On Minimal Valid Inequalities for Mixed Integer Conic Programs

Fatma Kılınç Karzan*

June 27, 2013

Abstract

We study mixed integer conic sets involving a general regular (closed, convex, full dimensional, and pointed) cone $\mathcal{K}$ such as the nonnegative orthant, the Lorentz cone or the positive semidefinite cone. In a unified framework, we introduce $\mathcal{K}$-minimal inequalities and show that under mild assumptions, these inequalities together with the trivial cone-implied inequalities are sufficient to describe the convex hull. We study the properties of $\mathcal{K}$-minimal inequalities by establishing necessary conditions for an inequality to be $\mathcal{K}$-minimal. This characterization leads to a broader class of $\mathcal{K}$-sublinear inequalities, which includes $\mathcal{K}$-minimal inequalities as a subclass. We establish a close connection between $\mathcal{K}$-sublinear inequalities and the support functions of sets with certain structure. This leads to practical ways of showing that a given inequality is $\mathcal{K}$-sublinear and $\mathcal{K}$-minimal. We provide examples to show how our framework can be applied.

Our framework generalizes the results from the mixed integer linear case, such as the minimal inequalities for mixed integer linear programs are generated by sublinear (positively homogeneous, subadditive and convex) functions that are also piecewise linear. So whenever possible we highlight the connections to the existing literature. However our study reveals that such a cut generating function view is not possible for in the conic case even when the cone involved is the Lorentz cone.

1 Introduction

A *Mixed Integer Conic Program* (MICP) is an optimization program of the form

$$\text{Opt} = \inf_{x\in\mathbb{R}^n} \left\{ c^T x : Ax - b \in \mathcal{K},\ x_i \in \mathbb{Z} \text{ for all } i \in I \right\} \qquad (MICP)$$

where $\mathcal{K}$ is a *regular cone* (full-dimensional, closed, convex and pointed) in a finite dimensional Euclidean space F with an inner product; $c \in \mathbb{R}^n$ is the *objective*; $b \in F$ is the *right hand side*; $A : \mathbb{R}^n \to F$ is a linear map, and $I \subset \{1, \ldots, n\}$ is the index set of integer variables. We assume that all of the data involved with MICP, i.e., c, b, A is rational. Examples of regular cones, $\mathcal{K}$, include the nonnegative orthant, $\mathbb{R}^m_+$, the Lorentz cone, $\mathcal{L}^m$, and the positive semidefinite cone, $\mathcal{S}^m_+$.

Mixed Integer Linear Programs (MILPs) are a particular case of MICPs with $\mathcal{K} = \mathbb{R}^m_+$. While MILPs offer an incredible representation power, various optimization problems involving risk constraints and discrete decisions lead to MICPs with conic constraints. These include many applications from decision making under uncertainty domain, i.e., robust optimization and stochastic programming paradigms, such as portfolio optimization with fixed transaction costs in finance (see [33, 49]), stochastic joint location-inventory models

*Carnegie Mellon University, Pittsburgh, Pennsylvania 15213, USA, fkilinc@andrew.cmu.edu

[5]. Conic constraints include various specific convex constraints such as linear, convex quadratic, eigenvalue, etc., and hence offer significant representation power (see [14] for a detailed introduction to conic programming and its applications in various domains). Clearly allowing discrete decisions further increases the representation power of MICPs. Moreover, the most powerful relaxations to many linear combinatorial optimization problems are based on conic (in particular semidefinite) relaxations (see [2, 36, 50] for earlier work and [34] for a survey on this topic). Besides, MILPs have been heavily exploited for approximating non-convex nonlinear optimization problems. For a wide range of these problems, MICPs offer tighter relaxations and thus potentially a better overall algorithmic performance. Therefore, MICPs have gained considerable interest and arise in several important applications in many diverse fields.

Currently, the cutting plane theory lies at the basis of most efficient algorithms for solving MILPs and has been developed and very well understood in the case of MILPs. While many commercial solvers such as CPLEX [30], Gurobi [40], and MOSEK [52] are expanding their features to include solvers for MICPs, the theory and algorithms for solving MICPs are still in their infancy (see [6]). In fact, the most promising approaches to solve MICPs are based on the extension of cutting plane techniques (see [6, 7, 17, 18, 21, 23, 32, 46, 62]) in combination with conic relaxations and branch-and-bound algorithm. Numerical performance of these techniques is still under investigation. Evidence from MILP setup indicates that adding a small yet essential set of strong cutting planes is key to the success of such a procedure. Yet, except very specific and simple cases, it is not possible to theoretically evaluate the strength of the corresponding valid inequalities, i.e., redundancy, domination, etc., for MICPs. This is in sharp contrast to the MILP case, where the related questions have been studied extensively. In particular, the feasible region of an MILP with rational data is a polyhedron and the facial structure of a polyhedron (its faces, and facets) is very well understood. Various ways of proving whether a given inequality is a facet or not for an MILP are well established in the literature (see [53]). In addition to this, a new framework is developing rapidly in terms of establishing minimal and extremal inequalities for certain generic semi-infinite relaxations of MILPs (see [26] and references therein). On the other hand, for general MICPs, there is no natural extension of certain important polyhedral notions such as facets. Therefore, establishing such a theoretical framework to measure the strength of cutting planes in the MICP context remains a natural and important question, and our goal in this paper is to address this question.

Our approach in this paper is based on extending the notion of minimal inequalities from the MILP context. In particular, we characterize minimal inequalities for general yet well-structured sets (obtained from the intersection of an affine halfspace with a regular cone) with additional integrality restrictions on some variables. In this respect, when the cone is taken as the nonnegative orthant, our approach ties back to the recent work of Conforti et al. [24] as well as the cornerstone paper of Johnson [45]. The MILP counterparts of our results and further developments for MILPs were studied extensively in the literature. We demonstrate that these results naturally extend to MICPs, and hence, whenever possible, we highlight these connections.

To the best of our knowledge minimal inequalities have not been defined and studied in the MICP setting. In this paper, we contribute to the literature by establishing properties associated with minimality for MICPs, i.e., necessary, and sufficient conditions, as well as practical tools for testing whether a given inequality is minimal or not. Minimal inequalities in fact are directly related to the facial structure of the convex hull of the feasible region of corresponding MICPs, and we demonstrate that such a study can be done in a unified manner for all regular cones $\mathcal{K}$. Our study also reveals that MICPs present new challenges, various useful tools from the MILP framework such as semi-infinite relaxations or cut generating functions[1)] do not

[1)] Cut generating functions are formally introduced in the MILP setting by [24]. Informally, a cut generating function generates the coefficient of a variable in a given cut using only information of the instance pertaining to this variable.

extend. Therefore our derivations are based on the actual finite dimensional problem instance, and hence this study does not rely on the majority of previous literature on minimal inequalities. In a practical cutting plane procedure for MILPs and/or MICPs, one is indeed faced with a finite dimensional problem, and thus we believe that this is not a limitation but rather a contribution to the corresponding MILP literature. In particular, in the linear case, i.e., $\mathcal{K} = \mathbb{R}^m_+$, our results partially cover the related results from [45] and [24] showing that minimal inequalities can be properly related to cut generating functions, and these functions are sublinear and piece-wise convex. For non-polyhedral regular cones, we show that there exists minimal inequalities, which cannot be associated with any cut generating function. Finally we believe this work is a step forward for the study of MICPs with $\mathcal{K} = \mathcal{S}^m_+$. Such problems are frequently encountered in the semidefinite relaxations of combinatorial optimization problems. While various recent papers study MICPs with $\mathcal{K} = \mathcal{L}^m$, besides the work of [23], we are not aware of any paper explicitly studying valid inequalities for general MICPs with $\mathcal{K} = \mathcal{S}^m_+$.

1.1 Preliminaries and Notation

Let $(E, \langle\cdot,\cdot\rangle)$ be a finite dimensional Euclidean space with inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{K} \subset E$ be a *regular* cone, i.e., full-dimensional, closed, convex and pointed. Note that if every $\mathcal{K}_i \subset E_i$ for $i = 1, \ldots, k$ is a regular cone, then their direct product $\widetilde{\mathcal{K}} = \mathcal{K}_1 \times \ldots \times \mathcal{K}_k$ is also a regular cone in the Euclidean space $\widetilde{E} = E_1 \times \ldots \times E_k$ with inner product $\langle\cdot,\cdot\rangle_{\widetilde{E}}$, which is the sum of the inner products $\langle\cdot,\cdot\rangle_{E_i}$. Therefore without loss of generality we only focus on the case with a single regular cone $\mathcal{K}$.

In this paper, given a linear map $A : E \to \mathbb{R}^m$, a regular cone $\mathcal{K} \subset E$, and a *nonempty* set of right hand side vectors, $b \in \mathbb{R}^m$, i.e., $\mathcal{B} := \{b^1, b^2, \ldots\}$, we consider the union of conic sets defined by A, $\mathcal{K}$, and the list of vectors from $\mathcal{B}$, i.e.,

$$\mathcal{S}(A, \mathcal{K}, \mathcal{B}) := \{x \in E : \ Ax = b \text{ for some } b \in \mathcal{B}, \ x \in \mathcal{K}\},$$

and we are interested in determining valid inequalities for the closed convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$. Note that $\mathcal{B}$ can be a list of finite or infinite set of points. Without loss of generality we assume that $\mathcal{B} \neq \emptyset$, and for all $b \in \mathcal{B}$, there exists $x_b \in \mathcal{K}$ satisfying $Ax_b = b$.

For a given set S, we denote its topological interior with $\text{int}(S)$, its closure with $\overline{S}$, its boundary with $\partial S = \overline{S} \setminus \text{int}(S)$. We use $\text{conv}(S)$ to denote the convex hull of S, $\overline{\text{conv}}(S)$ for its closed convex hull and $\text{cone}(S)$ to denote the cone generated by the set S. We define the kernel of a linear map $A : E \to \mathbb{R}^m$, as $\text{Ker}(A) := \{u \in E : \ Au = 0\}$ and its image as $\text{Im}(A) := \{Au : \ u \in E\}$. We use A^* to denote the conjugate linear map [2)] given by the identity

$$\langle y, Ax\rangle = \langle A^*y, x\rangle \ \ \forall (x \in E, y \in \mathbb{R}^m).$$

[2)] When $E = \mathbb{R}^n$ and a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ is just an $m \times n$ real-valued matrix, and its conjugate is given by its transpose, i.e., $A^* = A^T$.

When $E = \mathcal{S}^n$, i.e., space of symmetric $n \times n$ matrices, it is natural to specify a linear map $A : \mathcal{S}^n \to \mathbb{R}^m$ as a collection $A^1, \ldots, A^m$ of m matrices from $\mathcal{S}^n$ such that

$$AZ = (\text{Tr}(ZA^1); \ldots; \text{Tr}(ZA^m)) : \mathcal{S}^n \to \mathbb{R}^m,$$

where $\text{Tr}(\cdot)$ denotes the trace of a matrix, i.e., the sum of its diagonal entries. In such a case, the conjugate linear map $A^* : \mathbb{R}^m \to \mathcal{S}^n$ is given by

$$A^*y = \sum_{j=1}^m y_j A^j, \ \ y = (y_1; \ldots; y_m) \in \mathbb{R}^m.$$

For a given cone $\mathcal{K} \subset E$, we let Ext($\mathcal{K}$) denote the set of its extreme rays, and $\mathcal{K}^*$ to denote its dual cone given by

$$\mathcal{K}^* := \{y \in E : \langle x, y\rangle \geq 0 \ \forall x \in \mathcal{K}\}.$$

A relation $a - b \in \mathcal{K}$, where $\mathcal{K}$ is a regular cone, is often called *conic inequality* between a and b and is denoted by $a \succeq_{\mathcal{K}} b$; such a relation indeed preserves the major properties of the usual coordinate-wise vector inequality $\geq$. We denote the strict conic inequality by $a \succ_{\mathcal{K}} b$, to indicate that $a - b \in \text{int}(\mathcal{K})$. In the sequel, we refer to a constraint of the form $Ax - b \in \mathcal{K}$ as a *conic inequality constraint* or simply *conic constraint* and also use $Ax \succeq_{\mathcal{K}} b$ interchangeably in the same sense.

While our theory is general enough to cover all regular cones, there are three important examples of regular cones common to most MICPs, namely the nonnegative orthant, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : \ x_i \geq 0 \ \forall i\}$; the Lorentz cone, $\mathcal{L}^n := \{x \in \mathbb{R}^n : \ x_n \geq \sqrt{x_1^2 + \ldots + x_{n-1}^2}\ \}$; and the positive semidefinite cone $\mathcal{S}^n_+ := \{X \in \mathbb{R}^{n\times n} : \ a^T X a \geq 0 \ \forall a \in \mathbb{R}^n, \ X = X^T\}$. These important regular cones are also self-dual, i.e., $\mathcal{K}^* = \mathcal{K}$. In the first two cases, the corresponding Euclidean space E is just $\mathbb{R}^n$ with dot product as the corresponding inner product. In the last case, E becomes the space of symmetric $n \times n$ matrices with Frobenius inner product, $\langle x, y\rangle = \text{Tr}(xy^T)$.

Notation e_i is used for the i^{th} unit vector[3)] of $\mathbb{R}^n$, and Id for the identity map in E, i.e., when E is $\mathbb{R}^n$, Id is just the $n \times n$ identity matrix, I_n.

1.2 Motivation and Connections to MICPs

While the set of the form given by $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ can be of interest by itself, here we show that it naturally arises as a relaxation of any conic optimization problem with integer variables. Suppose that we start out with the following set, which can be viewed as the feasible region associated with a MICP

$$\{(y, v) \in \mathbb{R}^k_+ \times F : \ Wy + Hv \succeq_{\mathcal{K}} b, \ v \in Q\} \tag{1}$$

where E, F are finite dimensional Euclidean spaces, $K \subset E$ is a proper cone, $W : \mathbb{R}^n \to E$ and $H : F \to E$ are linear maps, and $Q = \{v^1, v^2, \ldots\} \subseteq F$. In the case of mixed integer problems with conic constraints, one can define Q as the corresponding set of values the integer variables can take, i.e., if $E = \mathbb{R}^m$, Q can be set to be $\mathbb{Z}^m \cap V$ where $V \subseteq \mathbb{R}^m$ is a given set. Note that, in the above representation (1), the nonnegativity restriction on y variables is not a limitation since we can always represent an unrestricted variable as the difference of two nonnegative variables.

Let us define $b^i := b - Hv^i$ for $i = 1, 2, \ldots$ and set

$$\mathcal{B} = \left\{b^1, b^2, \ldots\right\}, \ \ A = \left[\begin{array}{cc} W, & -\text{Id} \end{array}\right], \ \ \text{and} \ \ x = \left(\begin{array}{c} y \\ z \end{array}\right),$$

where Id is the identity map in the Euclidean space E. Then we arrive at

$$\{x \in E' := \mathbb{R}^k \times E : \ x \in \mathcal{K}' := \mathbb{R}^k_+ \times \mathcal{K}, \ Ax = b \text{ for some } b \in \mathcal{B}\}$$

which is of form $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$. In fact, one can work by enforcing only subsets of integrality restrictions in Q and thus arrive at relaxations of the desired form as follows. Suppose that $E = \mathcal{S}^q$, $\mathcal{K} = \mathcal{S}^q_+$, $F = \mathbb{R}^p$, and $Q = \{0, 1\}^p$ in (1). In such a setting, to represent the linear map $H : \mathbb{R}^p \to \mathcal{S}^q$, we can associate a matrix

[3)] Throughout the paper, we use Matlab notation to denote vectors and matrices and all vectors are to be understood in column form.

