On the Unique-lifting Property

Gennadiy Averkov[1] and Amitabh Basu[2]

[1] Institute of Mathematical Optimization, Faculty of Mathematics, University of Magdeburg
[2] Department of Applied Mathematics and Statistics, The Johns Hopkins University

Abstract. We study the uniqueness of minimal liftings of cut generating functions obtained from maximal lattice-free polytopes. We prove a basic invariance property of unique minimal liftings for general maximal lattice-free polytopes. This generalizes a previous result by Basu, Cornuéjols and Köppe [3] for *simplicial* maximal lattice-free polytopes, thus completely settling this fundamental question about lifting. We also extend results from [3] for minimal liftings in maximal lattice-free simplices to more general polytopes. These nontrivial generalizations require the use of deep theorems from discrete geometry and geometry of numbers, such as the Venkov-Alexandrov-McMullen theorem on translative tilings, and McMullen's characterization of zonotopes.

1 Introduction

Overview and Motivation. The idea of *cut generating functions* has emerged as a major theme in recent research on cutting planes for mixed-integer linear programming. The main object of study is the following family of mixed-integer sets:

$$X_f(R,P) = \{(s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ : f + Rs + Py \in \mathbb{Z}^n\},$$

where $f \in \mathbb{R}^n$, and $R \in \mathbb{R}^{n\times k}$, $P \in \mathbb{R}^{n\times \ell}$. We denote the columns of matrices R and P by r^i, $i = 1,\dots,k$ and p^j, $j = 1,\dots,\ell$ respectively. For a fixed $f \in \mathbb{R}^n \setminus \mathbb{Z}^n$, a *cut generating pair* (ψ,π) *for* f is a pair of functions $\psi,\pi : \mathbb{R}^n \to \mathbb{R}$ such that $\sum_{i=1}^k \psi(r^i)s_i + \sum_{j=1}^\ell \pi(p^j)y_j \geq 1$ is a valid inequality (cutting plane) for $X_f(R,P)$, for all matrices R and P. This model and the idea of cut generating pairs arose in the work of Gomory and Johnson from the 70s. We refer the reader to [5] for a survey of the intense research activity this area has seen in the last 5-6 years.

A very important class of cut generating pairs is obtained using the *gauge function* of *maximal lattice-free polytopes* in $\mathbb{R}^n$. These are convex polytopes $B \subseteq \mathbb{R}^n$ such that $\operatorname{int}(B) \cap \mathbb{Z}^n = \emptyset$ and B is inclusion-wise maximal with this property. Given a maximal lattice-free polytope B such that $f \in \operatorname{int}(B)$, one can express $B = \{x \in \mathbb{R}^n : a^i \cdot (x-f) \leq 1 \quad \forall i \in I\}$. One then obtains a cut generating pair for f by setting $\psi(r) = \max_{i\in I} a^i \cdot r$ for all $r \in \mathbb{R}^n$ (this is known as the gauge of $B - f$), and using any π such that (ψ,π) is a cut generating pair. One commonly used π is defined by $\pi(r) = \min_{w\in\mathbb{Z}^n} \psi(r+w)$ for all $r \in \mathbb{R}^n$. It can be shown that (ψ,π) thus defined forms a valid cut generating pair.

Given a particular maximal lattice-free polytope B with $f \in \mathrm{int}(B)$, it is generally possible to find many different functions π such that (ψ, π) is a cut generating pair for f, when ψ is fixed to be the gauge of $B - f$. The different possible π's are called *liftings* and the strongest cutting planes are obtained from *minimal liftings*, i.e., π such that for every lifting π', the inequality $\pi' \leq \pi$ implies $\pi' = \pi$.

Closed-form formulas for cut generating pairs (ψ, π) are highly desirable for computational purposes. This is the main motivation for considering the special class of cut generating pairs obtained from the gauge of maximal lattice-free polytopes and their liftings. The gauge is given by the very simple formula $\psi(r) = \max_{i \in I} a^i \cdot r$, and the hope is that simple formulas can be found for its minimal liftings as well. In this regard, the following results are particularly useful. Dey and Wolsey established the following theorem in [6] for $n = 2$.

Theorem 1. (Theorems 5 and 6 in [4], Theorem 4 in [2].) *Let ψ be the gauge of $B - f$, where B is a maximal lattice-free polytope and $f \in \mathrm{int}(B)$. There exists a compact subset $R'(f, B) \subseteq \mathbb{R}^n$ such that $R'(f, B)$ has nonempty interior, and for every minimal lifting π, $\pi(r) = \psi(r)$ if and only if $r \in R'(f, B)$. Moreover, for all minimal liftings π, $\pi(r) = \pi(r + w)$ for every $w \in \mathbb{Z}^n$, $r \in \mathbb{R}^n$.*

This theorem shows that for a "fat" periodic region $R'(f, B) + \mathbb{Z}^n$, we have a closed-form formula for all minimal liftings (using the formula for ψ). In particular, when all the columns of the matrix P are in $R'(f, B) + \mathbb{Z}^n$, we can efficiently find the cutting plane $\sum_{i=1}^k \psi(r^i)s_i + \sum_{j=1}^{\ell} \pi(p^j)y_j \geq 1$ from B. Moreover, the above theorem shows that if $R'(f, B) + \mathbb{Z}^n = \mathbb{R}^n$, then there is a unique minimal lifting π. The following theorem from [2] establishes the necessity of this condition.

Theorem 2. (Theorem 5 in [2].) *Let ψ be the gauge of $B - f$, where B is a maximal lattice-free polytope and $f \in \mathrm{int}(B)$. Then ψ has a unique minimal lifting if and only if $R'(f, B) + \mathbb{Z}^n = \mathbb{R}^n$. ($R'(f, B)$ is the region from Theorem 1)*

The above theorems provide a geometric perspective on *sequence independent lifting* and *monoidal strengthening* that started with the work of Balas and Jeroslow [1]. In this context, characterizing pairs f, B with unique minimal liftings becomes an important question in the cut generating function approach to cutting planes. Recent work and related literature can be found in [2–7].

Our Contributions We will denote the convex hull, affine hull, interior and relative interior of a set X using $\mathrm{conv}(X)$, $\mathrm{aff}(X)$, $\mathrm{int}(X)$ and $\mathrm{relint}(X)$ respectively. We call an n-dimensional polytope S in $\mathbb{R}^n$ a *spindle* if $S = (b^1 + C_1) \cap (b^2 + C_2)$ is the intersection of two translated polyhedral cones $b^1 + C_1$ and $b^2 + C_2$, such that the apex $b^1 \in \mathrm{int}(b_2 + C_2)$ and the apex $b^2 \in \mathrm{int}(b_1 + C_1)$.

