{t=i}^{\infty} \mathrm{conv}(\{q_{i,j} : j \in [t]\}) \qquad \forall i \in \mathbb{N}. \tag{5.2}$$

Note that (5.2) follows directly from Carathéodory's theorem (see [27, Theorem 1.1.4]).

Let $S' := \bigcup_{t=1}^{\infty} S_t$. By Lemma 5.3, one has $f(S') = f(S)$. By Theorem 2.2, $f(S') \le \liminf_{t\to+\infty} f(S_t)$. Applying the assertion obtained in Case 1, we get $f(S_t) \le 2^{\mathrm{rank}(\Lambda)}$ for every $t \in \mathbb{N}$. The latter yields $f(S) = f(S') \le 2^{\mathrm{rank}(\Lambda)}$. ∎

Proof of Theorem 2.5. Observing that $\mathbb{Z}^d \times \mathbb{R}^n = \mathbb{Z}^d \times \{o\} + \{o\} \times \mathbb{R}^n$, where $\mathbb{Z}^d \times \{o\}$ is a lattice and $\{o\} \times \mathbb{R}^n$ is a linear space (and thus, a convex set), we see that (2.7) is a direct consequence of (2.6). The equality case $f(\mathbb{Z}^d \times \mathbb{R}^n) = 2^d$ follows from (3.6) and $f(\mathbb{Z}^d) = 2^d$. ∎

Acknowledgments. I thankfully acknowledge the help of the anonymous referees in improving the structure of the manuscript and presentation of some of the arguments (most notably, the compactness argument in the proofs of Lemma 4.1 and Theorem 2.2). Moreover, the referees pointed out a mistake in the proof of (2.7) in an earlier version of this paper. This indication has been a source of motivation for my further research on the topic of this paper. As a result, the text of the manuscript has been extended with Theorems 2.1.II, 2.2, 2.3, and 2.4.

REFERENCES

[1] K. Andersen, Q. Louveaux, R. Weismantel, and L. A. Wolsey, *Inequalities from two rows of a simplex tableau*, in Integer Programming and Combinatorial Optimization, Lecture Notes in Comput. Sci. 4513, Springer, Berlin, 2007, pp. 1–15.

[2] G. Averkov, *A proof of Lovász's theorem on maximal lattice-free sets*, Beitr. Algebra Geom., 54 (2013), pp. 105–109.

[3] G. Averkov and Ch. Wagner, *Inequalities for the lattice width of lattice-free convex sets in the plane*, Beitr. Algebra Geom., 52 (2012), pp. 1–23.

[4] G. Averkov, Ch. Wagner, and R. Weismantel, *Maximal lattice-free polyhedra: Finiteness and an explicit description in dimension three*, Math. Oper. Res., 36 (2011), pp. 721–742.

[5] G. Averkov and R. Weismantel, *Transversal numbers over subsets of linear spaces*, Adv. Geom., 12 (2012), pp. 19–28.

[6] E. Balas, *Intersection cuts—A new type of cutting planes for integer programming*, Oper. Res., 19 (1971), pp. 19–39.

[7] I. Bárány and J. Matoušek, *A fractional Helly theorem for convex lattice sets*, Adv. Math., 174 (2003), pp. 227–235.

[8] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli, *Maximal lattice-free convex sets in linear subspaces*, Math. Oper. Res., 35 (2010), pp. 704–720.

[9] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli, *Minimal inequalities for an infinite relaxation of integer programs*, SIAM J. Discrete Math., 24 (2010), pp. 158–168.

[10] D. E. Bell, *A theorem concerning the integer lattice*, Stud. Appl. Math., 56 (1977), pp. 187–188.

[11] V. Boltyanski, H. Martini, and P. S. Soltan, *Excursions into Combinatorial Geometry*, Universitext, Springer-Verlag, Berlin, 1997.

[12] M. Conforti, G. Cornuéjols, and G. Zambelli, *Corner polyhedron and intersection cuts*, Surveys Oper. Res. Management Sci., 16 (2011), pp. 105–120.

[13] A. Del Pia and R. Weismantel, *On convergence in mixed integer programming*, Math. Program. Ser. A, 135 (2012), pp. 397–412.

[14] S. S. Dey and L. A. Wolsey, *Constrained infinite group relaxations of MIPs*, SIAM J. Optim., 20 (2010), pp. 2890–2912.

[15] J.-P. Doignon, *Convexity in crystallographical lattices*, J. Geometry, 3 (1973), pp. 71–85.

[16] R. Fukasawa and O. Günlük, *Strengthening lattice-free cuts using non-negativity*, Discrete Optim., 8 (2011), pp. 229–245.

[17] R. J. Gardner and P. Gronchi, *A Brunn-Minkowski inequality for the integer lattice*, Trans. Amer. Math. Soc., 353 (2001), pp. 3995–4024.

[18] R. J. Gardner, P. Gronchi, and Ch. Zong, *Sums, projections, and sections of lattice sets, and the discrete covariogram*, Discrete Comput. Geom., 34 (2005), pp. 391–409.

[19] P. M. Gruber, *Convex and Discrete Geometry*, Grundl. Math. Wiss. 336, Springer, Berlin, 2007.

[20] P. M. Gruber and C. G. Lekkerkerker, *Geometry of Numbers*, 2nd ed., North-Holland Math. Lib. 37, North–Holland, Amsterdam, 1987.

[21] P. M. Gruber and J. M. Wills, eds., *Handbook of Convex Geometry, Vol. A, B*, North–Holland, Amsterdam, 1993.

[22] L. Lovász, *Geometry of numbers and integer programming*, in Mathematical Programming (Tokyo, 1988), Math. Appl. (Japanese Ser.) 6, SCIPRESS, Tokyo, 1989, pp. 177–201.

[23] D. A. Morán R. and S. S. Dey, *On maximal S-free convex sets*, SIAM J. Discrete Math., 25 (2011), pp. 379–393.

[24] Sh. Onn, *On the geometry and computational complexity of Radon partitions in the integer lattice*, SIAM J. Discrete Math., 4 (1991), pp. 436–447.

[25] R. T. Rockafellar, *Convex Analysis*, Princeton Math. Ser. 28, Princeton University Press, Princeton, NJ, 1970.

[26] H. E. Scarf, *An observation on the structure of production sets with indivisibilities*, Proc. Natl. Acad. Sci. USA, 74 (1977), pp. 3637–3641.

[27] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia Math. Appl. 44, Cambridge University Press, Cambridge, 1993.

[28] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley-Intersci. Ser. Discrete Math., John Wiley & Sons, Chichester, UK, 1986.

[29] M. J. Todd, *The Number of Necessary Constraints in an Integer Program: A New Proof of Scarf's Theorem*, Technical Report 355, School of Operations Research and Industrial Engineering, Cornell University, Ithaca, NY, 1977.

[30] M. L. J. van de Vel, *Theory of Convex Structures*, North-Holland Math. Lib. 50, North–Holland, Amsterdam, 1993.

[31] R. Weismantel, *A cutting plane theory for mixed integer optimization*, in Proceedings of the International Congress of Mathematicians, Volume IV (New Delhi), Hindustan Book Agency, Gurgaon, India, 2010, pp. 2996–3007.