$H^i \in \mathcal{S}^q$ for each $i = 1, \ldots, p$. Let $(\bar{y}, \bar{v})$ be a solution which is feasible to the corresponding continuous relaxation of (1), and set $\bar{z} = b - W\bar{y} - H\bar{v}$. Suppose that for some $\emptyset \neq I \subseteq \{1, \ldots, p\}$, $\bar{v}_i \notin \{0, 1\}$ for $i \in I$. Without loss of generality, assume that $1 \in I$ and consider the following set which explicitly enforces integrality restriction of v_1:

$$\left\{ x = (y; v_2; \ldots; v_p; z) \in \mathbb{R}^k \times \mathbb{R}^{p-1} \times \mathcal{S}^q : \begin{array}{l} x \in \mathbb{R}^k_+ \times \mathbb{R}^{p-1}_+ \times \mathcal{S}^q_+, \\ Ax \equiv Wy + \sum_{i=2}^p H^i v^i - z = \hat{b} \text{ for some } \hat{b} \in \mathcal{B} \end{array} \right\},$$

where $\mathcal{B} = \{b; b - H^1\}$. Any valid inequality for this set will correspond to a valid inequality for (1), e.g., for some $\mu^y \in \mathbb{R}^k,\ \mu^v \in \mathbb{R}^{p-1},\ \mu^z \in \mathcal{S}^q$, and $\eta_0 \in \mathbb{R}$, let

$$\langle \mu^y, y \rangle + \sum_{i=2}^p \mu^v_{i-1} v_i + \langle \mu^z, z \rangle \geq \eta_0$$

be a valid inequality for the above set, then the following inequality will be valid for the closed convex hull of (1):

$$\langle \mu^y + W^* \mu^z, y \rangle + (H^1)^* \mu_z v_1 + \sum_{i=2}^p (\mu^v_{i-1} + (H^i)^* \mu_z) v_i \geq \eta_0 + \langle \mu^z, b \rangle$$

where we have used the identity $z = Wy + Hv - b$ in the transformation. Moreover if

$$\langle \mu^y, \bar{y} \rangle + \sum_{i=2}^p \mu^v_{i-1} \bar{v}_i + \langle \mu^z, \bar{z} \rangle \not\geq \eta_0$$

holds, then the inequality in the original space of variables separate $(\bar{y}, \bar{v})$ from the closed convex hull of (1). Similarly we can consider other integrality restrictions by properly splitting the variables into discrete versus continuous categories and defining the set $\mathcal{B}$ accordingly. For example, in order to consider the integrality restriction of v_1 and v_2 together, we need to set $\mathcal{B} = \{b,\ b - H^1,\ b - H^2,\ b - H^1 - H^2\}$, etc.

When $\mathcal{K} = \mathbb{R}^n_+$, relaxations of the above form have been studied in a number of other contexts, in particular for MILP and complementarity problems. In the MILP setting, i.e., $\mathcal{K} = \mathbb{R}^n_+$, clearly the above transformation is valid. Furthermore, when $\mathcal{B}$ is a finite list, Johnson [45] has studied this set under the name of *linear programs with multiple right hand side choice*. In another closely related recent work, Conforti et al. [24] study $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with $\mathcal{K} = \mathbb{R}^n_+$ and possibly an infinite list $\mathcal{B}$, and suggest that Gomory's *corner polyhedron* [37] as well as some other problems such as *linear programs with complementarity restrictions* can be viewed in this framework. Both [45] and [24] study characterizations of minimal inequalities, yet [24] views the topic almost entirely through cut generating functions and characterizing the properties of important ones.

1.3 Classes of Valid Inequalities and Our Goal

Recall that we are interested in the valid inequalities for the solution set

$$\mathcal{S}(A, \mathcal{K}, \mathcal{B}) := \{x \in E : \ Ax = b \text{ for some } b \in \mathcal{B},\ x \in \mathcal{K}\}.$$

Consider the set of all vectors $\mu \in E$ such that μ is not equal to zero, and μ_0 defined as

$$\mu_0 := \inf_x \{\langle \mu, x \rangle : \ x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})\} \tag{2}$$

is finite. We denote the set of such vectors μ with $\Pi(A, \mathcal{K}, \mathcal{B}) \subset E$. Any vector $\mu \in \Pi(A, \mathcal{K}, \mathcal{B})$ and a number $\eta_0 \leq \mu_0$ where μ_0 is as defined in (2), gives a *valid inequality* (v.i.) for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$, i.e.,

$$\langle \mu, x \rangle \geq \eta_0$$

is satisfied by all $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$. For a given vector μ and $\eta_0 \leq \mu_0$, we denote the corresponding valid inequality with $(\mu; \eta_0)$ for short hand notation. If both $(\mu; \eta_0)$ and $(-\mu; -\eta_0)$ are valid inequalities, then $\langle \mu, x \rangle = \eta_0$ holds for all $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$, and in this case, we refer to $(\mu; \eta_0)$ as a *valid equation* (v.e.) for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$.

Let $C(A, \mathcal{K}, \mathcal{B}) \subset E \times \mathbb{R}$ denote the convex cone of all valid inequalities given by $(\mu; \eta_0)$. $C(A, \mathcal{K}, \mathcal{B})$, being a convex cone in $E \times \mathbb{R}$, can always be written as the sum of a linear subspace L of $E \times \mathbb{R}$ and a pointed cone C, i.e., $C(A, \mathcal{K}, \mathcal{B}) = L + C$. Given L, the largest linear subspace contained in $C(A, \mathcal{K}, \mathcal{B})$, let $L^{\perp}$ denote the orthogonal complement of L, then a unique representation for $C(A, \mathcal{K}, \mathcal{B})$ can be obtained by letting $C = C^{\perp}$ where $C^{\perp} = C(A, \mathcal{K}, \mathcal{B}) \cap L^{\perp}$. A *generating set* (G_L, G_C) for a cone $C(A, \mathcal{K}, \mathcal{B})$ is a minimal set of elements $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$ such that $G_L \subseteq L$, $G_C \subseteq C$, and

$$C(A, \mathcal{K}, \mathcal{B}) = \left\{ \sum_{w \in G_L} \alpha_w w + \sum_{v \in G_C} \lambda_v v : \ \lambda_v \geq 0 \right\}.$$

Our study of $C(A, \mathcal{K}, \mathcal{B})$ will be based on characterizing the properties of the elements of its generating sets. We will refer to the vectors in G_L as *generating valid equalities* and the vectors in G_C as *generating valid inequalities* of $C(A, \mathcal{K}, \mathcal{B})$. An inequality $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$ is called *extreme v.i.* of $C(A, \mathcal{K}, \mathcal{B})$, if there exists a generating set for $C(A, \mathcal{K}, \mathcal{B})$ including $(\mu; \eta_0)$ as a generating v.i. either in G_L or in G_C. Note that any valid inequality $(\mu; \eta_0)$ with $\eta_0 < \mu_0$ does not belong to a generating set of $C(A, \mathcal{K}, \mathcal{B})$.

Clearly the inequalities in the generating set (G_L, G_C) of the cone $C(A, \mathcal{K}, \mathcal{B})$ are of great importance, they are necessary and sufficient for the description of the closed convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$. In this paper we are interested in identifying properties of these inequalities. It is easy to note that G_L is finite, as a basis of the subspace L can be taken as G_L. For nonpolyhedral cones such as $\mathcal{L}^n$ with $n \geq 3$, G_C need not be finite. In fact we provide an example demonstrating this in section 4.

1.4 Outline

The main body of this paper is organized as follows. We start by briefly summarizing the literature in section 2 and in the next section, we introduce the class of $\mathcal{K}$-minimal v.i. and show that under a mild assumption, this class of inequalities together with the trivial cone-implied inequalities is sufficient to describe the convex hull of the solution set. Moreover we establish a number of necessary conditions for $\mathcal{K}$-minimality. During our study, in a particular case, we identify a shortcoming of the $\mathcal{K}$-minimality definition, i.e., we establish that certain inequalities, despite being redundant, are always $\mathcal{K}$-minimal. Nonetheless, one of our necessary conditions leads us to our next class of inequalities, $\mathcal{K}$-sublinear v.i., studied in section 4. We study the relation between $\mathcal{K}$-sublinearity and $\mathcal{K}$-minimality and show that the set of extreme inequalities in the cone of $\mathcal{K}$-sublinear v.i. contains all of the extreme inequalities from the cone of $\mathcal{K}$-minimal inequalities. In section 5, we establish that every $\mathcal{K}$-sublinear v.i. can be associated with a convex set of certain structure defined by the linear map A and the dual cone $\mathcal{K}^*$. Also we show that any nonempty convex set with this structure leads to a valid inequality. Through this connection with structured convex sets, we provide necessary conditions for $\mathcal{K}$-sublinear v.i., as well as sufficient conditions for a valid inequality to be $\mathcal{K}$-sublinear and $\mathcal{K}$-minimal. We finish this section by examining the conic Mixed Integer Rounding (MIR) inequality from [6] in our framework. We characterize the linearity of $C(A, \mathcal{K}, \mathcal{B})$ in section 6, and finish by stating some further research questions.

2 Related Literature

The literature on solving MICPs is growing rapidly. On one hand clearly any method for general nonlinear integer programming applies to MICPs as well. A significant body of work has extended known techniques from MILPs to nonlinear integer programs, mostly only involving binary variables. These include Reformulation Linearization Technique (see [61, 59] and references therein), Lift-and-Project and Disjunctive Programming methods [9, 10, 22, 51, 56, 60, 62, 63], and the lattice-free set paradigm [15, 16]. In addition to these, several papers (see [47, 48, 58, 57]) introduce hierarchies of convex (semidefinite programming) relaxations in higher dimensional spaces. These relaxations quickly become impractical due their exponentially growing sizes and the difficulty of projecting them onto the original space of variables. Another stream of research [1, 19, 32, 54, 64, 65, 66, 67] is on the development of linear outer approximation based branch-and-bound algorithms for nonlinear integer programming. While they have the advantage of fast and easy to solve relaxations; the bounds from these approximations may not be as strong as desired.

Exploiting the conic structure when present, as opposed to general convexity, paves the way for developing algorithms with much better performance. Particularly in the special case of MILPs, this has led to very successful results. Despite the lack of effective warm-start techniques, efficient interior point methods exist for $\mathcal{K} = \mathcal{L}^n$ or $\mathcal{K} = \mathcal{S}^n_+$ (see [14]). Therefore supplying the branch-and-bound tree with the corresponding continuous conic relaxation at the nodes and deriving cutting planes to strengthen these relaxations have gained considerable interest recently. In particular, Çezik and Iyengar [23] developed valid inequalities for MICPs with general proper cones by extending Chvatal-Gomory (C-G) integer rounding cuts [53]. Several recent papers study MICPs involving Lorentz cones, $\mathcal{K} = \mathcal{L}^n$, and suggest valid inequalities. Atamtürk and Narayanan [6, 7] have introduced conic MIR cuts for a specific set involving $\mathcal{L}^2$ and then reduced the general case of $\mathcal{L}^n$ to this simple case in a systematic way. Outer approximation based inequalities are suggested in [32, 67]. Drewes and Pokutta [33] extended the Lift-and-Project paradigm to MICPs with convex quadratic constraints and binary variables. Belotti et al. [12] studied the intersection of a convex set with a linear disjunction and suggested a new conic cut for MICPs with $\mathcal{K} = \mathcal{L}^n$. Dadush et al. [31] studied split cuts for ellipsoids, a specific MICP with a bounded feasible region, and independently suggested a valid inequality, which overlaps with the one from [12]. Recently Andersen and Jensen [3], and Modaresi et al. [51] generalized split and intersection cuts to MICPs involving a Lorentz cone, and derived closed form conic quadratic expressions for the resulting valid inequalities.

The literature on cutting plane theory for MILP, i.e., when $\mathcal{K} = \mathbb{R}^n_+$, is extensive (see [27] for a recent survey). Most cutting planes used in MILP can be viewed in the context of Gomory's corner polyhedron, which dates back to [37]. A particular stream of research studies a related problem based upon building a semi-infinite relaxation of an MILP, i.e., the *mixed-integer group problem* of [38] and introduces the study of *minimal functions* and *extremal functions* as convenient ways of examining the properties of cut generating functions. In this literature, various papers (see [38, 39, 44] and references therein) have focused on characterizing *valid functions*, i.e., $\psi : \mathbb{R}^m \to \mathbb{R}$ such that the inequality

$$\sum_{j=1}^{n} \psi(A^j) x_j \geq 1$$

holds for all feasible solutions x for any possible number of variables and any choice of columns, A^j, corresponding to these variables. In this framework, a valid function ψ is said to be *minimal* if there is no valid function ψ' distinct from ψ such that $\psi' \leq \psi$ (the inequality relation between functions is stated in terms of pointwise relation). As non-minimal valid functions are implied by a combination of minimal valid functions, only minimal valid functions are needed to generate valid inequalities. Gomory and Johnson in [38, 39] studied a single row setting, i.e., $m = 1$, and characterized corresponding minimal functions. Johnson [44] extended it further by characterizing minimal functions for m-row relaxations. Since then many related relaxations have been investigated extensively for deriving valid inequalities from multiple rows of a simplex tableau (see [4, 11, 20, 28] and references therein). We refer the reader to [26] for a recent survey.

Perhaps the papers that are most closely connected to our study in this stream are from the "lattice-free cutting plane" theory. For the MILP case, a number of studies, e.g., [4, 20, 25, 26], establish an intimate connection between minimal functions and lattice-free (in our context $\mathcal{B}$-free) convex sets, i.e., convex sets which have a certain given $\hat{b} \notin \mathcal{B}$ in their interior, do not contain any point from the given set $\mathcal{B}$ and are maximal with respect to inclusion. In particular, Borozan and Cornuejols [20] established that minimal valid functions for the semi-infinite relaxation of an MILP correspond to maximal lattice-free convex sets, and thus they arise from nonnegative, piecewise linear, positively homogeneous, convex functions. There is a particular relation between these results and the intersection cuts of [8] as well as the multiple right hand side choice systems [45]. We refer the interested reader to [24, 25, 26, 29] for details and recent results. In the MILP framework, the relaxations of origins described in section 1.2 are studied extensively in the context of "lattice-free cutting plane" theory. Despite the extensive literature available in this area for $\mathcal{K} = \mathbb{R}^n_+$, to the best of our knowledge there is no literature on this topic in the general (non-polyhedral) conic case.

3 $\mathcal{K}$-Minimal Valid Inequalities

In the case of MILP, a minimal valid inequality is defined as a valid inequality $(\mu; \eta_0)$ such that if $\rho \leq \mu$ and $\rho \neq \mu$, then $(\rho; \eta_0)$ is not a v.i., i.e., reducing any μ_i for $i \in \{1, \ldots, n\}$ will lead to a strict reduction in the right hand side value. A natural extension of this definition to the MICP case leads us to:

Definition 3.1 ($\mathcal{K}$-minimal valid inequality) *An inequality $(\mu; \eta_0)$ is a minimal (with respect to cone $\mathcal{K}$) valid inequality if $\mu \in \Pi(A, \mathcal{K}, \mathcal{B})$, $\eta_0 \leq \mu_0$, and for all $\rho \in \Pi(A, \mathcal{K}, \mathcal{B})$, $\rho \neq \mu$, and $\rho \preceq_{\mathcal{K}^*} \mu$, we have $\rho_0 < \eta_0$.*

Note that $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$ implies $x \in \mathcal{K}$, and from the definition of the dual cone, for any $\delta \in \mathcal{K}^* \setminus \{0\}$, the inequality $\langle \delta, x \rangle \geq 0$ is a valid inequality for $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$, and thus $(\delta; 0) \in C(A, \mathcal{K}, \mathcal{B})$ for any $\delta \in \mathcal{K}^* \setminus \{0\}$. We refer to these valid inequalities as *cone-implied inequalities*. Also note that none of the cone-implied inequalities is $\mathcal{K}$-minimal. To see this, consider any v.i. $(\delta; 0)$ with $\delta \in \mathcal{K}^* \setminus \{0\}$, then for $\rho = \alpha\delta$ with $\alpha < 1$, we have $\delta - \rho = (1 - \alpha)\delta \in \mathcal{K}^* \setminus \{0\}$ implying $\rho \preceq_{\mathcal{K}^*} \delta$ and $(\rho; 0) \in C(A, \mathcal{K}, \mathcal{B})$. Moreover, according to our definition, a $\mathcal{K}$-minimal v.i. $(\mu; \eta_0)$ cannot be dominated by another v.i., which is a sum of a cone implied inequality and another valid inequality. The following example shows a simple set $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ together with the $\mathcal{K}$-minimal inequalities describing its convex hull.