Let B be a maximal lattice-free polytope in $\mathbb{R}^n$ and let $f \in \mathrm{int}(B)$. By $\mathrm{Fct}(B)$ we denote the set of all facets of B. With each F and f we associate the set $P_F(f) := \mathrm{conv}(\{f\} \cup F)$. With each F and each $z \in F \cap \mathbb{Z}^n$ we associate the spindle $S_{F,z}(f) := P_F(f) \cap (z + f - P_F(f))$. Furthermore, we define $R_F(f) :=$

$\bigcup_{z\in F\cap\mathbb{Z}^n} S_{F,z}(f)$, the union of all spindles arising from the facet F, and the *lifting region* $R(f,B) := \bigcup_{F\in\mathrm{Fct}(B)} R_F(f)$ associated with the point f.

One of the main results in [2] was to establish that $R(f,B) - f$ is precisely the region $R'(f,B)$ described in Theorem 1. In light of Theorem 2, we say B *has the unique-lifting property with respect to* f if $R(f,B)+\mathbb{Z}^n = \mathbb{R}^n$, and B *has the multiple-lifting property with respect to* f *if* $R(f,B)+\mathbb{Z}^n \neq \mathbb{R}^n$. We summarize our main contributions in this paper.

(i) A natural question arises: is it possible that B has the unique-lifting property with respect to one $f_1 \in \mathrm{int}(B)$, and has the multiple-lifting property with respect to another $f_2 \in \mathrm{int}(B)$? This question was investigated in [3] and the main result was to establish that this cannot happen when B is a *simplicial* polytope. We prove this for general maximal lattice-free polytopes without the simpliciality assumption:

Theorem 3. (Unique-lifting invariance theorem.) *Let B be any maximal lattice-free polytope. For all $f_1, f_2 \in \mathrm{int}(B)$, B has the unique-lifting property with respect to f_1 if and only if B has the unique-lifting property with respect to f_2.*

In view of this result, we can speak about the unique-lifting property of B, without reference to any $f \in \mathrm{int}(B)$.

(ii) To prove Theorem 3, we first show that the volume of $R(f,B)/\mathbb{Z}^n$ (the region $R(f,B)$ sent onto the torus $\mathbb{R}^n/\mathbb{Z}^n$) is an affine function of f (Theorem 4). This is also an extension of the corresponding theorem from [3] for simplicial B. Besides handling the general case, our proof is also significantly shorter and more elegant. We develop a tool for computing volumes on the torus, which enables us to circumvent a complicated inclusion-exclusion argument from [3] (see pages 349-350 in [3]). We view this volume computation tool as an important technical contribution of this paper.

(iii) A major contribution of [3] was to characterize the unique-lifting property for a special class of simplices. We generalize all the results from [3] to a broader class of polytopes, called *pyramids* in Sections 3 and 5 (see Remark 1 and Theorems 5 and 11). For our generalizations, we build tools in Section 4 that invoke non-trivial theorems from the geometry of numbers and discrete geometry, such as the Venkov-Alexandrov-McMullen theorem for translative tilings in $\mathbb{R}^n$, McMullen's characterizations of polytopes with centrally symmetric faces [9] and the combinatorial structure of zonotopes.

(iv) Our techniques give an iterative procedure to construct new families of polytopes with the unique-lifting property in every dimension $n \in \mathbb{N}$. This vastly expands the known list of polytopes with the unique-lifting property. See Remarks 1, 2 and 3.

2 Invariance Theorem on the Uniqueness of Lifting

We consider the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, equipped with the natural Haar measure that assigns volume 1 to $\mathbb{R}^n/\mathbb{Z}^n$. It is clear that a compact set $X \subseteq \mathbb{R}^n$ covers

$\mathbb{R}^n$ by lattice translations, i.e., $X + \mathbb{Z}^n = \mathbb{R}^n$, if and only if $\mathrm{vol}_{\mathbb{T}^n}(X/\mathbb{Z}^n) = 1$, where $\mathrm{vol}_{\mathbb{T}^n}(X/\mathbb{Z}^n)$ denotes the volume of $X/\mathbb{Z}^n$ in $\mathbb{T}^n$.

Theorem 3 will follow as a direct consequence of the following result.

Theorem 4. *The function $f \mapsto \mathrm{vol}_{\mathbb{T}_n}(R(f,B)/\mathbb{Z}^n)$, acting from $\mathrm{int}(B)$ to $\mathbb{R}$, is the restriction of an affine function.*

When B is clear from context, we write $R(f)$ instead of $R(f,B)$.

Lemma 1. *Let $F_1, F_2 \in \mathrm{Fct}(B)$ and let $z_i \in \mathrm{relint}(F_i) \cap \mathbb{Z}^n$ for $i \in \{1,2\}$. Suppose $x_1 \in \mathrm{int}(S_{F_1,z_1}(f))$ and $x_2 \in \mathrm{int}(S_{F_2,z_2}(f))$ be such that $x_1 - x_2 \in \mathbb{Z}^n$. Then $F_1 = F_2$ and the point $x_1 - x_2$ lies in the linear subspace parallel to the hyperplane $\mathrm{aff}(F_1) = \mathrm{aff}(F_2)$. Thus, $x_1 - x_2 \in \mathrm{aff}(F_1 - F_1)$.*

Proof. For $i \in \{1,2\}$, if f, x_i and z_i do not lie on a common line, we introduce the two-dimensional affine space $A_i := \mathrm{aff}\{f, x_i, z_i\}$. Otherwise choose A_i to be any two-dimensional affine space containing f, x_i and z_i. The set $T_i := P_{F_i}(f) \cap A_i$ is a triangle, whose one vertex is f. We denote the other two vertices by a_i and b_i. Observe that a_i, b_i are on the boundary of facet F_i such that the open interval $(a_i, b_i) \subseteq \mathrm{relint}(F_i)$. Since z_i lies on the line segment connecting a_i, b_i and $z_i \in \mathrm{relint}(F_i)$, there exists $0 < \lambda_i < 1$ such that $z_i = \lambda_i a_i + (1-\lambda_i) b_i$. Since $x_i \in \mathrm{int}(S_{F_i,z_i}(f))$, there exist $0 < \mu_i, \alpha_i, \beta_i < 1$ such that $x_i = \mu_i f + \alpha_i a_i + \beta_i b_i$ and $\mu_i + \alpha_i + \beta_i = 1$. Also observe that $x_i \in \mathrm{relint}(T_i \cap (z_i + f - T_i))$. Therefore, $\alpha_i < \lambda_i$ and $\beta_i < 1 - \lambda_i$.