Example 3.1 *Let $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ be defined with $\mathcal{K} = \mathcal{L}^3 = \{x \in \mathbb{R}^3 : \ x_3 \geq \sqrt{x_1^2 + x_2^2}\ \} = \mathcal{K}^*$, $A = [-1, 0, 1]$ and $\mathcal{B} = \{0, 2\}$, i.e.,*

$$\mathcal{S}(A, \mathcal{K}, \mathcal{B}) = \{x \in \mathcal{K} : \ -x_1 + x_3 = 0\} \bigcup \{x \in \mathcal{K} : \ -x_1 + x_3 = 2\}.$$

Then

$$\begin{aligned}\mathrm{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) &= \{x\in\mathbb{R}^3:\ x_3\geq\sqrt{x_1^2+x_2^2},\ 0\leq -x_1+x_3\leq 2\}\\ &= \{x\in\mathbb{R}^3:\ \langle x,\delta\rangle\geq 0\ \forall\delta\in\mathrm{Ext}(\mathcal{K}^*),\ x_1-x_3\geq -2\},\end{aligned}$$

and thus the cone of valid inequalities is given by

$$C(A,\mathcal{K},\mathcal{B})=\mathrm{cone}\left(\mathcal{K}^*\times\{0\},(1;0;-1;-2)\right).$$

The only $\mathcal{K}$-minimal inequality in this description is given by $\mu=(1;0;-1)$ with $\eta_0=-2=\mu_0$. It is easy to see that this inequality is valid and also necessary for the description of the convex hull. In order to verify that it is in fact $\mathcal{K}$-minimal, consider any $\delta\in\mathcal{K}^*\setminus\{0\}$, and set $\rho=\mu-\delta$. Then ρ_0, i.e., the right hand side value for which $\langle\rho,x\rangle\geq\rho_0$ is a v.i., is given by

$$\begin{aligned}\rho_0 &:= \inf_x\{\langle\rho,x\rangle:\ x\in\mathcal{S}(A,\mathcal{K},\mathcal{B})\}\\ &\leq \inf_x\{\langle\rho,x\rangle:\ x\in\mathcal{K},\ -x_1+x_3=2\}\\ &= \inf_x\{x_1-x_3-\langle\delta,x\rangle:\ x\in\mathcal{K},\ -x_1+x_3=2\}\\ &= \inf_x\{-2-\langle\delta,x\rangle:\ x\in\mathcal{K},\ -x_1+x_3=2\}\\ &= -2-\sup_x\{\langle\delta,x\rangle:\ x\in\mathcal{K},\ -x_1+x_3=2\}\\ &< -2=\mu_0,\end{aligned}$$

where the strict inequality follows from the fact that $u=(0;1;2)$ is in the interior of $\mathcal{K}$ and satisfies $-u_1+u_3=2$ (and thus is feasible) and moreover for any $\delta\in\mathcal{K}^*\setminus\{0\}$, $\langle\delta,u\rangle>0$. Clearly all of the other inequalities involved in the description of $\mathrm{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ *are of the form $\langle\delta,x\rangle\geq 0$ with $\delta\in\mathrm{Ext}(\mathcal{K}^*)$ and hence are not $\mathcal{K}$-minimal.*

On the other hand, there can be situations where none of the inequalities describing the convex hull of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, including the non-cone-implied inequalities, is $\mathcal{K}$-minimal. Let us consider a slightly modified version of Example 3.1 with a different $\mathcal{B}$ set:

Example 3.2 *Let $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ be defined with $\mathcal{K}=\mathcal{L}^3=\{x\in\mathbb{R}^3:\ x_3\geq\sqrt{x_1^2+x_2^2}\}$, $A=[-1,0,1]$ and $\mathcal{B}=\{0\}$. Then* $\mathrm{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))=\{x\in\mathbb{R}^3:\ x_3\geq\sqrt{x_1^2+x_2^2},\ -x_1+x_3=0\}=\{x\in\mathbb{R}^3:\ x_1=x_3,\ x_2=0,\ x_1,x_3\geq 0\}$

None of the inequalities in the description of $\mathrm{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$ *is $\mathcal{K}$-minimal. In fact the only inequality in the description of* $\mathrm{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$*, which is not a cone implied inequality is $\mu=(-1;0;1)$ with $\eta_0=0=\mu_0$. But this v.i. cannot be $\mathcal{K}$-minimal since $\rho=(-0.5;0;0.5)$ satisfies $\delta=\mu-\rho=(-0.5;0;0.5)\in\mathrm{Ext}(\mathcal{K}^*)$ and $(\rho;\eta_0)$ is a v.i. In fact, for any valid inequality $(\mu;\eta_0)$ that is in the description of* $\mathrm{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$*, there exists $\alpha>0$ such that we can subtract the vector $\delta=\alpha(-1;0;1)\in\mathrm{Ext}(\mathcal{K}^*)$ from μ, and still obtain $(\mu-\delta;\eta_0)$ as a valid inequality.*

Our main result in this section gives an exact characterization of the cases when the inequalities that form a generating set (G_L,G_C) are $\mathcal{K}$-minimal. Before we state our main result, we would like to remind a definition. Given two vectors, $u,v\in C$ where C is a cone with lineality space L, u is said to be an *L-multiple* of v if $u=\alpha v+\ell$ for some $\alpha>0$, and $\ell\in L$. From this definition, it is clear that if u is an L-multiple of v, then v is also an L-multiple of u.

Note that whenever the lineality L of the cone $C(A, \mathcal{K}, \mathcal{B})$ is nontrivial, i.e., $L \neq \{0\}$, the generating valid inequalities are only defined uniquely up to the L-multiples. In addition to this, the trivial v.i. given by $(\delta; 0)$ for some $\delta \in \mathcal{K}^*$ are not of particular interest as they can be extreme but they are never $\mathcal{K}$-minimal. Therefore we define G_C^+ to be the vectors from G_C, which are not L-multiples of any $(\delta; 0)$ where $\delta \in \mathcal{K}^* \setminus \{0\}$. Note that G_C^+ is again only uniquely defined up to L-multiples.

The peculiar situation of Example 3.2 is a result of the fact that $\mathcal{S}(A, \mathcal{K}, \mathcal{B}) \subset \{x \in \mathcal{K} : -x_1 + x_3 = 0\}$. The next Proposition states that this is precisely the situation in which the notion of $\mathcal{K}$-minimality of a v.i. is not well-defined. Note that this is a straightforward extension of the associated result from [45] given in the linear case to our conic case.

Proposition 3.1 *Every valid equation in G_L and every generating valid inequality in G_C^+ is $\mathcal{K}$-minimal if and only if there does not exist $\delta \in \mathcal{K}^* \setminus \{0\}$ such that $\langle \delta, x \rangle = 0$ for all $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$, i.e., $\nexists \delta \in \mathcal{K}^* \setminus \{0\}$ such that $(\delta; 0)$ is a v.e.*

Proof.

($\Rightarrow$) Suppose $\exists \delta \in \mathcal{K}^*$ such that $\delta \neq 0$ and $\langle \delta, x \rangle = 0$ holds for all $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$. Then $(\delta; 0)$ is v.e., and hence $(\delta; 0) \in L$. Suppose $(\mu; \mu_0)$ is a v.i., then $(\mu - \delta; \mu_0)$ is also valid. Since $\delta \in \mathcal{K}^*$, this implies that $(\mu; \mu_0)$ is not $\mathcal{K}$-minimal. Given that $(\mu; \mu_0)$ was arbitrary this implies that there is no $\mathcal{K}$-minimal valid inequality.

($\Leftarrow$) This part of the proof uses the following Lemma from [45]:

Lemma 3.1 *Suppose v is in a generating set for cone C and there exists $v^1, v^2 \in C$ such that $v = v^1 + v^2$, then v^1, v^2 are L-multiples of v.*

Suppose $(\mu; \mu_0) \in G_L \cup G_C^+$ is not $\mathcal{K}$-minimal. Then there exists a nonzero $\delta \in \mathcal{K}^*$ such that $(\mu - \delta; \mu_0) \in C(A, \mathcal{K}, \mathcal{B})$. Note that $(\delta; 0) \in C(A, \mathcal{K}, \mathcal{B})$, therefore $(\mu + \delta; \mu_0)$ is valid as well. Using the above Lemma, we obtain $(\delta; 0)$ is an L-multiple of $(\mu; \mu_0)$. Using the definition of G_C^+, we get $(\mu; \mu_0) \in G_L$. Given that $(\delta; 0)$ is an L-multiple of $(\mu; \mu_0)$ and G_L is uniquely defined up to L-multiples, we get that $(\delta; 0) \in G_L$. Hence $\langle \delta, x \rangle = 0$ is a valid equation, proving the result.

∎

In the remainder of this paper, we will assume that the $\mathcal{K}$-minimality definition of a valid inequality makes sense, i.e.,

Assumption 1: For each $\delta \in \mathcal{K}^* \setminus \{0\}$, there exists some $x_\delta \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$ such that $\langle \delta, x_\delta \rangle > 0$.

Note that under **Assumption 1**, Proposition 3.1 simply states that in order to describe the convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$, one only needs to include $\mathcal{K}$-minimal valid inequalities in addition to the trivial cone-implied inequalities. Let $C_m(A, \mathcal{K}, \mathcal{B})$ denote the set of $\mathcal{K}$-minimal valid inequalities. Note that $C_m(A, \mathcal{K}, \mathcal{B})$ is clearly closed under positive scalar multiplication and is thus a cone (but it is not necessarily a convex cone). Then using Proposition 3.1, we have the following

Corollary 3.1 *Suppose that **Assumption 1** holds. Then for any generating set (G_L, G_C) of $C(A, \mathcal{K}, \mathcal{B})$, (G_L, G_C^+) generates $C_m(A, \mathcal{K}, \mathcal{B})$.*

In light of Proposition 3.1 and Corollary 3.1, we arrive at the conclusion that the set of $\mathcal{K}$-minimal v.i., i.e., $C_m(A, \mathcal{K}, \mathcal{B})$, constitutes an important (in fact the only nontrivial) class of the generating inequalities under **Assumption 1**. This motivates us to further study their properties.

3.1 Necessary Conditions for $\mathcal{K}$-Minimal Inequalities

A necessary condition for the $\mathcal{K}$-minimality of a v.i. is given by

Proposition 3.2 *Let $(\mu;\eta_0)$ with $\mu \in \Pi(A,\mathcal{K},\mathcal{B})$ and $\eta_0 \in \mathbb{R}$, be a $\mathcal{K}$-minimal v.i. Then whenever $\mu \in \mathcal{K}^*$ or $\mu \in -\mathcal{K}^*$, we have $\eta_0 = \mu_0$ where μ_0 is as defined in* (2)*. Furthermore, $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$ and $\mu \in \mathcal{K}^*$ (respectively $\mu \in -\mathcal{K}^*$) implies $\eta_0 > 0$ (respectively $\eta_0 < 0$).*

Proof. Suppose $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$. Since $(\mu;\eta_0)$ is a v.i., $\eta_0 \le \mu_0$. Assume for contradiction that $\eta_0 < \mu_0$. We need to consider only two cases:

(i) $\mu \in \mathcal{K}^* \setminus \{0\}$: Then we should have $\eta_0 > 0$, otherwise it is either a cone implied inequality or is dominated by a cone-implied inequality, both of which are not possible. Let $\beta = \frac{\eta_0}{\mu_0}$, and consider $\rho = \beta \cdot \mu$. Then $(\rho;\eta_0)$ is a valid inequality since $0 < \beta < 1$, $(\mu;\mu_0) \in C(A,\mathcal{K},\mathcal{B})$ and $C(A,\mathcal{K},\mathcal{B})$ is a cone. But $\mu - \rho = (1-\beta)\mu \in \mathcal{K}^* \setminus \{0\}$ since μ is nonzero and $\beta < 1$, this is a contradiction. Thus we conclude $\eta_0 = \mu_0$.

(ii) $-\mu \in \mathcal{K}^* \setminus \{0\}$: If $\mu_0 > 0$, then $\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \emptyset$ since $(-\mu;0)$ is trivially valid and we cannot satisfy both $(-\mu;0)$ and $(\mu;\mu_0)$ when $\mu_0 > 0$. But $\mathcal{S}(A,\mathcal{K},\mathcal{B}) = \emptyset$ contradicts with our assumption that $\mathcal{B} \neq \emptyset$ and for all $b \in \mathcal{B}$, there exist $x \in \mathcal{K}$ with $Ax = b$. Moreover if $\mu_0 = 0$, then $\mathcal{S}(A,\mathcal{K},\mathcal{B}) \subset \{x \in \mathcal{K} :\ \langle \mu, x\rangle = 0\}$, which contradicts **Assumption 1**. Hence we conclude that $\eta_0 < \mu_0 < 0$. Once again let $\beta = \frac{\eta_0}{\mu_0}$, and consider $\rho = \beta \cdot \mu$. Then $(\rho;\eta_0)$ is a valid inequality since $\beta > 1$, $(\mu;\mu_0) \in C(A,\mathcal{K},\mathcal{B})$ and $C(A,\mathcal{K},\mathcal{B})$ is a cone. But $\mu - \rho = (1-\beta)\mu \in \mathcal{K}^* \setminus \{0\}$ since $\mu \in -\mathcal{K}^* \setminus \{0\}$ and $\beta > 1$. But this is a contradiction to the $\mathcal{K}$-minimality of $(\mu;\eta_0)$, hence we conclude that $\eta_0 = \mu_0$.

∎

Clearly Proposition 3.2 does not cover all possible cases for μ, i.e., it is possible to have $\mu \notin \{\mathcal{K}^*, -\mathcal{K}^*\}$ leading to a $\mathcal{K}$-minimal v.i. While one is naturally inclined to believe that for a $\mathcal{K}$-minimal inequality $(\mu;\eta_0)$, we should always have $\eta_0 = \mu_0$ where μ_0 is as defined by (2), the following example shows that this is not necessarily true.

Example 3.3 *Consider the following solution set $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ defined with $A = [-1, 1]$, $\mathcal{B} = \{-2, 1\}$ and $\mathcal{K} = \mathbb{R}^2_+$. Clearly*

$$\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B})) = \{x \in \mathbb{R}^2 :\ -x_1 + x_2 \ge -2,\ x_1 - x_2 \ge -1,\ x_1 + 2x_2 \ge 2,\ x_1, x_2 \ge 0\},$$

and one can easily show that each of the nontrivial inequalities in this description is in fact $\mathcal{K}$-minimal.

*Let us consider the v.i. given by $(\mu;\eta_0) = (1;-1;-2)$ with $\mu_0 = -1$ which is clearly dominated by $x_1 - x_2 \ge -1$. We will show that according to our $\mathcal{K}$-minimality definition (in this case, it is the usual minimality definition), $(\mu;\eta_0)$ is also $\mathcal{K}$-minimal. Note that **Assumption 1** holds since $\{(0;1),(2;0)\} \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$, so minimality makes sense.*

Suppose that $(\mu;\eta_0)$ is not $\mathcal{K}$-minimal, then there exists $\rho = \mu - \delta$ with $\delta \in \mathcal{K}^* = \mathbb{R}^2_+$, and $\delta \neq 0$ such that $(\rho;\eta_0)$ is a valid inequality. This implies

$$\begin{aligned}
-2 = \eta_0 \ &\le\ \inf_x\{\langle \rho, x\rangle :\ x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})\} = \min_x\{\langle \rho, x\rangle :\ x \in \text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))\} \\
&=\ \min_x\{\langle \rho, x\rangle :\ -x_1 + x_2 \ge -2,\ x_1 - x_2 \ge -1,\ x_1 + 2x_2 \ge 2,\ x_1, x_2 \ge 0\} \\
&=\ \max_\lambda\{-2\lambda_1 - \lambda_2 + 2\lambda_3 :\ -\lambda_1 + \lambda_2 + \lambda_3 \le \rho_1,\ \lambda_1 - \lambda_2 + 2\lambda_3 \le \rho_2,\ \lambda \in \mathbb{R}^3_+\} \\
&=\ \max_\lambda\{-2\lambda_1 - \lambda_2 + 2\lambda_3 :\ -\lambda_1 + \lambda_2 + \lambda_3 \le 1 - \delta_1,\ \lambda_1 - \lambda_2 + 2\lambda_3 \le -1 - \delta_2,\ \lambda \in \mathbb{R}^3_+\}
\end{aligned}$$

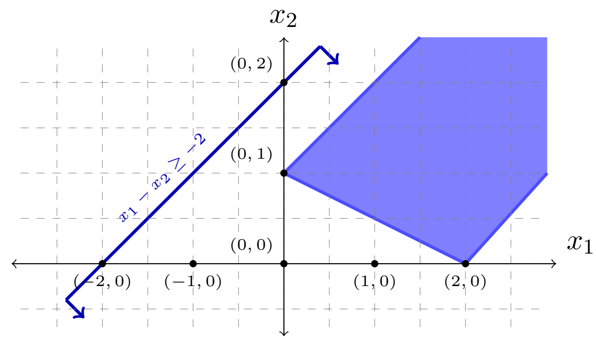

Figure 1: Convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ for Example 3.3

where the third equation follows from strong duality (clearly the primal problem is feasible), and the fourth equation follows from the definition of $\rho = \mu - \delta$. On the other hand the following system

$$\begin{array}{rcl} \lambda & \geq & 0 \\ \lambda_1 - \lambda_2 - \lambda_3 & \geq & \delta_1 - 1 \\ -\lambda_1 + \lambda_2 - 2\lambda_3 & \geq & 1 + \delta_2 \end{array}$$

implies that $0 \geq -3\lambda_3 \geq \delta_1 + \delta_2$. Considering that $\delta \in \mathbb{R}^2_+$, this leads to $\delta_1 = \delta_2 = 0$, which is a contradiction to $\delta \neq 0$. Therefore we conclude that $(\mu; \eta_0) = (1; -1; -2) \in C_m(A, \mathcal{K}, \mathcal{B})$ yet $\eta_0 \neq \mu_0$.

In fact we can generalize the situation of Example 3.3, and prove the following Proposition, which states that under a special condition, $\text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset$, any valid inequality with $\mu \in \text{Im}(A^*)$ is a $\mathcal{K}$-minimal inequality.