Consider first the case $\mu_1 \geq \mu_2$. In this case, we consider the integral point $z_2 + x_1 - x_2 \in \mathbb{Z}^n$.

$$\begin{aligned} z_2 + x_1 - x_2 &= \lambda_2 a_2 + (1-\lambda_2) b_2 + \mu_1 f + \alpha_1 a_1 + \beta_1 b_1 - \mu_2 f - \alpha_2 a_2 - \beta_2 b_2 \\ &= (\mu_1 - \mu_2) f + (\lambda_2 - \alpha_2) a_2 + (1 - \lambda_2 - \beta_2) b_2 + \alpha_1 a_1 + \beta_1 b_1 \end{aligned}$$

Since $(\mu_1 - \mu_2) + (\lambda_2 - \alpha_2) + (1 - \lambda_2 - \beta_2) + \alpha_1 + \beta_1 = 1$, and each of the terms in the sum are nonnegative, $z_2 + x_1 - x_2$ is a convex combination of points in B. Since B has no point from $\mathbb{Z}^n$ in its interior, and $f \in \mathrm{int}(B)$, the coefficient of f in the above expression must be 0. Thus, $\mu_1 = \mu_2$. So, $z_2 + x_1 - x_2 = (\lambda_2 - \alpha_2) a_2 + (1 - \lambda_2 - \beta_2) b_2 + \alpha_1 a_1 + \beta_1 b_1$, where all coefficients are strictly positive. Since $(a_i, b_i) \subseteq \mathrm{relint}(F_i)$, if $F_1 \neq F_2$, then $z_2 + x_1 - x_2 \in \mathrm{int}(B)$ leading to a contradiction to the fact that B is lattice-free. Thus, $F_1 = F_2$. $\mu_1 = \mu_2$ implies that $x_1 - x_2 \in \mathrm{aff}(F_1 - F_1)$.

The case $\mu_1 \leq \mu_2$ is similar with the same analysis performed on $z_1 + x_2 - x_1$.

An analytical formula for volume on the torus. Let R be a compact subset of $\mathbb{R}^n$. Analytically we can represent R by its indicator function $\mathbf{1}_R$ (which is defined to be equal to 1 on R and equal to 0 outside R). So, the volume of R in $\mathbb{R}^n$ is just the integral $\int_{\mathbb{R}^n} \mathbf{1}_R(x)\,\mathrm{d}\,x$. Of course, $\int_{\mathbb{R}^n} \mathbf{1}_R(x)$ is in general not an appropriate expression for $\mathrm{vol}_{\mathbb{T}^n}(R/\mathbb{Z}^n)$ because in this integral we overcount those points $x \in R$ for which there exists another point $y \in R$ with $x - y \in \mathbb{Z}^n$, i.e., $x - y$ is a point in $(R - R) \cap \mathbb{Z}^n$. Now, the function

$$c_R := \sum_{z \in (R-R)\cap \mathbb{Z}^n} \mathbf{1}_{R-z} \tag{1}$$

is precisely the function which describes whether or not we do overcounting and also how much overcounting we actually do. It can be checked that

$$c_R(x) = |\,\{y \in R \,:\, x \equiv y \,(\operatorname{mod} \mathbb{Z}^n)\}\,| = |(R-x)\cap \mathbb{Z}^n| = |R \cap (x+\mathbb{Z}^n)|.$$

Since $\mathbb{Z}^n$ is discrete and R is compact, $|c_R(x)|$ is a finite natural number for all $x \in R$. If $c_R(x) = 1$ for some $x \in R$, then there is no overcounting. If $c_R(x) > 1$, then there is an overcounting and the number shows how much overcounting has been done for the particular point. To remove the effect of overcounting, we need to divide by $c_R(x)$. Thus, we have

Lemma 2. *Let $R \subseteq \mathbb{R}^n$ be a compact set with nonempty interior. Then*

$$\operatorname{vol}_{\mathbb{T}^n}(R/\mathbb{Z}^n) = \int_R \frac{\mathrm{d}\,x}{c_R(x)}. \tag{2}$$

That is, for every $t \in R/\mathbb{Z}^n$, each element of $X_t := \{x \in R : x \equiv t \,(\operatorname{mod} \mathbb{Z}^n)\}$ is counted with the weight $1/|X_t|$ and by this $t \in R/\mathbb{Z}^n$ is counted exactly once (no overcounting!).

We first observe the following property of the function c_R.

Lemma 3. *Let $F \in \operatorname{Fct}(B)$ and let $R = R_F(f)$. Let T be an invertible affine linear map, i.e., $Tx = Lx + t$ for some invertible linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ and $t \in \mathbb{R}^n$. Suppose L leaves the linear subspace parallel to F unchanged, i.e., $Lz = z$ for all $z \in \operatorname{aff}(F-F)$. Then for all $y \in \mathbb{R}^n$,*

$$c_{TR}(Ty) \geq c_R(y).$$

Proof. For any $z \in \operatorname{aff}(F-F)$ and any $y \in \mathbb{R}^n$, $\mathbf{1}_{TR-z}(Ty) = 1 \Leftrightarrow Ty \in TR - z \Leftrightarrow y \in R - L^{-1}z \Leftrightarrow y \in R - z \Leftrightarrow \mathbf{1}_{R-z}(y) = 1$. Therefore,

$$\mathbf{1}_{TR-z}(Ty) = \mathbf{1}_{R-z}(y). \tag{3}$$

Also, by Lemma 1, the set $(R-R)\cap\mathbb{Z}^n$ is contained $\operatorname{aff}(F-F)$. Thus, $(R-R)\cap\mathbb{Z}^n \subseteq (TR-TR)\cap\mathbb{Z}^n$. Therefore,

$$\begin{aligned} c_{TR}(Ty) &= \textstyle\sum_{z\in(TR-TR)\cap\mathbb{Z}^n} \mathbf{1}_{TR-z}(Ty) & \\ &\geq \textstyle\sum_{z\in(R-R)\cap\mathbb{Z}^n} \mathbf{1}_{TR-z}(Ty) & \text{We are dropping nonnegative terms} \\ &= \textstyle\sum_{z\in(R-R)\cap\mathbb{Z}^n} \mathbf{1}_{R-z}(y) & \text{Using (3)} \\ &= c_R(y) & \end{aligned}$$

An easy technical lemma:

Lemma 4. *Let $g : D \to \mathbb{R}$ be a function defined on a subset $D \subseteq \mathbb{R}^n$. Let $M : D \to \mathbb{R}$ be an affine linear map from D to $\mathbb{R}$ that is not identically* 0*, i.e., there exists $a \in \mathbb{R}^n, b \in \mathbb{R}$ such that $Mx = a \cdot x + b$ for all $x \in D$. Further,*

$$g(x)Mx' = g(x')Mx$$

for all $x, x' \in D$. Then g is an affine linear map on D.