Proposition 3.3 *Suppose $\text{Ker}(A) \cap \text{int}(\mathcal{K}) \neq \emptyset$. Then for any $\mu \in \text{Im}(A^*)$ and any $-\infty < \eta_0 \leq \mu_0$ where μ_0 is defined by (2), we have $(\mu; \eta_0) \in C_m(A, \mathcal{K}, \mathcal{B})$.*

Proof. Consider $d \in (\text{Ker}(A) \cap \text{int}(\mathcal{K})) \neq \emptyset$, note that $d \neq 0$. Then for any $b \in \mathcal{B}$, d is in the recession cone of the set $\mathcal{S}_b := \{x \in E : \ Ax = b, \ x \in \mathcal{K}\})$. Let $x_b \in \mathcal{S}_b$, then $P_b := \{x_b + \tau d : \ \tau \geq 0\} \subseteq \mathcal{S}_b$ holds for any $b \in \mathcal{B}$. Moreover $P_b \cap \text{int}(\mathcal{K}) \neq \emptyset$ and thus **Assumption 1** holds here, so $\mathcal{K}$-minimality makes sense.

Assume for contradiction that the statement is not true, i.e., there exists $\mu \in \text{Im}(A^*)$ together with $\eta_0 \leq \mu_0$, such that $(\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B})$. Then there exists $\delta \in \mathcal{K}^* \setminus \{0\}$ such that $(\mu - \delta; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$, which implies

$$\begin{array}{rcl} -\infty < \eta_0 & \leq & \inf_x \{\langle \mu - \delta, x \rangle : \ x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})\} \\ & \leq & \inf_{b \in \mathcal{B}} \inf_x \{\langle \mu - \delta, x \rangle : \ Ax = b, \ x \in \mathcal{K}\} \\ & \leq & \inf_{b \in \mathcal{B}} \inf_x \{\langle \mu - \delta, x \rangle : \ x \in P_b\} \\ & \leq & \inf_{b \in \mathcal{B}} \left[\underbrace{\langle \mu - \delta, x_b \rangle}_{\in \mathbb{R}} + \inf_\tau \{\langle \mu - \delta, \tau d \rangle : \ \tau \geq 0\} \right] \end{array}$$

Also note that $\inf_\tau\{\langle \mu-\delta, \tau d\rangle : \ \tau \geq 0\} = -\infty$ when $\langle \mu-\delta, d\rangle < 0$. But $\langle \mu-\delta, d\rangle < 0$ is impossible since it would have implied $-\infty < \eta_0 \leq -\infty$. Therefore we conclude that $\langle \mu-\delta, d\rangle \geq 0$.

Finally $\mu \in \text{Im}(A^*)$ implies that there exists λ such that $\mu = A^*\lambda$. Taking this into account, we arrive at

$$0 \leq \langle \mu-\delta, d\rangle = \langle A^*\lambda, d\rangle - \langle \delta, d\rangle = \langle \lambda, \underbrace{Ad}_{=0}\rangle - \langle \delta, d\rangle = -\langle \delta, d\rangle$$

where we used the fact that $d \in \text{Ker}(A)$. But $d \in \text{int}(\mathcal{K})$ and $\delta \in \mathcal{K}^* \setminus \{0\}$ implies that $\langle \delta, d\rangle > 0$, which is a contradiction. ∎

Example 3.3 and Proposition 3.3 indicate a weakness of the $\mathcal{K}$-minimality definition. This weakness can be easily fixed by requiring that $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$ is $\mathcal{K}$-minimal only if η_0 *cannot be increased without making the current inequality invalid*, i.e., requiring $\eta_0 = \mu_0$ where μ_0 is as defined in (2). However, in order to be consistent with the original minimality definition for $\mathcal{K} = \mathbb{R}^n_+$, we do not add such a restriction in our $\mathcal{K}$-minimality definition. As will be clear from the rest of the paper, such a restriction will make minimal change in our analysis. Moreover any v.i. $(\mu; \eta_0)$ with $\eta_0 < \mu_0$ clearly will not be of much interest since it will never belong to a generating set for $C(A, \mathcal{K}, \mathcal{B})$.

Proposition 3.4 *Let $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$ and suppose that there exists a linear map $Z : \mathcal{K} \to \mathcal{K}$ such that $AZ^* = A$ where Z^* denotes the conjugate linear map of Z, and $\mu - Z\mu \in \mathcal{K}^* \setminus \{0\}$, then $(\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B})$.*

Proof. Given $(\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B})$, suppose the condition in the statement of the proposition holds for such a linear map Z, we shall prove that $(\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B})$. Let $\delta = \mu - Z\mu$, then $\delta \in \mathcal{K}^* \setminus \{0\}$ and define $\rho := \mu - \delta$. Consider any $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$, then $Z^*x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$. (Note that $Z : \mathcal{K} \to \mathcal{K}$, and hence for any $x \in \mathcal{K}$, we have $Z^*x \in \mathcal{K}$. Moreover, $AZ^*x = Ax$ due to $AZ^* = A$ and $AZ^*x = Ax = b$ for some $b \in \mathcal{B}$ since $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$). In addition to this note that

$$\langle \rho, x\rangle = \langle \mu - \delta, x\rangle = \langle Z\mu, x\rangle = \langle \mu, Z^*x\rangle \geq \eta_0,$$

where the last inequality follows from the definition of η_0 and the fact that $Z^*x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$. Hence we have

$$\inf_x\{\langle \rho, x\rangle : \ Ax = b \text{ for some } b \in \mathcal{B}, x \in \mathcal{K}\} \geq \eta_0,$$

which implies that $(\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B})$. ∎

Proposition 3.4 states an involved necessary condition for a valid inequality to be $\mathcal{K}$-minimal. Let $\mathcal{K} \subset E$ and Id be the identity map on E, and define $\mathcal{F}_\mathcal{K}$ as

$$\mathcal{F}_\mathcal{K} := \{(Z : E \to E) : \ Z \text{ is linear, and } Z^*v \in \mathcal{K}, \ \forall v \in \mathcal{K}\}.$$

Then $(\mu; \mu_0)$ is $\mathcal{K}$-minimal inequalities only if the following holds:

$$(\text{Id} - Z)\mu \notin \mathcal{K}^* \setminus \{0\} \ \ \forall Z \in \mathcal{F}_\mathcal{K} \text{ such that } AZ^* = A.$$

Based on this result, the set $\mathcal{F}_\mathcal{K}$ has certain importance. In fact $\mathcal{F}_\mathcal{K}$ is the cone of $\mathcal{K} - \mathcal{K}$ positive maps, which also appear in applications of robust optimization, quantum physics, etc. (see [13]). When $\mathcal{K} = \mathbb{R}^n_+$, $\mathcal{F}_\mathcal{K} = \{Z \in \mathbb{R}^{n\times n} : \ Z_{ij} \geq 0 \ \forall i, j\}$. However, in general, the description of $\mathcal{F}_\mathcal{K}$ can be quite nontrivial for different cones $\mathcal{K}$. In fact, in [13], it is shown that deciding whether a given linear map takes $\mathcal{S}^n_+$ to itself is an NP-Hard optimization problem. In another particular case of interest, when $\mathcal{K} = \mathcal{L}^n$, a quite nontrivial explicit description of $\mathcal{F}_\mathcal{K}$ via a semidefinite representation is given by Hildebrand in [41, 42]. Due to the general difficulty of characterizing $\mathcal{F}_\mathcal{K}$, in the next section, we study a relaxed version of the condition from Proposition 3.4 and focus on a larger class of valid inequalities, which subsumes the class of $\mathcal{K}$-minimal inequalities. We refer this larger class as *$\mathcal{K}$-sublinear inequalities*.

4 $\mathcal{K}$-Sublinear Valid Inequalities

Definition 4.1 *For any $(\mu; \eta_0)$ with $\mu \in \Pi(A, \mathcal{K}, \mathcal{B})$ and η_0 finite, we will say that $(\mu; \eta_0)$ is a $\mathcal{K}$-*sublinear *v.i. if it satisfies both of the following conditions:*

(***A.1***) $\quad 0 \leq \langle \mu, u\rangle$ *for all u such that $Au = 0$ and $\langle \alpha, v\rangle u + v \in \mathcal{K} \ \ \forall v \in \text{Ext}(\mathcal{K})$ holds for some $\alpha \in \text{Ext}(\mathcal{K}^*)$,*

(***A.2***) $\quad \eta_0 \leq \langle \mu, x\rangle$ *for all $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$.*

We will denote $C_a(A, \mathcal{K}, \mathcal{B})$ as the cone of $\mathcal{K}$-sublinear valid inequalities (one can easily verify that the set of $(\mu; \eta_0)$ satisfying conditions (**A.1**)-(**A.2**) in fact leads to a cone in the space $E \times \mathbb{R}$).

Condition (**A.2**) is simply true for every v.i. On the other hand, condition (**A.1**) is not very intuitive. There is a particular and simple case of (**A.1**) that is of interest and deserves a separate treatment. Let $(\mu; \eta_0)$ satisfy (**A.1**), then it also satisfies the following condition:

$$(\textbf{A.0}) \qquad 0 \leq \langle \mu, u\rangle \text{ for all } u \in \mathcal{K} \text{ such that } Au = 0.$$

In order to see that in fact (**A.0**) is a special case of (**A.1**), consider any u such that $u \in \mathcal{K}$ and $Au = 0$, and together with any $\alpha \in \text{Ext}(\mathcal{K}^*)$. Since $\mathcal{K}$ is a cone, we can see that for any such pair of u and α condition (**A.1**) is automatically satisfied. While (**A.1**) already implies (**A.0**), treating (**A.0**) separately seems to be handy as some of our results depend solely on conditions (**A.0**) and (**A.2**).

While condition (**A.0**) is not as strong as (**A.1**), we have the following Remark and Proposition on condition (**A.0**).

Remark 4.1 *From the definitions of* $\text{Ker}(A)$ *and* $\text{Im}(A^*)$*, it is clear that if $\mu \in \text{Im}(A^*)$, then μ satisfies condition (**A.0**).*

Moreover, we have the following relation

Proposition 4.1 *Suppose $\mu \in \Pi(A, \mathcal{K}, \mathcal{B})$, then μ satisfies condition (**A.0**).*

Proof. Suppose condition (**A.0**) is violated by some $\mu \in \Pi(A, \mathcal{K}, \mathcal{B})$. Then there exists $u \in \mathcal{K}$ such that $Au = 0$ and $\langle \mu, u\rangle < 0$. Note that for any $\beta > 0$ and $x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$, $x + \beta u \in \mathcal{K}$ and $A(x + \beta u) = Ax \in \mathcal{B}$, hence $x + \beta u \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$. On the other hand,

$$\langle \mu, x + \beta u\rangle = \langle \mu, x\rangle + \beta\langle \mu, u\rangle$$

can be made arbitrarily small by increasing β, which implies that $\mu_0 = -\infty$ where μ_0 is as defined in (2). But then this is a contradiction since we assumed $\mu \in \Pi(A, \mathcal{K}, \mathcal{B})$, which implies $\mu_0 \neq -\infty$. ∎

Our next Theorem simply states that every $\mathcal{K}$-sublinear valid inequality is a $\mathcal{K}$-minimal v.i.

Theorem 4.1 *If $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$, then $(\mu;\eta_0) \in C_a(A,\mathcal{K},\mathcal{B})$.*

Proof. Consider any $\mathcal{K}$-minimal v.i. $(\mu;\eta_0)$. Since $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$, then $(\mu;\eta_0)$ is a valid inequality and hence condition (**A.2**) of $\mathcal{K}$-sublinearity is automatically satisfied.

Assume for contradiction that condition (**A.1**) is violated by $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$, i.e., there exists u such that $Au = 0$, $\langle \mu, u\rangle < 0$ and an $\alpha \in \text{Ext}(\mathcal{K}^*)$ such that $\langle \alpha, v\rangle u + v \in \mathcal{K} \ \ \forall v \in \text{Ext}(\mathcal{K})$ holds. Based on u and α, let us define a linear map $Z : E \to E$ as

$$Zx = \langle x, u\rangle \alpha + x \ \text{ for any } x \in E.$$

Note that $A : E \to \mathbb{R}^m$ and thus its conjugate $A^* : \mathbb{R}^m \to E$. Without loss of generality let $A^* e_i =: A^i \in E$ for $i = 1,\ldots,m$, where e_i is the i^{th} unit vector in $\mathbb{R}^m$. This way, for all $i = 1,\ldots,m$, we have $ZA^*e^i = \langle A^i, u\rangle + A^i = A^i$ where we have used $u \in \text{Ker}(A)$ implies $\langle A^i, u\rangle = 0$. Therefore we have $ZA^* = A^*$, moreover since $A : E \to \mathbb{R}^m$ and $Z : E \to E$ are linear maps, we have ZA^* is a linear map and its conjugate is given by $AZ^* = A$ as desired.

Moreover, for all $w \in \mathcal{K}^*$ and $v \in \text{Ext}(\mathcal{K})$, we have

$$\langle Zw, v\rangle = \langle(\langle w,u\rangle\alpha + w), v\rangle = \langle w,u\rangle\langle\alpha,v\rangle + \langle w,v\rangle = \langle w, \underbrace{\langle\alpha,v\rangle u + v}_{\in\mathcal{K}}\rangle \geq 0.$$

As any $v \in \mathcal{K}$ can be written as a convex combination of points from $\text{Ext}(\mathcal{K})$, we conclude that $Z \in \mathcal{F}_{\mathcal{K}}$. Finally by recalling that $\alpha \in \mathcal{K}^*$ and is nonzero, we get

$$\mu - Z\mu = -\underbrace{\langle\mu,u\rangle}_{<0}\alpha \in \mathcal{K}^* \setminus \{0\},$$

which is a contradiction to the necessary condition for $\mathcal{K}$-minimal inequalities given by Proposition 3.4. ∎

The proof of Theorem 4.1 reveals that a specific case of the condition from Proposition 3.4 underlies the definition of $\mathcal{K}$-sublinear inequalities, in particular condition (**A.1**). Next, we intend to show that conditions (**A.0**)-(**A.2**) underlie the definition of *subadditive inequalities* from [45] in the MILP case.

Remark 4.2 *When the cone $\mathcal{K}$ is known and simple enough, i.e., it has finitely many and orthogonal to each other extreme rays, the interesting cases of condition (**A.1**), i.e., the ones that are not covered by condition (**A.0**), can be simplified.*

Suppose $\mathcal{K} = \mathbb{R}^n_+$, then the extreme rays of $\mathcal{K}$ as well as $\mathcal{K}^*$ are just the unit vectors, e_i. Let us first consider the case of $\alpha = e_i$. In such a case, for any $v \in \text{Ext}(\mathcal{K})$, we have the requirement on u as

$$v_i u + v \in \mathcal{K} \ \ \forall v \in \text{Ext}(\mathcal{K}) := \{e_1,\ldots,e_n\}.$$

*Considering that we only have unit vectors as the extreme rays, we see that this requirement may have an effect only for the extreme rays with a nonzero v_i value, which is just the case of $v = e_i$. And hence we can equivalently decompose condition (**A.1**) into n easier to deal with pieces and rephrase it as follows:*

$$(\textbf{\textit{A.1i}}) \qquad 0 \leq \langle\mu,u\rangle \text{ for all } u \text{ such that } Au = 0 \text{ and } u + e_i \in \mathbb{R}^n_+.$$

Let A^i denote the i^{th} column of the matrix A. By a change of variables, one easily notices that this requirement is equivalent to the following subaddtivity relation on the coefficients of μ (imagine that $\mu_i =$

$\sigma(A^i)$ for some subadditive function $\sigma(\cdot)$, then the relation below would exactly represent the subadditivity property of $\sigma(\cdot)$ over the columns of A):

$$(\textbf{\textit{A.1i}}) \qquad \mu_i \leq \langle \mu, w \rangle \text{ for all } w \in \mathbb{R}^n_+ \text{ such that } Aw = A^i, \text{ and for all } i = 1, \ldots, n.$$

*In fact, in the simplest case of $\mathcal{K} = \mathbb{R}^n_+$, this refinement of condition (**A.1**), i.e., (**A.1i**), together with the conditions (**A.0**) and (**A.2**) is defined as the class of subadditive valid inequalities in [45].*

In the situations when $\mathcal{K}$-minimality notion is well-defined, i.e., under **Assumption 1**, one can show that a precise relation between the generators of the cones of $\mathcal{K}$-sublinear v.i. and $\mathcal{K}$-minimal v.i. We state this in Theorem 4.2 below. Note that this is an extension of the corresponding result from [45] to the conic case, and most of the original proof remains intact but we decided to include the result for completeness.