Proof. Fix $x_0 \in D$ such that $Mx_0 \neq 0$. Then for any $x \in D$, $g(x) = \frac{g(x_0)}{Mx_0} Mx$. Since M is affine, this shows that g is affine.

We finally give the proof of Theorem 4.

Proof (Proof of Theorem 4). First we observe that

$$\mathrm{vol}_{\mathbb{T}^n}(R(f)/\mathbb{Z}^n) = \sum_{F \in \mathrm{Fct}(B)} \mathrm{vol}_{\mathbb{T}^n}(R_F(f)/\mathbb{Z}^n)$$

because for two distinct facets F and F', $\mathrm{int}(R_F(f))$ and $\mathrm{int}(R_{F'}(f))$ do not intersect modulo $\mathbb{Z}^n$ by Lemma 1. Therefore, it suffices to show that for a fixed F, $\mathrm{vol}_{\mathbb{T}^n}(R_F(f)/\mathbb{Z}^n)$ is an affine function in f. Consider $f, f' \in \mathrm{int}(B)$. Fix $v_1, \ldots, v_n \in F$ that are affinely independent. Let A be the matrix formed by the columns $v_1 - f, \ldots, v_n - f$ and A' be the matrix formed by the columns $v_1 - f', \ldots, v_n - f'$. Let $L = A^{-1}A'$ and let $t = -Lf + f'$. Then the invertible affine linear map $T' : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T'x = Lx + t$ maps v_i to v_i for all $i = 1, \ldots, n$, and f to f'. Thus, $T'R_F(f) = R_F(f')$. Moreover, L (therefore L^{-1}) leaves $\mathrm{aff}(F - F)$ unchanged. Note that the Jacobian of T' is given by $|\det(L)|$. Applying Lemma 3 with $R = R_F(f)$ and $T = T'$, and again with $R = R_F(f')$ and $T = T'^{-1}$, we get $c_{T'R_F(f)}(T'y) = c_{R_F(f)}(y)$ for all $y \in \mathbb{R}^n$. Then,

$$\begin{aligned}
\mathrm{vol}_{\mathbb{T}^n}(R_F(f')/\mathbb{Z}^n) &= \mathrm{vol}_{\mathbb{T}^n}(T'R_F(f)/\mathbb{Z}^n) \\
&= \textstyle\int_{T'R_F(f)} \frac{\mathrm{d}\,x}{c_{T'R_F(f)}(x)} \\
&= \textstyle\int_{R_F(f)} |\det(L)| \frac{\mathrm{d}\,y}{c_{T'R_F(f)}(T'y)} \\
&= \textstyle\int_{R_F(f)} |\det(L)| \frac{\mathrm{d}\,y}{c_{R_F(f)}(y)} \\
&= |\det(L)| \textstyle\int_{R_F(f)} \frac{\mathrm{d}\,y}{c_{R_F(f)}(y)} \\
&= |\det(L)|\, \mathrm{vol}_{\mathbb{T}^n}(R_F(f)/\mathbb{Z}^n)
\end{aligned}$$

where the second and last equalities follow from Lemma 2, the third equality follows from the change of variable $y := T'^{-1}x$. Since $|\det(L)| = \frac{|\det(A')|}{|\det(A)|}$, we have

$$\mathrm{vol}_{\mathbb{T}^n}(R_F(f')/\mathbb{Z}^n)|\det(A)| = \mathrm{vol}_{\mathbb{T}^n}(R_F(f)/\mathbb{Z}^n)|\det(A')|. \tag{4}$$

Finally, observe that $\det(A)$ is equal to the determinant of the $(n+1) \times (n+1)$ matrix whose first n rows are formed by the column vectors $v_1, \ldots, v_n, f$ and the last row is the row vector of all 1's. Similarly, $\det(A')$ is given by the determinant of the $(n+1) \times (n+1)$ matrix whose first n rows are formed by the column vectors $v_1, \ldots, v_n, f'$ and the last row is the row vector of all 1's. Hence, there exists a affine linear map $M : \mathbb{R}^n \to \mathbb{R}$ such that $\det(A) = Mf$ and $\det(A') = Mf'$. Thus, (4) and Lemma 4 combine to show that $\mathrm{vol}_{\mathbb{T}^n}(R_F(f)/\mathbb{Z}^n)$ is an affine function of $f \in \mathrm{int}(B)$.

Theorem 4 implies the following.

Corollary 1 *Let B be a maximal lattice-free polytope in $\mathbb{R}^n$. Then the set $\{\, f \in B : \mathrm{vol}_n(R(f)/\mathbb{Z}^n) = 1 \,\}$ is a face of B.*

Proof. Since $\mathrm{vol}_n(R(f)/\mathbb{Z}^n)$ is always at most 1, the value 1 is a maximum value for the function $\mathrm{vol}_n(R(f)/\mathbb{Z}^n)$. By Theorem 4, optimizing this function over B is a linear program and hence the optimal set is a face of B.

Proof (Proof of Theorem 3). Corollary 1 implies Theorem 3. Indeed, if the set $\{ f \in B : \mathrm{vol}_n(R(f)/\mathbb{Z}^n) = 1 \}$ is B, then $R(f) + \mathbb{Z}^n = \mathbb{R}^n$ for all $f \in B$ and for all $f \in \mathrm{int}(B)$, B has the unique-lifting property with respect to f. Otherwise, $R(f) + \mathbb{Z}^n \neq \mathbb{R}^n$ for all $f \in \mathrm{int}(B)$ and for all $f \in \mathrm{int}(B)$, B has the multiple-lifting property with respect to f.

3 Unique Lifting in Pyramids

The Construction. We study an iterative procedure for creating higher dimensional maximal lattice-free polytopes. Let $n \geq 1$. Consider any polytope $B \subseteq \mathbb{R}^{n+1}$ such that $B \subseteq \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$, and a point $v^0 \in \mathbb{R}^{n+1}$ such that $v^0_{n+1} > 0$. Let $C(B, v^0)$ be the cone formed with $B - v^0$ as base. We define

$$\mathrm{Pyr}(B, v^0) = (C(B, v^0) + v^0) \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} \geq -1\}.$$

Informally speaking, we put a translated cone through v^0 with B as the base and "cut it off" by the hyperplane $\{x \in \mathbb{R}^{n+1} : x_{n+1} = -1\}$, to create the pyramid $\mathrm{Pyr}(B, v^0)$ (see page 9, Ziegler [12] for a related construction). We will use the terminology that $\mathrm{Pyr}(B, v^0)$ is a *pyramid over* B. The facet of $\mathrm{Pyr}(B, v^0)$ induced by $\{x \in \mathbb{R}^{n+1} : x_{n+1} \geq -1\}$ will be called the *base* of $\mathrm{Pyr}(B, v^0)$. v^0 will be called the *apex* of $\mathrm{Pyr}(B, v^0)$.

Lifting Properties for Pyramids. We say $\mathrm{Pyr}(B, v^0)$ is *2-partitionable* if the integer hull of $\mathrm{Pyr}(B, v^0)$ is contained in $\{x \in \mathbb{R}^{n+1} : -1 \leq x_{n+1} \leq 0\}$. We show that if B is a maximal lattice-free body with the multiple-lifting property, then $\mathrm{Pyr}(B, v^0)$ is also a body with multiple-lifting for any v^0 such that $\mathrm{Pyr}(B, v^0)$ is a 2-partitionable maximal lattice-free pyramid.

Proposition 2 *Let $B \subseteq \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ be a maximal lattice-free polytope (when viewed as an n-dimensional polytope in $\mathbb{R}^n$), and $v^0 \in \mathbb{R}^{n+1}$ such that* $\mathrm{Pyr}(B, v^0)$ *is a 2-partitionable maximal lattice-free polytope.*

If B is a body with the multiple-lifting property, then $\mathrm{Pyr}(B)$ *is a body with the multiple-lifting property.*

Proof. Since B is a body with multiple-lifting, there exists a vertex v of B such that the lifting region $R(v)$ satisfies $\mathrm{vol}_{\mathbb{T}^n}(R(v)/\mathbb{Z}^n) < 1$. Consider the edge of $\mathrm{Pyr}(B, v^0)$ passing through v^0 and v and let $\hat{v}$ be the vertex of this edge that lies on the base of $\mathrm{Pyr}(B, v^0)$. The lifting region $R(\hat{v})$ for $\mathrm{Pyr}(B, v^0)$ is a cylinder over $R(v)$ of height 1. Thus, $\mathrm{vol}_{\mathbb{T}^{n+1}}(R(\hat{v})/\mathbb{Z}^{n+1}) < 1$ and therefore, by Theorem 4, $\mathrm{Pyr}(B, v^0)$ is a body with multiple-lifting.

We now show that the unique-lifting property is preserved under the pyramid operation with a special property. This will give us a tool to iteratively construct bodies with unique-lifting in every dimension $n \in \mathbb{N}$.

Proposition 3 *Let* $B \subseteq \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ *be a maximal lattice-free polytope (when viewed as an* n*-dimensional polytope in* $\mathbb{R}^n$*), and* $v^0 \in \mathbb{R}^{n+1}$ *such that* $\mathrm{Pyr}(B, v^0)$ *is a maximal lattice-free polytope. Suppose further that the base* F_0 *of* $\mathrm{Pyr}(B, v^0)$ *contains an integer translate of* B*.*

If B *is a body with unique-lifting, then* $\mathrm{Pyr}(B, v^0)$ *is a body with unique-lifting.*

Proof. For all the vertices $\hat{v}$ of $\mathrm{Pyr}(B, v^0)$ on F_0, the lifting region $R(\hat{v})$ for $\mathrm{Pyr}(B, v^0)$ contains a cylinder of height 1 over the lifting $R(v)$ for B with respect to the vertex v that lies on the edge connecting v^0 and $\hat{v}$. ($R(\hat{v})$ might contain other spindles that come from integer points in $\mathrm{Pyr}(B, v^0)$ that are not in B, but this can only help with the unique-lifting property.)

So we need to look at the vertex v^0. Let S be the set of integer points in $B \cap \mathbb{Z}^{n+1}$. By our hypothesis, after a unimodular transformation, we can assume that $S - e^{n+1} \subseteq B - e^{n+1} \subseteq F_0$, where $e^{n+1} = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ is the standard unit vector perpendicular to F_0. Let $\bar{v}$ be the projection of v^0 onto the hyperplane $\{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$. Since $B - e^{n+1} \subseteq F_0$, we have $\bar{v} \in B$. Let $R(\bar{v})$ be the lifting region in B with respect to $\bar{v}$ (when B is viewed as an n-dimensional polytope in $\mathbb{R}^n$).

We show that for every point $\bar{x} \in R(\bar{v})$, there is an interval of height 1 over $\bar{x}$ that is contained in $R(v^0)$ in $\mathrm{Pyr}(B, v^0)$. By Lemma 1, this will suffice to show that $\mathrm{vol}_{\mathbb{T}^{n+1}}(R(v^0)/\mathbb{Z}^{n+1}) = 1$ since $\mathrm{vol}_{\mathbb{T}^n}(R(\bar{v})/\mathbb{Z}^n) = 1$ because B is a body with the unique-lifting property.

Let B, when viewed as embedded in $\mathbb{R}^n$, be described by $\{x \in \mathbb{R}^n : a^i \cdot x \leq b_i \;\; i \in I\}$ where I is the index set for the facets. Then $\mathrm{Pyr}(B, v^0)$ can be described by $\{(x, x_{n+1}) \in \mathbb{R}^{n+1} : a^i \cdot x + \delta^i x_{n+1} \leq b_i \;\; \forall i \in I, \;\; x_{n+1} \geq -1\}$. $B - e^{n+1} \subseteq F_0$ implies that $\delta^i \geq 0$ for all $i \in I$.