Theorem 4.2 *Suppose that **Assumption 1** holds. Then any generating set of $C_a(A, \mathcal{K}, \mathcal{B})$ will be of form (G_L, G_a) where $G_a \supseteq G_C^+$ and (G_L, G_C) is a generating set of $C(A, \mathcal{K}, \mathcal{B})$. Moreover, if $(\mu; \eta_0) \in G_a \setminus G_C^+$, then $(\mu; \eta_0)$ is not a $\mathcal{K}$-minimal v.i.*

Proof. Let (G_ℓ, G_a) be a generating set of $C_a(A, \mathcal{K}, \mathcal{B})$ and (G_L, G_C) be a generating set of $C(A, \mathcal{K}, \mathcal{B})$. Note that $C_a(A, \mathcal{K}, \mathcal{B}) \subseteq C(A, \mathcal{K}, \mathcal{B})$, and hence the linear subspace of $C_a(A, \mathcal{K}, \mathcal{B})$ is contained in that of $C(A, \mathcal{K}, \mathcal{B})$, therefore without loss of generality we may assume $G_\ell \subseteq G_L$. Moreover under the assumption of Theorem 4.2, we can use Corollary 3.1, which implies that $C_m(A, \mathcal{K}, \mathcal{B})$ has a generating set of the form (G_L, G_C^+). By Theorem 4.1, we have $C_m(A, \mathcal{K}, \mathcal{B}) \subseteq C_a(A, \mathcal{K}, \mathcal{B})$ and hence we can conclude that the subspace spanned by G_L is contained in that of G_ℓ. Together with $G_\ell \subseteq G_L$, we arrive at $G_\ell = G_L$.

Let Q be the orthogonal complement to the subspace generated by G_L and define $C' = C(A, \mathcal{K}, \mathcal{B}) \cap Q$, $C'_m = C_m(A, \mathcal{K}, \mathcal{B}) \cap Q$ and $C'_a = C_a(A, \mathcal{K}, \mathcal{B}) \cap Q$. Clearly C', C'_m and C'_a are pointed cones and satisfy $C'_m \subseteq C'_a \subseteq C'$. Without loss of generality we can assume that $C' = \text{cone}(G_C)$, and $C'_m = \text{cone}(G_C^+)$. If not, we can always get to this case by adding and/or subtracting vectors from the subspace spanned by G_L. Note that when we add/subtract a v.e. from a $\mathcal{K}$-minimal v.i., we always end up with a $\mathcal{K}$-minimal v.i. again. Given that the elements of G_C^+ are extreme in both C' and C'_m, they remain extreme in C'_a as well. Therefore $G_C^+ \subseteq G_a$.

Finally, consider any $(\mu; \eta_0) \in G_a \setminus G_C^+$. We need to show that $(\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B})$. Suppose not, then $(\mu; \eta_0) \in C_m(A, \mathcal{K}, \mathcal{B})$ but not in G_C^+, which implies that $(\mu; \eta_0)$ is not extreme in $C_m(A, \mathcal{K}, \mathcal{B})$. Noting $C_m(A, \mathcal{K}, \mathcal{B}) \subseteq C_a(A, \mathcal{K}, \mathcal{B})$, we conclude that $(\mu; \eta_0)$ is not extreme in $C_a(A, \mathcal{K}, \mathcal{B})$ as well but this is a contradiction to the fact that $(\mu; \eta_0) \in G_a$ and (G_L, G_a) is a generating set for $C_a(A, \mathcal{K}, \mathcal{B})$. Therefore we conclude that for any $(\mu; \eta_0) \in G_a \setminus G_C^+$, $(\mu; \eta_0) \notin C_m(A, \mathcal{K}, \mathcal{B})$. ∎

The above theorem implicitly describes a way of obtaining all of the nontrivial v.i. of $C(A, \mathcal{K}, \mathcal{B})$: first identify a generating set (G_L, G_a) for $C_a(A, \mathcal{K}, \mathcal{B})$ and then test its elements for $\mathcal{K}$-minimality to identify G_C^+. On one hand, this is good news, as we seem to have a better handle on $C_a(A, \mathcal{K}, \mathcal{B})$ via the conditions given by **(A.0)**-**(A.2)**. On the other hand, testing these conditions as stated in **(A.0)**-**(A.2)**, seems to be a nontrivial task. Moreover we need to establish further ways of characterizing $\mathcal{K}$-minimality. Both of these tasks constitute our next section. As we proceed, we also demonstrate how to apply our framework via a few examples.

5 Relations to Support Functions

In this section, we relate our characterization of $\mathcal{K}$-sublinear v.i. to the support functions of sets with certain structure. Recall that a *support function* of a nonempty set $D \subseteq \mathbb{R}^m$ is defined as

$$\sigma_D(z) := \sup_\lambda \{\langle z, \lambda\rangle : \ \lambda \in D\}$$

for any $z \in \mathbb{R}^m$.

For any nonempty set D, it is well known that its support function, $\sigma_D(\cdot)$, satisfies the following properties (see [43, 55] for an extended exposure to the topic):

(S.1) $\sigma_D(0) = 0$, in fact $\sigma_D(0) \in \{0, \infty\}$ holds for any sublinear function (nonnegative)

(S.2) $\sigma_D(z^1 + z^2) \leq \sigma_D(z^1) + \sigma_D(z^2)$ (subadditive)

(S.3) $\sigma_D(\beta z) = \beta\sigma_D(z) \ \ \forall \beta > 0$ and for all $z \in \mathbb{R}^m$ (positively homogeneous)

In particular support functions are positively homogeneous and subadditive, therefore sublinear and thus convex.

There is a particular connection between our $\mathcal{K}$-sublinear v.i. and support functions of convex sets with certain structure. This connection leads to a number of necessary conditions for $\mathcal{K}$-sublinear v.i. We state this connection in a series of results as follows:

Theorem 5.1 *Consider any $(\mu; \eta_0) \in C(A, \mathcal{K}, \mathcal{B})$ that satisfies condition (**A.0**), and define*

$$D = \{\lambda \in \mathbb{R}^m : \ \exists \gamma \in \mathcal{K}^* \text{ such that } A^*\lambda + \gamma = \mu\}. \tag{3}$$

Then $D \neq \emptyset$, $\sigma_D(0) = 0$ and $\sigma_D(Az) \leq \langle \mu, z\rangle$ for all $z \in \mathcal{K}$.

Proof. Suppose $D = \emptyset$. Let $\zeta \in \text{int}(\mathcal{K}^*)$ and consider $D_\zeta := \{(\lambda, \gamma, t) \in \mathbb{R}^m \times \mathcal{K}^* \times \mathbb{R}_+ : \ A^*\lambda + \gamma = \mu + t\zeta\}$. Note that $\zeta \in \text{int}(\mathcal{K}^*)$ implies D_ζ is strictly feasible. Now consider the strictly feasible conic optimization problem

$$\text{Opt}(P) := \sup_{\lambda,\gamma,t} \{-t : \ A^*\lambda + \gamma = \mu + t\zeta, \ \gamma \in \mathcal{K}^*, \ t \geq 0\},$$

and its dual given by

$$\text{Opt}(D) := \inf_z \{\langle \mu, z\rangle : \ Az = 0, \ \langle \zeta, z\rangle \leq 1, \ z \in \mathcal{K}\}.$$

Since D_ζ is strictly feasible, strong duality applies here and $\text{Opt}(P) = \text{Opt}(D)$. Moreover since we have assumed that $D = \emptyset$, $D_\zeta \cap (\mathbb{R}^m \times \mathcal{K}^* \times \{0\}) = \emptyset$, implying that $0 > \text{Opt}(P) = \text{Opt}(D)$. Using strong duality, we have a feasible dual solution $\bar{z} \in \mathcal{K}$ such that $A\bar{z} = 0$, $\langle \zeta, \bar{z}\rangle \leq 1$, and $\langle \mu, \bar{z}\rangle < 0$. But this contradicts condition (**A.0**) and thus $\mathcal{K}$-minimality of $(\mu; \eta_0)$, hence we conclude that $D \neq \emptyset$.

Given that $\sigma_D(\cdot)$ is the support function of D and $D \neq \emptyset$, we have $\sigma_D(0) = 0$.

Finally, for any $z \in \mathcal{K}$, we have

$$\begin{aligned} \sigma_D(Az) &= \sup_\lambda \{\langle Az, \lambda\rangle : \ \lambda \in D\} = \sup_\lambda \{\langle z, A^*\lambda\rangle : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\} \\ &\leq \sup_\lambda \{\langle z, \mu\rangle : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\} = \langle z, \mu\rangle, \end{aligned}$$

where the last inequality follows from the fact that $z \in \mathcal{K}$ and for any $\lambda \in D$, we have $\mu - A^*\lambda \in \mathcal{K}^*$, implying $\langle \mu - A^*\lambda, z\rangle \geq 0$. Therefore we arrive at $\sigma_D(Az) \leq \langle \mu, z\rangle$. ∎

Proposition 5.1 *Suppose μ satisfies condition (**A.0**) and define*

$$\perp_z := \{\gamma \in \mathcal{K}^* : \langle \gamma, z\rangle = 0\} \text{ for any } z \in \mathcal{K}. \tag{4}$$

Then for all $z \in \mathcal{K}$ such that $\perp_z \cap(\mu - \text{Im}(A^*)) \neq \emptyset$, we have $\sigma_D(Az) = \langle \mu, z\rangle$ where D is defined by (3).

Proof. Consider any $z \in \mathcal{K}$, then we have

$$\begin{aligned} \sigma_D(Az) &= \sup_{\lambda}\{\langle Az, \lambda\rangle : \lambda \in D\} \\ &= \sup_{\gamma,\lambda}\{\langle z, A^*\lambda\rangle : A^*\lambda = \mu - \gamma, \gamma \in \mathcal{K}^*\} \\ &= \langle z, \mu\rangle - \inf_{\gamma}\{\langle z, \gamma\rangle : \gamma \in \mu - \text{Im}(A^*), \gamma \in \mathcal{K}^*\} = \langle z, \mu\rangle \end{aligned}$$

where the last equation follows from the fact that $\langle z, \gamma\rangle \geq 0$ for all $z \in \mathcal{K}$ and $\gamma \in \mathcal{K}^*$, and there exists $\bar{\gamma} \in \perp_z \cap(\mu - \text{Im}(A^*))$, i.e., $\bar{\gamma} \in \mathcal{K}^* \cap (\mu - \text{Im}(A^*))$ and $\langle \mu, \bar{\gamma}\rangle = 0$. ∎

Noting that when $\mu \in \text{Im}(A^*)$, we have $0 \in \mathcal{K}^* \cap (\mu - \text{Im}(A^*))$, and by Remark 4.1, we have the following

Corollary 5.1 *For any $\mu \in \text{Im}(A^*)$, we have $D \neq \emptyset$ and $\sigma_D(Az) = \langle \mu, z\rangle$ for all $z \in \mathcal{K}$ where D is defined as in* (3).

We illustrate these necessary conditions for $\mathcal{K}$-sublinearity via the following simple

Example 5.1 *Let $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ be defined with $\mathcal{K} = \mathcal{L}^3 = \{x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\}$, $A = [0, 0, 1]$ and $\mathcal{B} = \{1, 2\}$. Then $\text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) = \{x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}, 1 \leq x_3 \leq 2\}$ (see Figure 2).*

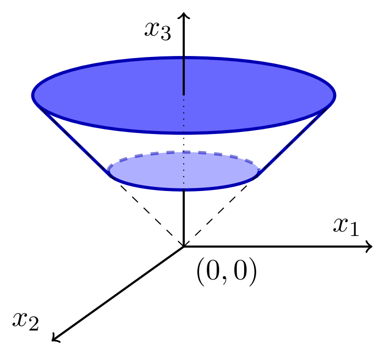

Figure 2: The set $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ corresponding to Example 5.1

We claim that the inequalities given by $\mu^{(1)} = (0; 0; 1)$ with $\eta_0^{(1)} = 1$ and $\mu^{(2)} = (0; 0; -1)$ with $\eta_0^{(2)} = -2$ are both $\mathcal{K}$-sublinear (in fact also $\mathcal{K}$-minimal). Here we will show that the necessary conditions for $\mathcal{K}$-sublinearity established so far are satisfied. We will revisit this example after establishing the sufficient conditions for $\mathcal{K}$-sublinearity and $\mathcal{K}$-minimality.

Clearly both $\mu^{(1)}, \mu^{(2)} \in \text{Im}(A^*)$ and $\eta_0^{(i)} = \inf_x\{\langle \mu^{(i)}, x\rangle : x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})\}$. Also one can easily check that for the corresponding sets D_i, $i = 1, 2$ associated with these inequalities are given by

$$\begin{aligned} D_1 &= \{\lambda : \exists \gamma \in \mathcal{K}^* \text{ s.t. } \gamma_1 = 0;\ \gamma_2 = 0;\ \lambda + \gamma_3 = 1\} = \{\lambda : \lambda \leq 1\} \\ D_2 &= \{\lambda : \lambda \leq -1\}, \end{aligned}$$

and are nonempty. Moreover, we have $\sigma_{D_i}(Az) = \sigma_{D_i}(z_3) = \langle \mu^{(i)}, z\rangle$ *for all* $z \in \mathcal{K}$ *and hence* $\inf_{b\in\mathcal{B}} \sigma_{D_i}(b) = \eta_0^{(i)}$ *for* $i = 1, 2$.

When $\mathcal{K} = \mathbb{R}^n_+$, the relationship between $\mathcal{K}$-sublinear valid inequalities and support functions of sets can be further enchanced, i.e., every $\mathcal{K}$-sublinear v.i. can be viewed as the support function of a set with certain structure. This result is given by

Proposition 5.2 *Suppose that* $\mathcal{K} = \mathbb{R}^n_+$. *Let* $\mu \in E = \mathbb{R}^n$ *satisfy condition* ***(A.1i)*** *for all* $i = 1, \ldots, n$. *Then* $\perp_{e_i} \cap(\mu - \text{Im}(A^*)) \neq \emptyset$, *and thus* $\sigma_D(A^i) = \mu_i$ *for all* $i = 1, \ldots, n$ *where* A^i *is the* i^{th} *column of the matrix* A.

Proof. Suppose that the statement is not true. Then there exist i such that $\perp_{e_i} \cap(\mu - \text{Im}(A^*)) = \emptyset$. Note that $\perp_{e_i} = \{\gamma \in \mathbb{R}^n_+ : \gamma_i = 0\} = \text{cone}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}$. Therefore we arrive at the following system of linear inequalities in γ, λ being infeasible:

$$\begin{aligned} \gamma + A^*\lambda &= \mu \\ \gamma_j &\geq 0 \qquad \forall j \neq i \\ \gamma_i &= 0 \end{aligned}$$

Using Farkas' Lemma, we conclude that $\exists u, v$ such that $u + v = 0$, $v_j \geq 0$ for all $j \neq i$, $Au = 0$ and $\langle u, \mu\rangle \geq 1$. By eliminating u, this implies that $\exists v$ such that $v_j \geq 0$ for all $j \neq i$, $Av = 0$ and $\langle v, \mu\rangle \leq -1$. Hence if $v_i < -1$, we can scale v so that $v_i \geq -1$, and arrive at the conclusion that there exists v such that $v + e_i \in \mathbb{R}^n_+ = \mathcal{K}$, $Av = 0$ and $\langle v, \mu\rangle < 0$, which is a contradiction to the condition **(A.1i)**.

By noting that condition **(A.1i)** is necessary for the $\mathcal{K}$-sublinearity (and also $\mathcal{K}$-minimality) of $(\mu; \eta_0)$, we conclude that for all i, we have $\perp_{e_i} \cap(\mu - \text{Im}(A^*)) \neq \emptyset$, and $\sigma_D(A^i) = \mu_i$ for all $i = 1, \ldots, n$ follows from Corollary 5.1. ∎

In fact when $\mathcal{K} = \mathbb{R}^n_+$, the result of Proposition 5.2, together with basic facts on support functions of sets, suggests a functional view point. We can interpret the cut coefficients μ_i of a $\mathcal{K}$-sublinear v.i. as $\sigma(A^i)$, the corresponding value of a subadditive function $\sigma(\cdot)$ evaluated at the column A^i. Furthermore this subadditive function has a very specific form, i.e., it is the support function of the sets of form

$$D_\mu := \{\lambda \in \mathbb{R}^m : A^*\lambda \leq \mu\}.$$

Note that the support functions of these sets are automatically sublinear (subadditive and positively homogeneous), and in fact piecewise linear and convex. When $\mathcal{K} = \mathbb{R}^n_+$, this connection was already known previously (see [45]). In fact, the work of Johnson in [45] goes further by showing that in order to verify $\mathcal{K}$-sublinearity, only finitely many of the requirements (**A.0**), (**A.1i**), and (**A.2**) (those satisfying a minimal linear dependence condition) are needed to be verified. Moreover, the authors in [24] also consider a similar setting with $\mathcal{K} = \mathbb{R}^n_+$, introduce cut generating functions, and establish their relations with so-called S-free (in our context $\mathcal{B}$-free) sets. [24] has the additional assumption that $0 \notin \text{conv}(\mathcal{B})$, and is only interested in cuts (or cut generating functions) that separate 0 from $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$. Clearly the corresponding cuts are subset of $C(A, \mathcal{K}, \mathcal{B})$, and whenever they are necessary they will be $\mathcal{K}$-minimal and thus $\mathcal{K}$-sublinear as well, and hence the support functions of the corresponding sets D_μ will be the corresponding cut-generating functions of interest from [24].