Consider $\bar{x} \in R(\bar{v})$. Let $\overline{F}$ be the facet of B and $\bar{z} \in \overline{F} \cap \mathbb{Z}^{n+1}$ such that $\bar{x} \in S_{\overline{F},\bar{z}}(\bar{v})$ (defined with respect to B). Let $z^0 = \bar{z} - e^{n+1} \in F_0$. Now $S_{F_0,z^0}(v^0)$ is given by

$$\begin{array}{rl} a^i \cdot x + \delta^i x_{n+1} \leq b_i & \forall i \in I \\ -a^i \cdot x - \delta^i x_{n+1} \leq -a^i \cdot \bar{z} + \delta^i & \forall i \in I \end{array} \tag{5}$$

Let $j \in I$ be such that $\overline{F}$ is given by $a^j \cdot x \leq b_j$. We now show that the following two points $x_1 = (\bar{x}, \frac{b_j - a^j \cdot \bar{x}}{\delta^j})$ and $x_2 = (\bar{x}, \frac{b_j - a^j \cdot \bar{x}}{\delta^j} - 1)$ are both in $S_{F_0,z^0}(v^0)$. Since $S_{F_0,z^0}(v^0)$ is convex, this will imply that the entire segment of height 1 connecting x_1, x_2 lies inside $S_{F_0,z^0}(v^0) \subseteq R(v^0)$.

Observe that $a^i \cdot \bar{v} + \delta^i v^0_{n+1} = b_i$ for all $i \in I$ and therefore

$$\frac{b_i - a^i \cdot \bar{v}}{\delta^i} = v^0_{n+1} \quad \forall i \in I \tag{6}$$

Since $\bar{x} \in R(\bar{v})$, there exists $a \in \overline{F}$ such that $\bar{x} = \mu\bar{v} + (1 - \mu)a$ for some $0 \leq \mu \leq 1$. We now check the first set of inequalities in (5) for x_1. For any $i \in I$, the following inequalities are true.

$$
\begin{aligned}
a^i \cdot \bar{x} + \delta^i\left(\tfrac{b_j - a^j\cdot\bar{x}}{\delta^j}\right) &= a^i \cdot (\mu\bar{v} + (1-\mu)a) + \delta^i\left(\tfrac{b_j - a^j\cdot(\mu\bar{v}+(1-\mu)a)}{\delta^j}\right) \\
&= \mu(a^i\cdot\bar{v}) + (1-\mu)(a^i\cdot a) + \delta^i\left(\tfrac{\mu b_j - \mu(a^j\cdot\bar{v})}{\delta^j}\right) \\
&\quad \text{(because } a^j\cdot a = b_j \text{ since } a\in\overline{F}) \\
&= \mu(a^i\cdot\bar{v}) + (1-\mu)(a^i\cdot a) + \mu\delta^i\left(\tfrac{b_i - a^i\cdot\bar{v}}{\delta^i}\right) \quad \text{using (6)} \\
&= \mu(a^i\cdot\bar{v}) + (1-\mu)(a^i\cdot a) + \mu(b_i - a^i\cdot\bar{v}) \\
&\le b_i \qquad \text{since } a^i\cdot a \le b^i \text{ because } a\in B
\end{aligned}
$$

Since the last coordinate of x_2 is less than the last coordinate of x_1 and $\delta^i \geq 0$ for all $i \in I$, we see that the first set of inequalities in (5) are also satisfied for x_2.

We now check the second set of inequalities in (5) for x_2. Consider $\bar{v}, \bar{z}$ and $\bar{x}$ and consider the two dimensional affine hyperplane A passing through all these 3 points. The set $T := A \cap P_{\overline{F}}(\bar{v})$ is a triangle with $\bar{v}$ as one vertex and the other two vertices a and b lying on $\overline{F}$. This is the same construction that was used in the proof of Lemma 1. Since $\bar{z}$ lies on the line segment connecting a, b, there exists $0 \leq \lambda \leq 1$ such that $\bar{z} = \lambda a + (1-\lambda)b$. Since $\bar{x} \in T$, there exist $0 \leq \mu, \alpha, \beta \leq 1$ such that $\bar{x} = \mu\bar{v} + \alpha a + \beta b$ and $\mu + \alpha + \beta = 1$. Also observe that $\bar{x} \in T \cap (\bar{z} + \bar{v} - T)$. Therefore, $\alpha \leq \lambda$ and $\beta \leq 1 - \lambda$. Now we do the computations. For any $i \in I$,

$$
\begin{aligned}
-a^i\cdot\bar{x} - \delta^i\left(\tfrac{b_j - a^j\cdot\bar{x}}{\delta^j} - 1\right) &= -a^i\cdot(\mu\bar{v} + \alpha a + \beta b) - \delta^i\left(\tfrac{b_j - a^j\cdot(\mu\bar{v}+\alpha a+\beta b)}{\delta^j}\right) + \delta^i \\
&= \mu(-a^i\cdot\bar{v}) + \alpha(-a^i\cdot a) + \beta(-a^i\cdot b) - \delta^i\left(\tfrac{\mu b_j - \mu(a^j\cdot\bar{v})}{\delta^j}\right) + \delta^i \\
&\quad \text{(because } a^j\cdot a = a^j\cdot b = b_j \text{ since } a,b\in\overline{F}) \\
&= \mu(-a^i\cdot\bar{v}) + \alpha(-a^i\cdot a) + \beta(-a^i\cdot b) - \mu\delta^i\left(\tfrac{b_i - a^i\cdot\bar{v}}{\delta^i}\right) + \delta^i \\
&\quad \text{(using (6))} \\
&= \alpha(-a^i\cdot a) + \beta(-a^i\cdot b) - \mu b_i + \delta^i \\
&= -a^i\cdot\bar{z} + a^i\cdot(\lambda a + (1-\lambda)b) + \alpha(-a^i\cdot a) + \beta(-a^i\cdot b) - \mu b_i + \delta^i \\
&= -a^i\cdot\bar{z} + (\lambda-\alpha)(a^i\cdot a) + (1-\lambda-\beta)(a^i\cdot b) - \mu b_i + \delta^i \\
&\le -a^i\cdot\bar{z} + (\lambda-\alpha)b_i + (1-\lambda-\beta)b_i - \mu b_i + \delta^i \\
&\quad \text{(since } \alpha\le\lambda,\ \beta\le 1-\lambda,\ a^i\cdot a\le b_i \text{ and } a^i\cdot b\le b_i \text{ because } a,b\in B) \\
&= -a^i\cdot\bar{z} + \delta^i
\end{aligned}
$$

Finally since x_1 has a higher value for the last coordinate than x_2 and $\delta^i \geq 0$, x_1 satisfies the second set of constraints in (5) also.

Remark 1. Propositions 2 and 3 provide strict generalizations of all the results on 2-partitionable simplices from Section 4 in [3]. Furthermore, we can use these propositions to iteratively construct bodies in every dimension $n \in \mathbb{N}$ with or without the unique-lifting property. See Remarks 2 and 3.

Axis-parallel Simplices. We use Proposition 3 to show that a certain class of simplices has unique-lifting. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of positive reals such that $\frac{1}{a_1} + \ldots + \frac{1}{a_n} = 1$. Then $S(a) := \operatorname{conv}\{0, a_1e^1, a_2e^2, \ldots, a_ne^n\}$ is a maximal lattice-free simplex (where $e^1, \ldots, e^n$ form the standard basis for $\mathbb{R}^n$). The following theorem is a generalization of results in [3, 4], where it was proved for the special case when $a_i = n$ for all $i = 1, \ldots, n$.

Theorem 5. *$S(a)$ is a body with unique-lifting for any n-tuple $a = (a_1, \ldots, a_n)$ such that $\frac{1}{a_1} + \ldots + \frac{1}{a_n} = 1$.*

Proof. Observe that $B = S(a) \cap \{x \in \mathbb{R}^n : x_i = 1\}$ can be expressed as $S(a') \subseteq \mathbb{R}^{n-1}$, where $a' = (a'_1, \ldots, a'_{n-1})$ is an $n-1$-tuple where $a'_i = a_i - \frac{a_i}{a_n}$ for $i = 1, \ldots, n-1$. Thus, we can use induction to prove the theorem. The case $n = 1$ is trivial, since $S(a)$ is simply an interval of length 1 and is easily seen to be a body with unique-lifting. For the induction step, we observe that $S(a)$ is a pyramid over $B = S(a) \cap \{x \in \mathbb{R}^n : x_i = 1\}$ with base $S(a) \cap \{x \in \mathbb{R}^n : x_i = 0\}$, and B is an integer translate of $S(a')$: $B = S(a') + e^n$. Further, the base contains $S(a')$ and thus contains an integer translate of B. By the induction hypothesis, $S(a')$ is a body with unique-lifting. Therefore, so is B and by Proposition 3, $S(a)$ is a body with unique-lifting.

Remark 2. We can iteratively build pyramids over $S(a)$ to get more general simplices with the unique-lifting property by repeatedly applying Proposition 3. We simply need to make sure that the base of the pyramid $\mathrm{Pyr}(S(a), v^0)$ we create contains an integer translate of $S(a)$.

4 Spindles that Translatively Tile $\mathbb{R}^n$

In this section, we build some tools from discrete geometry and geometry of numbers. We will apply these tools to gain further insight into pyramids with the unique-lifting property in Section 5.

For any full dimensional polytope $P \subseteq \mathbb{R}^n$, a *ridge* is a face of dimension $n-2$. Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric full dimensional polytope with centrally symmetric facets. Let G be any ridge of P. The *belt* corresponding to G is the set of all facets which contain a translate of G or $-G$. Observe that every centrally symmetric polytope P with centrally symmetric facets has belts of even size greater than or equal to 4.

A *zonotope* is a polytope given by a finite set of vectors $V = \{v^1, \ldots, v^k\} \subseteq \mathbb{R}^n$ in the following way:

$$Z(V) = \{\lambda_1 v^1 + \ldots + \lambda_k v^k : -1 \leq \lambda_i \leq 1 \quad \forall i = 1, \ldots, k\}.$$

We begin with a technical lemma about the combinatorial structure of zonotopes in $\mathbb{R}^n$. For space constraints, the proof appears in the appendix.

Lemma 5. *Let Z be a full dimensional zonotope in $\mathbb{R}^n$ such that every belt of Z is of size 4. Then Z is the image of the n-dimensional hypercube under an invertible affine transformation.*

Theorem 6. *(McMullen [10]) Let $S \subseteq \mathbb{R}^n$ be a full-dimensional centrally symmetric spindle with centrally symmetric facets. Then S is the image of the n-dimensional hypercube under an invertible affine transformation.*

Proof. Let a and $-a$ be the apexes of the spindle S. Consider any belt of S. Since S is centrally symmetric, each belt is even length, i.e., of length k where k is an even natural number. Now there are $k/2$ facets $F_1, \dots, F_{k/2}$ involved in this belt that are incident on a (and the remaining $k/2$ facets are incident on $-a$). Let G be the $n-2$ dimensional ridge that defines this belt. We project S onto the 2 dimensional space perpendicular to G to get a polygon P. The facets $F_1, \dots, F_{k/2}$ are all projected onto edges of the polygon. Moreover, observe that a is projected onto all these edges. This implies that $k/2 \leq 2$, otherwise, we have three edges of a polygon incident on the same point in $\mathbb{R}^2$. Thus, $k \leq 4$.

Since every belt has length 4, each $n-2$ ridge in S is centrally symmetric. Therefore, by a theorem of McMullen [9], S is a zonotope. Since S is a zonotope whose belts are length 4, by Lemma 5, S is the image of the n-dimensional hypercube under an invertible affine transformation.

Theorem 6 was communicated to us by Peter McMullen via personal email. We include a complete proof here as the result does not appear explicitly in the literature. The above proof is based on a proof sketch by Prof. McMullen.

We say that a set $S \subseteq \mathbb{R}^n$ *translatively tiles* $\mathbb{R}^n$ if $\mathbb{R}^n$ is the union of translates of S whose interiors are disjoint. We now state the celebrated Venkov-Alexandrov-McMullen theorem on translative tilings.

Theorem 7. *[Venkov-Alexandrov-McMullen (see Theorem 32.2 in [11])] Let P be a compact convex set with non-empty interior that translatively tiles $\mathbb{R}^n$. Then (i) P is a centrally symmetric polytope, (ii) All facets of P are centrally symmetric, and (iii) every belt of P is either length 4 or 6.*

Theorem 8. *Let $S \subseteq \mathbb{R}^n$ be a full-dimensional spindle that translatively tiles space. Then S is the image of the n-dimensional hypercube under an invertible affine transformation.*

Proof. Follows from the Venkov-Alexadrov-McMullen theorem (Theorem 7) and Theorem 6.