Motivated by the positive result of Proposition 5.2 in the case of $\mathcal{K} = \mathbb{R}^n_+$, one inclines to think that a similar result will hold for general proper cones $\mathcal{K}$, which unfortunately is not true. On the other hand, in the next Proposition, we state that when $(\mu; \eta_0)$ is a $\mathcal{K}$-sublinear v.i. even if $\mu \notin \text{Im}(A^*)$, there exists $z \in \text{Ext}(\mathcal{K})$ such that $\sigma_D(Az) = \langle \mu, z\rangle$. Before we prove Proposition 5.3, we need the following

Lemma 5.1 *Suppose that $\mu \in E$ satisfies condition (**A.0**), and $\perp_z \cap(\mu - \mathrm{Im}(A^*)) = \emptyset$ holds for all $z \in \mathrm{Ext}(\mathcal{K})$ where $\perp_z$ is as defined by* (4)*. Then there exists $\bar{\gamma} \in \mathrm{int}(\mathcal{K}^*)$ such that $\bar{\gamma} \in \mu - \mathrm{Im}(A^*)$. Moreover $\inf_{b \in \mathcal{B}} \sigma_D(b) = \mu_0$ where D is defined by* (3) *and μ_0 is defined by* (2).

Proof. First note that since μ satisfies condition (**A.0**), by Theorem 5.1, the set D defined in (3) is nonempty, which implies that $\{\gamma : \exists \lambda \text{ s.t. } \gamma + A^*\lambda = \mu, \ \gamma \in \mathcal{K}^*\} \neq \emptyset$.

In addition to this, $\mathbf{0} \in \bigcap_{z \in \mathrm{Ext}(\mathcal{K})} \perp_z$ and therefore together with the assumption of the Lemma that $\perp_z \cap(\mu - \mathrm{Im}(A^*)) = \emptyset$, we conclude that $\mathbf{0} \notin \mu - \mathrm{Im}(A^*)$. Moreover by rephrasing the statement of Lemma and definition of $\perp_z$, we get

$$\begin{aligned} 0 &< \inf_{z \in \mathrm{Ext}(\mathcal{K})} \inf_{\gamma,\lambda} \{\langle \gamma, z \rangle : \ \gamma + A^*\lambda = \mu, \ \gamma \in \mathcal{K}^*\} \\ &= \inf_{\gamma,\lambda} \left\{ \inf_{z} \{\langle \gamma, z \rangle : \ z \in \mathrm{Ext}(\mathcal{K})\} : \ \gamma + A^*\lambda = \mu, \ \gamma \in \mathcal{K}^* \right\}. \end{aligned}$$

Now assume for contradiction that the set $\{\gamma : \ \exists \lambda \text{ s.t. } \gamma + A^*\lambda = \mu, \ \gamma \in \mathcal{K}^*\} \subseteq \partial \mathcal{K}^*$. This together with the above inequality implies that there exists $\bar{\gamma} \in \partial \mathcal{K}^*$ such that $\langle \bar{\gamma}, z \rangle > 0$ for all $z \in \mathrm{Ext}(\mathcal{K})$, which implies that $\langle \bar{\gamma}, z \rangle > 0$ for all $z \in \mathcal{K} \setminus \{0\}$. Since $\mathcal{K}^*$ is a closed convex cone, $\langle \bar{\gamma}, z \rangle > 0$ for all $z \in \mathcal{K} \setminus \{0\}$ implies that $\bar{\gamma} \in \mathrm{int}(\mathcal{K}^*)$, which is a contradiction. Thus we conclude that there exists $\bar{\gamma} \neq \mathbf{0}$ such that $\bar{\gamma} \in \mathrm{int}(\mathcal{K}^*) \cap (\mu - \mathrm{Im}(A^*))$.

To finish the proof note that

$$\begin{aligned} \mu_0 &:= \inf_{x} \{\langle \mu, x \rangle : \ x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})\} \\ &= \inf_{b \in \mathcal{B}} \inf_{x} \{\langle \mu, x \rangle : \ Ax = b, \ x \in \mathcal{K}\} \\ &= \inf_{b \in \mathcal{B}} \sup_{\lambda} \{\langle b, \lambda \rangle : \ A^*\lambda + \gamma = \mu, \ \gamma \in \mathcal{K}^*\} = \inf_{b \in \mathcal{B}} \sigma_D(b), \end{aligned}$$

where the third equality follows from strong duality, which holds due to the existence of a feasible $\bar{\gamma} \in \mathrm{int}(\mathcal{K}^*)$. Therefore we have $\mu_0 = \inf_{b \in \mathcal{B}} \sigma_D(b)$.

∎

Proposition 5.3 *Suppose that $\mu \in E$ satisfies condition (**A.0**), and $\perp_z \cap(\mu - \mathrm{Im}(A^*)) = \emptyset$ holds for all $z \in \mathrm{Ext}(\mathcal{K})$ where $\perp_z$ is as defined by* (4)*. If $\mu \in \Pi(A, \mathcal{K}, \mathcal{B})$, and $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ is closed, then there exists $z \in \mathrm{Ext}(\mathcal{K})$ such that $\sigma_D(Az) = \langle \mu, z \rangle$ where D is defined by* (3).

Proof. Assume for contradiction that $\sigma_D(Az) < \langle \mu, z \rangle$ for all $z \in \mathrm{Ext}(\mathcal{K})$. Then by Lemma 5.1, we conclude that there exists $\bar{\gamma} \in \mathrm{int}(\mathcal{K}^*) \cap (\mu - \mathrm{Im}(A^*))$ and $\inf_{b \in \mathcal{B}} \sigma_D(b) = \mu_0$ where μ_0 is as defined in (2).

Since $\inf_{b \in \mathcal{B}} \sigma_D(b) = \mu_0 = \inf_x \{\langle \mu, x \rangle : \ x \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})\}$ and $\mu_0 \neq -\infty$, thus the linear function $\langle \mu, x \rangle$ attains its infimum over the closed set $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$, i.e., there exists $\bar{x} \in \mathcal{S}(A, \mathcal{K}, \mathcal{B})$, i.e., $\exists \bar{b} \in \mathcal{B}$ satisfying $A\bar{x} = \bar{b}$ and $\bar{x} \in \mathcal{K}$ such that $\mu_0 = \langle \mu, \bar{x} \rangle$. By expanding the definition of μ_0, we get

$$\begin{aligned} \langle \mu, \bar{x} \rangle = \mu_0 &= \inf_{x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})} \{\langle \mu, x \rangle\} = \inf_{x} \{\langle \mu, x \rangle : \ Ax = \bar{b}, \ x \in \mathcal{K}\} \\ &= \sup_{\lambda,\gamma} \{\langle \bar{b}, \lambda \rangle : \ A^*\lambda + \gamma = \mu, \ \gamma \in \mathcal{K}^*\} = \sigma_D(\bar{b}) = \sigma_D(A\bar{x}), \end{aligned}$$

where the second inequality follows from strong duality due to the existence of $\bar{\gamma}$. Therefore we conclude that when μ is $\mathcal{K}$-minimal, $\exists \bar{x} \in \mathcal{K}$ such that $A\bar{x} = \bar{b} \in \mathcal{B}$ and $\sigma_D(A\bar{x}) = \langle \mu, \bar{x} \rangle$. Since $\bar{x} \in \mathcal{K}$, there exists $z^1, \ldots, z^\ell \in \text{Ext}(\mathcal{K})$ with $\ell \leq n$ such that $\bar{x} = \sum_{i=1}^{\ell} z^i$, which leads to

$$\langle \mu, \bar{x} \rangle = \sigma_D(A\bar{x}) \underbrace{\leq}_{(*)} \sum_{i=1}^{\ell} \sigma_D(Az^i) \underbrace{<}_{(**)} \sum_{i=1}^{\ell} \langle \mu, z^i \rangle = \langle \mu, \bar{x} \rangle$$

where the inequality $(*)$ follows from the fact that $\sigma_D(\cdot)$ is a support function, and thus is subadditive, and $(**)$ follows from the assumption that $\sigma_D(Az) < \langle \mu, z \rangle$ for all $z \in \text{Ext}(\mathcal{K})$. But this is a contradiction, hence we conclude that there exists $z \in \text{Ext}(\mathcal{K})$ such that $\sigma_D(Az) = \langle \mu, z \rangle$. ∎

To summarize whenever $\mu \in \Pi(A, \mathcal{K}, \mathcal{B})$, and $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ is closed, Propositions 5.1 and 5.3 together cover all possible cases and indicate that for a $\mathcal{K}$-sublinear inequality, there exists at least one $z \in \text{Ext}(\mathcal{K})$ such that $\sigma_D(Az) = \langle \mu, z \rangle$. While one is inclined to think that the above property of support functions of D can only be valid for a $\mathcal{K}$-minimal v.i., the following simple example shows that this property also holds for some valid inequalities, which are $\mathcal{K}$-sublinear but not $\mathcal{K}$-minimal.

Example 5.2 *Consider set $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ with $\mathcal{K} = \mathcal{L}^3 = \{x \in \mathbb{R}^3 : x_3 \geq \sqrt{x_1^2 + x_2^2}\}$, $A = [1, 0, 0]$ and $\mathcal{B} = \{-1, 1\}$. In this case,* $\text{conv}(\mathcal{S}(A, \mathcal{K}, \mathcal{B})) = \{x \in \mathbb{R}^3 : x_3 \geq \sqrt{1 + x_2^2},\ -1 \leq x_1 \leq 1\}$ *(see Figure 3).*

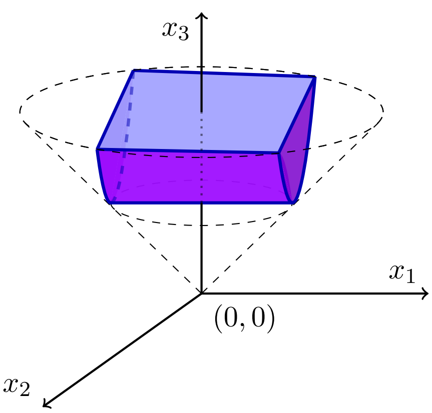

Figure 3: Convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ corresponding to Example 5.2

Note that the above description of the convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ involves the following inequalities:

(a) $\mu^{(+)} = (1; 0; 0)$ *with* $\eta_0^{(+)} = -1$ *and* $\mu^{(-)} = (-1; 0; 0)$ *with* $\eta_0^{(-)} = -1$;

(b) $\mu^{(t)} = (0; t; \sqrt{t^2 + 1})$ *with* $\eta_0^{(t)} = 1$ *for all* $t \in \mathbb{R}$.

Here, we aim to show that these inequalities satisfy the necessary conditions for $\mathcal{K}$-sublinear inequalities; later on we will in fact show that all of these inequalities are $\mathcal{K}$-minimal.

In the first case (a), it is easily seen that for $i \in \{+, -\}$, the corresponding sets D_i associated with these inequalities are given by

$$\begin{aligned} D_+ &= \{\lambda : \exists \gamma \in \mathcal{K}^* \text{ s.t. } \lambda + \gamma_1 = 1;\ \gamma_2 = 0;\ \gamma_3 = 0\} = \{\lambda : \lambda = 1\} \\ D_- &= \{\lambda : \lambda = -1\}. \end{aligned}$$

Also, both $\mu^{(+)}, \mu^{(-)} \in \text{Im}(A^*)$, and thus Corollary 5.1 implies, $\sigma_{D_i}(Az) = \sigma_{D_i}(z_1) = \langle \mu^{(i)}, z\rangle$ for all $z \in \mathcal{K}$ for $i \in \{+,-\}$. In addition to this, $\inf_{b\in\mathcal{B}} \sigma_{D_i}(b) = -1 = \eta_0^{(i)}$ for $i \in \{+,-\}$.

In the second case (b), for any given t, we have the associated sets D_t given by

$$D_t = \{\lambda : \ \exists \gamma \in \mathcal{K}^* \text{ s.t. } \lambda + \gamma_1 = 0; \ \gamma_2 = t; \ \gamma_3 = \sqrt{t^2+1}\} = \{\lambda : \ -1 \le \lambda \le 1\}.$$

Moreover, for all t, by considering $z^t \in \{(1; -t; \sqrt{t^2+1}), (-1; -t; \sqrt{t^2+1})\} \subset \text{Ext}(\mathcal{K})$, we have $\langle \mu^{(t)}, z^t\rangle = 1$ and $\sigma_{D_t}(Az^t) = \sigma_{D_t}(z_1^t) = \sigma_{D_t}(1) = 1$, proving $\langle \mu^{(t)}, z^t\rangle = \sigma_{D_t}(Az^t)$. Additionally, $\sigma_{D_t}(1) = 1 = \sigma_{D_t}(-1)$ implying $\inf_{b\in\mathcal{B}} \sigma_{D_t}(b) = 1 = \eta_0^{(t)}$ for all t.

Let us also consider another valid inequality $(\nu;\nu_0)$ given by $\nu = (0;1;2)$ and $\nu_0 = 1$. Note that the associated D_ν set is given by

$$D_\nu = \left\{\lambda : \ -\sqrt{3} \le \lambda \le \sqrt{3}\right\},$$

and is nonempty. Furthermore for any $z_\nu \in \left\{(\frac{1}{\sqrt{3}}; -\frac{1}{3}; \frac{2}{3}), (-\frac{1}{\sqrt{3}}; -\frac{1}{3}; \frac{2}{3})\right\} \subset \text{Ext}(\mathcal{K})$ we have $\sigma_{D_\nu}(Az_\nu) = \sigma_{D_\nu}(\pm\frac{1}{\sqrt{3}}) = 1 = \langle \nu, z_\nu\rangle$. Also $\inf_{b\in\mathcal{B}} \sigma_{D_\nu}(b) = \sqrt{3} > 1 = \nu_0$. Therefore in terms of the necessary conditions established so far for $\mathcal{K}$-sublinearity, there seems to be no difference between $(\nu;\nu_0)$ and the previous inequalities from above. When we revisit this example, we will show that $(\nu;\nu_0) \in C_a(A,\mathcal{K},\mathcal{B})$ is true, but $(\nu;\nu_0)$ is not $\mathcal{K}$-minimal. In fact, we can easily show that $(\nu;\nu_0)$ is dominated by $\mu^{(1)} = (0;1;\sqrt{2})$ and $\eta^{(1)} = 1$. And since $\delta = \nu - \mu^{(1)} = (0;0;2-\sqrt{2})$ is in $\mathcal{K}^* \setminus \{0\}$, we conclude that $(\nu;\nu_0) \notin C_m(A,\mathcal{K},\mathcal{B})$.

5.1 Cut Generating Sets

Given a linear map A and regular cone $\mathcal{K}$, the convex sets of the form (3) can be used to derive valid inequalities for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ as follows:

Proposition 5.4 *Consider any $\mu \in \text{Im}(A^*) + \mathcal{K}^*$, then the set defined by*

$$D_\mu = \{\lambda \in \mathbb{R}^m : \ \exists \gamma \in \mathcal{K}^* \text{ such that } A^*\lambda + \gamma = \mu\} \tag{5}$$

is nonempty; $\sigma_{D_\mu}(Az) \le \langle \mu, z\rangle$ holds for all $z \in \mathcal{K}$; and the inequality given by $(\mu;\eta_0)$ with $\eta_0 \le \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$ is valid for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$.

Proof. Let μ, D_μ, η_0 be as defined in the statement of the proposition. The non-emptiness of $D_\mu \neq \emptyset$ is evident from the condition on μ. By the fact that $D_\mu \neq \emptyset$, and following the same reasoning from the last part of proof of Theorem 5.1, we have $\sigma_{D_\mu}(Az) \le \langle \mu, z\rangle$.

In order to prove that $(\mu;\eta_0)$ is a v.i., we need to prove $\eta_0 \le \mu_0$ where μ_0 is as defined in (2). But this simply follows from the associated definitions, i.e.,

$$\begin{aligned}\eta_0 &\le \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b) = \inf_x \left\{\sigma_{D_\mu}(Ax) : \ x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})\right\} \\ &\le \inf_{x,\beta} \{\langle \mu, x\rangle : \ x \in \mathcal{K}, \ Ax = b \text{ for some } b \in \mathcal{B}\} = \mu_0,\end{aligned}$$

where the last inequality follows from the fact that for all $z \in \mathcal{K}$, we have $\sigma_{D_\mu}(Az) \le \langle \mu, z\rangle$. ∎

Given that $D_\mu \neq \emptyset$ is a necessary condition for $\mathcal{K}$-sublinearity and thus $\mathcal{K}$-minimality (see Theorem 5.1), and the fact that under **Assumption 1**, $\mathcal{K}$-minimal inequalities together with trivial cone-implied inequalities are sufficient to describe the convex hull of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ (see Proposition 3.1), Proposition 5.4 states that there is no point in studying and/or separating inequalities with $\mu \notin \text{Im}(A^*) + \mathcal{K}^*$.

5.2 Sufficient Conditions for $\mathcal{K}$-Sublinearity and $\mathcal{K}$-Minimality

Given any $(\mu;\eta_0)$ with $D_\mu \neq \emptyset$, we can easily test whether it is a $\mathcal{K}$-sublinear v.i. or not with the help of the following

Proposition 5.5 *Let $(\mu;\eta_0)$ be such that $D_\mu \neq \emptyset$ where D_μ is as defined in* (5) *and $\eta_0 \leq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$ (or it is known that $(\mu;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$). In addition to this, suppose that there exists $x^i \in \operatorname{Ext}(\mathcal{K})$ such that $\sigma_D(Ax^i) = \langle \mu, x^i\rangle$ for all $i \in I$ and $\sum_{i\in I} x^i \in \operatorname{int}(\mathcal{K})$, then $(\mu;\eta_0) \in C_a(A,\mathcal{K},\mathcal{B})$.*

Proof. If $\eta_0 \leq \inf_{b\in\mathcal{B}} \sigma_{D_\mu}(b)$, then by Proposition 5.4, we have $(\mu;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$, which automatically implies that condition (**A.2**) is satisfied. To verify condition (**A.1**), consider any $\alpha \in \operatorname{Ext}(\mathcal{K}^*)$ and u such that $Au = 0$ and $\langle \alpha, v\rangle u + v \in \mathcal{K}\ \forall v \in \operatorname{Ext}(\mathcal{K})$. Let $\mathcal{V}_\alpha = \{v \in \operatorname{Ext}(\mathcal{K}) :\ \langle \alpha, v\rangle = 1\}$, it is clear that $\langle u+v,\gamma\rangle \geq 0$ holds for all $v \in \mathcal{V}_\alpha$ and $\gamma \in \mathcal{K}^*$. Also, since $D_\mu \neq \emptyset$, we can find $\bar\lambda$ and $\bar\gamma \in \mathcal{K}^*$ satisfying $A^*\bar\lambda + \bar\gamma = \mu$. In fact for any such $\bar\lambda, \bar\gamma$, we have

$$\begin{aligned}
\langle \mu, u\rangle &= \langle A^*\bar\lambda + \bar\gamma, u\rangle \\
&= \langle \bar\lambda, \underbrace{Au}_{=0}\rangle + \langle \bar\gamma, u\rangle \\
&\geq \langle \bar\gamma, -v\rangle \quad \forall v \in \mathcal{V}_\alpha
\end{aligned}$$

Note that $\langle \gamma, -v\rangle \leq 0$ for all $\gamma \in \mathcal{K}^*$ and $v \in \mathcal{V}_\alpha \subset \mathcal{K}$. In order to finish the proof, all we need to show is that there exists $\bar v \in \mathcal{V}_\alpha$ such that $\langle \bar\gamma, \bar v\rangle = 0$. Clearly when $\mu \in \operatorname{Im}(A^*)$, we can take $\bar\gamma = 0$, and hence conclude that $\langle \mu, u\rangle \geq -\langle \bar\gamma, \bar v\rangle = 0$ holds for all such u. In the more general case, we have

$$\begin{aligned}
& \inf_{\gamma,\lambda} \inf_{v} \{\langle \gamma, v\rangle :\ v \in \mathcal{V}_\alpha,\ A^*\lambda + \gamma = \mu,\ \gamma \in \mathcal{K}^*\} \\
=\ & \inf_{v} \left\{ \inf_{\gamma,\lambda} \{\langle \mu - A^*\lambda, v\rangle :\ A^*\lambda + \gamma = \mu,\ \gamma \in \mathcal{K}^*\} :\ v \in \mathcal{V}_\alpha \right\} \\
=\ & \inf_{v} \left\{ \langle \mu, v\rangle - \underbrace{\sup_{\gamma,\lambda} \{\langle \lambda, Av\rangle :\ A^*\lambda + \gamma = \mu,\ \gamma \in \mathcal{K}^*\}}_{=\sigma_D(Av)} :\ v \in \mathcal{V}_\alpha \right\}
\end{aligned}$$

Since there exists $x^i \in \operatorname{Ext}(\mathcal{K})$ such that $\sigma_D(Ax^i) = \langle \mu, x^i\rangle$ for all $i \in I$ and $\sum_{i\in I} x^i \in \operatorname{int}(\mathcal{K})$, for any $\alpha \in \operatorname{Ext}(\mathcal{K}^*)$, at least one of these x^i's will be in $\mathcal{V}_\alpha$, which implies that the above infimum is zero. Thus we get the desired conclusion that $\langle \mu, u\rangle \geq 0$, which proves that the condition (**A.1**) is also satisfied. ∎

Note that considering the results obtained from Theorem 5.1, and Propositions 5.1 and 5.3, we conclude that the conditions stated in Proposition 5.5 are almost necessary. This is upto the fact that we can prove the existence of at least one $x \in \operatorname{Ext}(\mathcal{K})$ satisfying $\sigma_D(Ax) = \langle \mu, x\rangle$ when $(\mu;\eta_0) \in C_a(A,\mathcal{K},\mathcal{B})$, yet the sufficient condition in Proposition 5.5 requires a number of such extreme rays summing up to an interior point of $\mathcal{K}$. When $\mathcal{K} = \mathbb{R}^n_+$, Proposition 5.2 together with Theorem 5.1 states that the conditions stated in Proposition 5.5 are necessary and sufficient for $\mathcal{K}$-sublinearity. We conjecture that this is also the case for general regular cones $\mathcal{K}$.

In addition to Proposition 5.5, under **Assumption 1**, we can state a sufficient condition for $\mathcal{K}$-minimality as follows:

Proposition 5.6 *Suppose that **Assumption 1** holds and $(\mu;\eta_0) \in C_a(A,\mathcal{K},\mathcal{B})$ is such that $-\infty < \eta_0 = \inf_{b\in\mathcal{B}} \sigma_D(b)$. Let $\hat{\mathcal{B}} = \{b \in \mathcal{B} :\ \sigma_D(b) \le \eta_0\}$. If the following condition holds,*

(I) *there exists $b^i \in \hat{\mathcal{B}}$ and $x^i \in \mathcal{K}$ such that $\sum_i x^i \in \text{int}(\mathcal{K})$, $Ax^i = b^i$ and $\langle \mu, x^i\rangle = \eta_0$,*

then $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$.

Proof. Suppose $(\mu;\eta_0) \in C_a(A,\mathcal{K},\mathcal{B})$ is such that $\eta_0 = \inf_{b\in\mathcal{B}} \sigma_D(b)$. Assume for contradiction that $\exists \delta \in \mathcal{K}^* \setminus \{0\}$ such that $(\mu - \delta;\eta_0) \in C(A,\mathcal{K},\mathcal{B})$.

Suppose that condition **(I)** holds for some $b^i \in \hat{\mathcal{B}}$ and $x^i \in \mathcal{K}$ such that $\sum_i x^i \in \text{int}(\mathcal{K})$, $Ax^i = b^i$ and $\langle \mu, x^i\rangle = \eta_0$. Note that for $\beta_i > 0$ with $\sum_i \beta_i = 1$, we have $\bar{x} := \sum_i \beta_i x^i \in \text{int}(\mathcal{K})$ and moreover by definition $\bar{x} \in \text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, and $\langle \mu, \bar{x}\rangle = \eta_0$. Since any valid inequality for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, in particular $(\mu - \delta;\eta_0)$ has to be valid for $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, we arrive at the contradiction

$$\eta_0 \le \langle \mu - \delta, \bar{x}\rangle < \eta_0,$$

where the last inequality follows from $\bar{x} \in \text{int}(\mathcal{K})$ and $\delta \in \mathcal{K}^* \setminus \{0\}$ implying $\langle \delta, \bar{x}\rangle > 0$ together with $\langle \mu, \bar{x}\rangle = \eta_0$.

∎

In particular, Proposition 5.6 states that in order for a $\mathcal{K}$-sublinear inequality to be $\mathcal{K}$-minimal, the inequality needs to be tight for a point at the intersection of $\text{int}(\mathcal{K})$ and $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. In the MILP case, clearly, this resembles a sufficient condition for an inequality to be facet defining. Nonetheless our minimality notion in general is much weaker. In the MILP case, all of the facets are necessary and sufficient; yet in general one does not need all of the $\mathcal{K}$-minimal inequalities for the description of $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, only a generating set for $C_m(A,\mathcal{K},\mathcal{B})$ together with the trivial cone-implied inequalities is needed.

Corollary 5.2 *For any $\mu \in \text{Im}(A^*)$, we have $D \neq \emptyset$, where D is defined as in* (3) *and $\mu_0 = \inf_{b\in\mathcal{B}} \sigma_D(b)$ where μ_0 is defined in* (2)*, and $(\mu,\mu_0) \in C_a(A,\mathcal{K},\mathcal{B})$.*

Proof. From the definition of D in (3), for any $\mu \in \text{Im}(A^*)$, we have $D \neq \emptyset$. Therefore in the light of Proposition 5.5, we have any valid inequality $(\mu;\eta_0)$ with $\mu \in \text{Im}(A^*)$ is $\mathcal{K}$-sublinear. Moreover, Corollary 5.1 states that for any $\mu \in \text{Im}(A^*)$, we have $\sigma_D(Az) = \langle \mu, z\rangle$ for all $z \in \mathcal{K}$. Therefore for any given b, and for any x_b such $x_b \in \mathcal{K}$ and $Ax_b = b$, we have $\sigma_D(b) = \sigma_D(Ax_b) = \langle \mu, x_b\rangle$, which implies,

$$\inf_{b\in\mathcal{B}} \sigma_D(b) = \inf_{b\in\mathcal{B}} \inf_{x} \{\langle \mu, x\rangle :\ x \in \mathcal{K},\ Ax = b\} = \mu_0,$$

(see the definition of μ_0 in (2)). ∎

Proposition 5.7 *Let $(\mu;\eta_0)$ be a $\mathcal{K}$-minimal v.i. such that $\mu \in \text{int}(\mathcal{K}^*)$. Then $\eta_0 = \mu_0 = \inf_{b\in\mathcal{B}} \sigma_D(b)$.*

Proof. Since $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$ and $\mu \in \mathcal{K}^*$, by Proposition 3.2, we have

$$\eta_0 = \mu_0 = \inf_{x} \{\langle \mu, x\rangle :\ x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})\}.$$

Moreover $(\mu;\eta_0) \in C_m(A,\mathcal{K},\mathcal{B})$, $(\mu;\eta_0)$ is a $\mathcal{K}$-sublinear valid inequality and therefore D_μ as defined in (5) is nonempty. Also by Proposition 5.4, $\mu \in \mathcal{K}^*$ implies that $\inf_{b\in\mathcal{B}} \sigma_D(b) \leq \mu_0$. Assume for contradiction that $\mu_0 > \inf_{b\in\mathcal{B}} \sigma_D(b)$, which implies

$$\begin{aligned} \mu_0 &> \inf_{b\in\mathcal{B}} \sigma_D(b) = \inf_{b\in\mathcal{B}} \sup_{\lambda} \{\langle b,\lambda\rangle : \; A^*\lambda + \gamma = \mu, \; \gamma \in \mathcal{K}^*\} \\ &= \inf_{b\in\mathcal{B}} \inf_{x} \{\langle \mu, x\rangle : \; Ax = b, \; x \in \mathcal{K}\} \\ &= \inf_{x} \{\langle \mu, x\rangle : \; x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})\} = \mu_0, \end{aligned}$$

where the second equality follows from strong duality, which holds due to the fact that $\mu \in \text{int}(\mathcal{K}^*)$, and at the last equation μ_0 is as defined in (2). Hence we have reached a contradiction, therefore we have $\eta_0 = \mu_0 = \inf_{b\in\mathcal{B}} \sigma_D(b)$. ∎

To demonstrate the proper uses of Propositions 5.5, 5.6 and 5.7, let us return to our previous examples.

Example 5.1 (cont.) First note that the convex hull of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ is full dimensional[4)] and thus there is no valid equation for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ implying that the lineality space of $C(A,\mathcal{K},\mathcal{B})$ is just the zero vector. Moreover, $\hat{z} = (0;0;1) \in \text{int}(\mathcal{K}) \cap \mathcal{S}(A,\mathcal{K},\mathcal{B})$ and hence **Assumption 1** is satisfied.

We claim that $(\mu^{(i)};\eta_0^{(i)})$ with $i = 1,2$ are both $\mathcal{K}$-minimal inequalities. We have already seen that for $i = 1,2$, $\mu^{(i)} \in \text{Im}(A^*)$, the sets D_i associated with them are nonempty, there are tight extreme points, i.e., $\sigma_{D_i}(Az^{(i)}) = \langle \mu^{(i)}, z^{(i)}\rangle$ satisfying the requirement of Proposition 5.5, and $\inf_{b\in\mathcal{B}} \sigma_{D_i}(b) = \eta_0^{(i)}$ holds, and hence by Proposition 5.5 $(\mu^{(i)};\eta_0^{(i)})$ belong to $C_a(A,\mathcal{K},\mathcal{B})$. Finally for $i = 1,2$, the points $z^{(i)} = (0;0;i)$ satisfies $z^{(i)} \in \text{int}(\mathcal{K}) \cap \mathcal{S}(A,\mathcal{K},\mathcal{B})$ and $\langle \mu, z^{(i)}\rangle = \eta_0^{(i)}$ and therefore using Proposition 5.6, we conclude that these inequalities are also $\mathcal{K}$-minimal.

Example 5.2 (cont.) Once again the convex hull of $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ is full dimensional and thus there is no valid equation for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ implying that the lineality space of $C(A,\mathcal{K},\mathcal{B})$ is just the zero vector. Moreover, $\hat{z} = (1;0;2) \in \text{int}(\mathcal{K}) \cap \mathcal{S}(A,\mathcal{K},\mathcal{B})$ and hence **Assumption 1** is satisfied.

We claim that

(a) $\mu^{(+)} = (1;0;0)$ with $\eta_0^{(+)} = -1$ and $\mu^{(-)} = (-1;0;0)$ with $\eta_0^{(-)} = -1$;

(b) $\mu^{(t)} = (0;t;\sqrt{t^2+1})$ with $\eta_0^{(t)} = 1$ for all $t \in \mathbb{R}$.

are all $\mathcal{K}$-minimal inequalities. We have already seen that the associated sets D_i are nonempty, $\inf_{b\in\mathcal{B}} \sigma_{D_i}(b) = \eta_0^{(i)}$ holds and there are tight extreme points, i.e., $\sigma_{D_i}(Az^{(i)}) = \langle \mu^{(i)}, z^{(i)}\rangle$ satisfying the requirement of Proposition 5.5, and hence all of them are in $C_a(A,\mathcal{K},\mathcal{B})$ by Proposition 5.5. Moreover, in case (a), by considering the points $z^{(+)} = (1;0;2) \in \text{int}(\mathcal{K}) \cap \mathcal{S}(A,\mathcal{K},\mathcal{B})$ and $z^{(-)} = (-1;0;2) \in \text{int}(\mathcal{K}) \cap \mathcal{S}(A,\mathcal{K},\mathcal{B})$, we get $\langle \mu^{(i)}, z^{(i)}\rangle = \eta_0^{(i)}$ holds for all $i \in \{+,-\}$ and therefore using part **(I)** of Proposition 5.6, we conclude that these inequalities are also $\mathcal{K}$-minimal. In case (b), for any $t \in \mathbb{R}$, consider $z_+^{(t)} = (1;-t;\sqrt{t^2+1}) \in \mathcal{K} \cap \mathcal{S}(A,\mathcal{K},\mathcal{B})$ and $z_-^{(t)} = (-1;-t;\sqrt{t^2+1}) \in \mathcal{K} \cap \mathcal{S}(A,\mathcal{K},\mathcal{B})$. Note that we have $\langle \mu^{(t)}, z_+^{(t)}\rangle = \eta_0^{(t)} = \langle \mu^{(t)}, z_-^{(t)}\rangle$ for all $t \in \mathbb{R}$, and hence $z^{(t)} := \frac{1}{2}(z_+^{(t)} + z_-^{(t)}) = (0;-t;\sqrt{t^2+1}) \in \text{int}(\mathcal{K}) \cap \text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. Therefore using Proposition 5.6, we conclude that $(\mu^{(t)};\eta_0^{(t)}) \in C_m(A,\mathcal{K},\mathcal{B})$ for all $t \in \mathbb{R}$.