5 Maximal Lattice-free Pyramids with Exactly One Integer Point in the Relative Interior of the Base

Theorem 9. *Let $P \subseteq \mathbb{R}^n$ be a maximal lattice-free pyramid, such that its base contains exactly one integer point in its relative interior. If P is a body with unique-lifting, then P is a simplex.*

The proof of Theorem 9 is a simple application of Theorem 8, and is given in the appendix.

Remark 3. Using Propositions 2 and 3, one can construct pyramids that are *not* simplices in arbitrarily high dimensions with (or without) the unique-lifting property. For example, start with $B \subseteq \mathbb{R}^2$ as a quadrilateral with (or without) unique-lifting (see [6]) and construct $\text{Pyr}(B, v^0) \subseteq \mathbb{R}^3$; iterate this procedure to get higher dimensional pyramids. The base of such pyramids with unique-lifting will have multiple integer points in its relative interior by Theorem 9.

We recall the following theorem proved in [3].

Theorem 10. *Let Δ be a maximal lattice-free simplex in $\mathbb{R}^n$ ($n \geq 2$) such that each facet of Δ has exactly one integer point in its relative interior. Then Δ has the unique-lifting property if and only if Δ is an affine unimodular transformation of* $\text{conv}\{0, ne^1, \ldots, ne^n\}$.

We can now generalize this result to pyramids.

Theorem 11. *Let P be a maximal lattice-free pyramid in $\mathbb{R}^n$ ($n \geq 2$) such that every facet of P contains exactly one integer point in its relative interior. P has the unique-lifting property if and only if P is an affine unimodular transformation of* $\text{conv}\{0, ne^1, \ldots, ne^n\}$.

Proof. Sufficiency follows from Theorem 5. If P has the unique-lifting property, by Theorem 9, P is a simplex. The result then follows from Theorem 10.

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6 Appendix

Given $V = \{v_1, \ldots, v_k\}$, the combinatorial structure of a zonotope $Z(V)$ is described completely by the hyperplane arrangement $\mathcal{A}_V := \{H_1, \ldots, H_k\}$ where $H_i := \{c \in \mathbb{R}^n : c \cdot v^i = 0\}$.

Theorem 12 (see Corollary 7.17 in [12]). *There is a natural bijection between the nonempty faces of of $Z(V)$ and the faces of the hyperplane arrangement $\mathcal{A}_V$. Moreover, if $Z(V)$ is full dimensional (i.e., the set V spans $\mathbb{R}^n$), this bijection establishes a one-to-one correspondence between facets of $Z(V)$ and the one-dimensional rays of $\mathcal{A}_V$.*

In the light of the above theorem, a belt of $Z(V)$ corresponds to the one-dimensional rays contained in a two-dimensional linear subspace L formed by the intersection of $n-2$ hyperplanes from $\mathcal{A}_V$.

Proof (Proof of Lemma 5). We assume without loss of generality that $Z = Z(V)$ is given by a *simple vector configuration* V, i.e., no v_i, v_j are parallel when $i \neq j$ (see page 206 in [12]). It suffices to show that $k \leq n$.

Suppose to the contrary and $k > n$. We consider a minimal linearly dependent set $V' = \{v^{i_1}, \ldots, v^{i_m}\} \subseteq V$. Since V is simple, $m \geq 3$. Let $V'' \subseteq V$ be a linearly independent subset such that $\text{span}(V'') \cap \text{span}(V') = \{0\}$ and $\text{span}(V' \cup V'') = \mathbb{R}^n$ (V'' exists because V is full rank). Since V' is minimally dependent, this implies $V'' \cup \{v^{i_1}, \ldots, v^{i_{m-3}}\}$ is linearly independent and $|V''| = n - m + 1$. Thus, $\overline{V} = V'' \cup \{v^{i_1}, \ldots, v^{i_{m-3}}\}$ is a linearly independent set of size $n-2$ (and if $m = 3$, then $\overline{V} = V''$.) Let L be the two-dimensional subspace formed by the intersection of the hyperplanes $\{H_i \in \mathcal{A}_V : v^i \in \overline{V}\}$. Since V' is minimally dependent, the one-dimensional lines formed by $L \cap H_{i_{m-2}}$, $L \cap H_{i_{m-1}}$ and $L \cap H_{i_m}$ are pairwise non-parallel. Thus, in the hyperplane arrangement $\mathcal{A}_{V' \cup V''} := \{H_i \in \mathcal{A}_V : v^i \in V' \cup V''\}$, we have a two-dimensional L which 3 nonparallel lines lying on L. Therefore, in the hyperplane arrangement $\mathcal{A}_V$, we have at least 3 non parallel lines on the two dimensional subspace L (the number of rays lying on L can only increase going from $\mathcal{A}_{V' \cup V''}$ to $\mathcal{A}_V$). But this corresponds to a belt of length at least 6. This is a contradiction.

Proof (Proof of Theorem 9). Let v be the apex and F be the base of P. Since P has unique-lifting, $R(v) + \mathbb{Z}^n = \mathbb{R}^n$. Moreover, $R(v)$ contains exactly one full dimensional spindle $S = S_{F,z}(v)$ since the base F has a unique integer point z in its relative interior (the remaining spindles in $R(v)$ are lower dimensional and hence can be neglected). Thus, $S + \mathbb{Z}^n = \mathbb{R}^n$. We claim that for every $w \in \mathbb{Z}^n$, $\text{int}(S) \cap \text{int}(S + w) = \emptyset$. By Lemma 1 this is true when $w \notin \text{aff}(F - F)$. If $w \in \text{aff}(F - F)$, then $z + w \notin \text{relint}(F)$ by our assumption that F has a unique integer point z in its relative interior. Thus, there exists $c \in \mathbb{R}^n$, such that $c \cdot x \leq c \cdot v$ is facet defining for P, and $c \cdot (z + w) \geq c \cdot v$. Moreover, by construction of S, $c \cdot x \geq c \cdot z$ is a defining inequality of S. Hence, $c \cdot x \geq c \cdot (z + w)$ is a valid inequality for $S + w$. Thus, for every $x \in S \subseteq P$, $c \cdot x \leq c \cdot v$. But, for every $x \in S + w$, $c \cdot x \geq c \cdot (z + w) \geq c \cdot v$. Thus, $\{x \in \mathbb{R}^n : c \cdot x = c \cdot v\}$ is a separating hyperplane for $\text{int}(S)$ and $\text{int}(S + w)$.

Thus, S translatively tiles space. By Theorem 8, we get that S is the image of the n-dimensional hypercube under an invertible affine transformation. This in turn implies that P is a simplex.