4) To see this, one can demonstrate the existence of $n+1$ affinely independent points from $\mathcal{S}(A,\mathcal{K},\mathcal{B}) \subseteq \mathbb{R}^n$ where $n = 3$.

We can also show that the system of infinitely many linear inequalities corresponding to $(\mu^{(t)};\eta_0^{(t)}) = (0;t;\sqrt{t^2+1};1)$ for all $t\in\mathbb{R}$ in fact has a compact conic representation as follows: For all $x \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$, we have

$$\begin{aligned}
& 1 \le 0x_1 + tx_2 + \sqrt{t^2+1}x_3 \quad \forall t\in\mathbb{R} \\
\iff \quad & 1 \le \inf_t \{0x_1 + tx_2 + \sqrt{t^2+1}x_3 : \ t\in\mathbb{R}\} \\
\iff \quad & 1 \le \inf_{t,\tau} \{tx_2 + \tau x_3 : \ t\in\mathbb{R}, \ \tau \ge \sqrt{t^2+1}\} \\
\iff \quad & 1 \le \inf_{t,\tau} \{tx_2 + \tau x_3 : \ t\in\mathbb{R}, \ \tau \ge \sqrt{t^2+1}\} \\
\iff \quad & 1 \le \inf_{t,\tau} \{tx_2 + \tau x_3 : \ t\in\mathbb{R}, \ (1;t;\tau)\in\mathcal{L}^3\} \\
\iff \quad & 1 \le \sup_{\alpha} \{-\alpha_1 : \ \alpha_2 = x_2, \ \alpha_3 = x_3, \ (\alpha_1;\alpha_2;\alpha_3)\in\mathcal{L}^3\} \qquad \text{due to } (*) \\
\iff \quad & (-1;x_2;x_3)\in\mathcal{L}^3
\end{aligned}$$

where $(*)$ is due to the fact that the primal conic problem is strictly feasible and hence strong duality applies here. Note that we have arrived at the single conic inequality $x_3 \ge \sqrt{1+x_2^2}$, which is valid for all $x\in\mathcal{S}(A,\mathcal{K},\mathcal{B})$ and this conic inequality represents all of the constraints $(\mu^{(t)};\eta_0^{(t)})$ for all $t\in\mathbb{R}$.

Finally note that we have seen the valid inequality $(\nu;\nu_0)$ given by $\nu = (0;1;2)$ and $\nu_0 = 1$ has an associated D_ν set which is nonempty and there are tight extreme points, i.e., $\sigma_{D_\nu}(Az^{(i)}) = \langle \nu, z^{(i)}\rangle$ satisfying the requirement of Proposition 5.5 and $\nu_0 = 1 < \sqrt{3} = \inf_{b\in\mathcal{B}}\sigma_{D_\nu}(b)$, hence by Proposition 5.5 $(\nu;\nu_0)\in C_a(A,\mathcal{K},\mathcal{B})$. While $\sigma_{D_\nu}(Az_\nu) = \langle \nu, z_\nu\rangle = \nu_0 = 1$ holds for any (and only) $z_\nu \in \left\{(\frac{1}{\sqrt{3}};-\frac{1}{3};\frac{2}{3}),(-\frac{1}{\sqrt{3}};-\frac{1}{3};\frac{2}{3})\right\} \subset \text{Ext}(\mathcal{K})$ and the mid point of these two points is in the interior of $\mathcal{K}$, this mid point is not in $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, i.e., condition **(I)** of Proposition 5.6 fails. In fact, this can be easily seen via Proposition 5.7. Note that $\nu\in\text{int}(\mathcal{K}^*)$ fails the necessary condition for $\mathcal{K}$-minimality given in Proposition 5.7, i.e., $\inf_{b\in\mathcal{B}}\sigma_{D_\nu}(b) = \sigma_{D_\nu}(1) = \sigma_{D_\nu}(-1) = \sqrt{3} > 1 = \nu_0$. Hence we conclude that $(\nu;\nu_0)\notin C_m(A,\mathcal{K},\mathcal{B})$.

Finally, it is clear that any valid conic inequality where at least one of the associated linear inequalities is not $\mathcal{K}$-minimal, will not be necessary for the description of $\text{conv}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$. For instance, in Example 5.2, $x_3 \ge \sqrt{1+\frac{1}{2}x_2^2}$ is a valid conic inequality but it is not necessary.

Remark 5.1 *It is recently shown in [29] that for $\mathcal{K} = \mathbb{R}^n_+$, when $\{0\}\notin\mathcal{B}$ and $\mathcal{B}$ is contained in the convex cone generated by $\{A^1,\ldots,A^n\}$ where A^i is the i^{th} column of the linear map A, for any $\mathcal{K}$-minimal v.i. separating 0 from $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$, there exists a corresponding cut generating function $\rho:\mathbb{R}^m\to\mathbb{R}$, which maps each A^i to a cut coefficient μ_i. This states that under a mild requirement, in order to generate all valid inequalities for $\mathcal{S}(A,\mathcal{K},\mathcal{B})$, it is sufficient to consider all v.i. obtained from cut generating functions. But the situation seems to be much more complex for general $\mathcal{K}$. Example 5.2 reveals an important fact: Unlike the case with $\mathcal{K} = \mathbb{R}^n_+$, for general cones $\mathcal{K}$ one cannot hope to find a cut generating function corresponding to each $\mathcal{K}$-minimal inequality. In particular, we have seen in this example that, for any $t\in\mathbb{R}$, $(\mu^{(t)};\eta_0^{(t)}) = (0;t;\sqrt{t^2+1};1)$ is a $\mathcal{K}$-minimal v.i. whereas the corresponding linear map is given by $A = [1,0,0]$. Note that for this example, any cut generating function, $\rho(\cdot)$, corresponding to the v.i. $(\mu;\eta_0)$ inevitably needs to satisfy $\mu_2 = \rho(A^2) = \rho(0) = \rho(A^3) = \mu_3$, yet the $\mathcal{K}$-minimal v.i. $(\mu^{(t)};\eta_0^{(t)}) = (0;t;\sqrt{t^2+1};1)$ for any $t\in\mathbb{R}$ violate this. This suggests that for general cones $\mathcal{K}$, we cannot hope to find cut generating functions corresponding to all of the $\mathcal{K}$-minimal v.i. In fact if we just rely on cut* *generating functions, we can completely miss large classes of essential valid inequalities for the description of* $\overline{\text{conv}}(\mathcal{S}(A,\mathcal{K},\mathcal{B}))$.

5.3 Connections to Conic Mixed Integer Rounding Cuts

Remark 5.2 *In the simple case of the polyhedral cone* $\mathcal{K} = \mathcal{L}^2 = \{x \in \mathbb{R}^2 : \ x_2 \geq |x_1|\}$*, there are only two extreme rays* $\alpha^{(1)} = (1;1)$ *and* $\alpha^{(2)} = (-1;1)$*. These extreme rays are orthogonal to each other, and hence condition* ***(A.1)*** *reduces to*

$$(\textbf{\textit{A.1(i)}}) \qquad 0 \leq \sum_{i=1}^{n} \mu(A^i)u_i \text{ for all } u \text{ such that } Au = 0 \text{ and } u + \alpha^{(i)} \in \mathcal{L}^2$$

for $i = 1,2$*. Following the same reasoning of Proposition 5.2, one can easily deduce that for any* $\mathcal{K}$*-minimal valid inequality* $(\mu;\eta_0)$ *and any extreme ray* z *of* $\mathcal{K} = \mathcal{L}^2$*, we have* $\sigma_D(Az) = \langle \mu, z\rangle$*.*

Example 5.3 *Using Proposition 5.2 and Remark 5.2, we can now analyze the conic mixed integer rounding cuts introduced in [6]. In particular, in [6], the following simple mixed integer set is studied*

$$\mathcal{S}_0 := \{(x,y,w,t) \in \mathbb{Z} \times \mathbb{R}^3_+ : \ |x + y - w - b| \leq t\}, \tag{6}$$

and it is shown that when $b = \lfloor b \rfloor + f$ *with* $f \in (0,1)$*, the following valid inequality*

$$(1-2f)(x - \lfloor b \rfloor) + f \leq t + y + w \tag{7}$$

together with the original inequality in $\mathcal{S}_0$ *gives* $\overline{\text{conv}}(\mathcal{S}_0)$*.*

Here we will prove that (7) *is in fact a* $\mathcal{K}$*-sublinear inequality. The first step in this analysis is to transform* $\mathcal{S}_0$ *into our normal form as*

$$\mathcal{S} := \left\{(y,w,t,\gamma) \in \mathbb{R}^3_+ \times \mathcal{L}^2 : \ \begin{bmatrix} y - w \\ t \end{bmatrix} - \gamma = \begin{bmatrix} b - x \\ 0 \end{bmatrix}\right\}, \tag{8}$$

which leads to $\mathcal{K} = \mathbb{R}^3_+ \times \mathcal{L}^2$*, which is a closed convex pointed cone with nonempty interior, and*

$$A = \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \underbrace{\begin{bmatrix} f \\ 0 \end{bmatrix}}_{:=b_1^+}, \underbrace{\begin{bmatrix} 1+f \\ 0 \end{bmatrix}}_{:=b_2^+}, \ldots, \underbrace{\begin{bmatrix} f-1 \\ 0 \end{bmatrix}}_{:=b_1^-}, \underbrace{\begin{bmatrix} f-2 \\ 0 \end{bmatrix}}_{:=b_2^-}, \ldots \right\}.$$

Before we proceed first note that ***Assumption 1*** *is satisfied, i.e., for any* $\epsilon_1, \epsilon_2 > 0$*,* $(y;w;t;\gamma_1;\gamma_2) = (f+\epsilon_1;\epsilon_1;\epsilon_2;0;\epsilon_2) \in \text{int}(K)$ *and also in* $\mathcal{S}(A,\mathcal{K},\mathcal{B})$*, therefore* $\mathcal{K}$*-minimality is well defined. Moreover* $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ *is not full dimensional,* $t - \gamma_2 = 0$ *is a valid equation, and one can easily prove that it is* $\mathcal{K}$*-minimal.*[5)]

We can use the results of Section 5 to check whether the inequality (7) *satisfies the conditions of* $\mathcal{K}$*-minimal inequalities. Using the first equation in* (8)*, we get* $y - w - \gamma_1 = b - x$*, which implies that*

5) The set D_e corresponding to this valid equation is just $D_e = \{(\lambda_1,\lambda_2) : \ \lambda_1 = 1, \lambda_2 = 0\} = \{(1,0)\}$ and the point $\bar{z}$ defined in the rest of this example works for this valid equation as well to show that it satisfies the necessary condition for $\mathcal{K}$-minimality.

$x - \lfloor b \rfloor = -y + w + \gamma_1 + f$. By substituting $x - \lfloor b \rfloor$ with $-y + w + \gamma_1 + f$, in (7)*, we can rewrite it in terms of the variables in our representation as follows:*

$$\begin{aligned}&(1-2f)(-y+w+\gamma_1+f)+f \leq t+y+w\\&(2-2f)y+2fw+t+(2f-1)\gamma_1+0\gamma_2 \geq f(2-2f),\end{aligned}$$

which means, $\eta_0 = f(2-2f)$, $\mu_1 = 2-2f$, $\mu_2 = 2f$, $\mu_3 = 1$, $\mu_4 = 2f-1$ and $\mu_5 = 0$ in our usual notation. The necessary conditions for $\mathcal{K}$-sublinearity state that for D given by (3)*, we should have $D \neq \emptyset$, and $\sigma_D(Az) = \langle \mu, z\rangle$ for all $z \in$* Ext$(\mathcal{K})$ *(since all of the extreme rays of $\mathcal{K}$ are orthogonal to each other).*

In our specific case, we have

$$\begin{aligned}D &= \{\lambda \in \mathbb{R}^2 : \ \exists \gamma \in \mathcal{K}^* \text{ such that } A^*\lambda + \gamma = \mu\}\\&= \left\{\lambda \in \mathbb{R}^2 : \ \lambda_1 \leq \mu_1, \ -\lambda_1 \leq \mu_2, \ \lambda_2 \leq \mu_3, \ \begin{bmatrix} -\lambda_1 \\ -\lambda_2 \end{bmatrix} \preceq_{\mathcal{L}^2} \begin{bmatrix} \mu_4 \\ \mu_5 \end{bmatrix}\right\}\\&= \left\{\lambda \in \mathbb{R}^2 : \ \lambda_1 \leq 2-2f, \ -\lambda_1 \leq 2f, \ \lambda_2 \leq 1, \ |2f-1+\lambda_1| \leq \lambda_2\right\},\end{aligned}$$

and the corresponding feasible region of D is plotted in Figure 4.

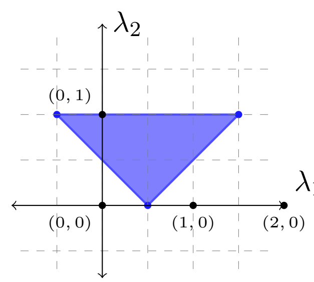

Figure 4: Feasible region corresponding to D for $f = 0.25$ in conic mixed integer rounding cut of [6].

As $f \in (0,1)$, we have $D \neq \emptyset$, proving that $(\mu;\eta_0)$ is $\mathcal{K}$-sublinear. Also the extreme rays of $\mathcal{K}$ are precisely Ext$(\mathcal{K}) = \{e_1, e_2, e_3, -e_4+e_5, e_4+e_5\}$ *where e_i stands for the i^{th} unit vector in $\mathbb{R}^5$. Moreover,*

$$\sigma_D(Ae_1) = \sigma_D(A^1) = 2-2f = \mu_1 = \langle \mu, e_1\rangle$$

where A^i denotes the i^{th} column of the matrix A. Similarly we can show that $\sigma_D(Ae_i) = \mu_i = \langle \mu, e_i\rangle$ for $i = 1, \ldots, 3$. Moreover, we have $\sigma_D(A(-e_4+e_5)) = \sigma_D(\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = 1-2f = -\mu_4+\mu_5 = \langle \mu, (-e_4+e_5)\rangle$ and $\sigma_D(A(e_4+e_5)) = \sigma_D(\begin{bmatrix} -1 \\ -1 \end{bmatrix}) = 2f-1 = \mu_4+\mu_5 = \langle \mu, (e_4+e_5)\rangle$.

Finally note that $\sigma_D(b_1^+) = f \cdot \sigma_D(e_1) = f(2-2f)$ and for $i = 1, 2, \ldots$, we have $\sigma_D(b_{i+1}^+) = (f+i)(2-2f) = (2-2f)i + 2f - 2f^2$. Considering $f \in (0,1)$, we conclude $\sigma_D(b_1^+) < \sigma_D(b_2^+) < \ldots$. Similarly $\sigma_D(b_1^-) = (1-f)\sigma_D(-e_1) = (1-f)(-2f) = 2f(f-1)$ and for $i = 1, 2, \ldots$, we have $\sigma_D(b_i^-) = (f-i)(-2f) = 2fi - 2f^2$, which implies $\sigma_D(b_1^-) < \sigma_D(b_2^-) < \ldots$ holds, and hence

$$\inf_{b \in \mathcal{B}} \sigma_D(b) = \min\left\{\sigma_D(b_1^+), \ \sigma_D(b_1^-)\right\} = f(2-2f) = \eta_0.$$

Finally consider the following set of points

$$\{z_1 := (f;0;0;0;0), z_2 := (0;1-f;0;0;0), z_3 := (0;0;f;-f;f), z_4 := (0;0;1-f;1-f;1-f)\}.$$

Given $f \in (0,1)$, one can easily see that for $i = 1,\dots,4$, we have $z_i \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ and $\langle \mu, z_i\rangle = \eta_0 = 2f - 2f^2$. Moreover $\bar{z} := \frac{1}{4}\sum_{i=1}^{4} z_i$ is in the interior of $\mathcal{K} = \mathbb{R}^3_+ \times \mathcal{L}^2$. Therefore using Proposition 5.6, we have shown that the valid inequality given by $(\mu;\eta_0) = (2-2f;2f;1;2f-1;0;2f-2f^2)$, which is equivalent to (7)*, is a $\mathcal{K}$-minimal inequality.*

6 Characterization of Valid Equations

Although our results with regard to the existence of $\mathcal{K}$-minimal inequalities assume that for all $\delta \in \mathcal{K}^* \setminus \{0\}$, there exists $z_\delta \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ such that $\langle \delta, z_\delta\rangle > 0$ (**Assumption 1**), with a slightly stronger assumption, i.e.,

Assumption 2: There exists $\hat{z} \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ such that $\hat{z} \in \text{int}(\mathcal{K})$ and $A\hat{z} = \hat{b}$ for some $\hat{b} \in \mathcal{B}$,

we can provide the following precise characterization of the valid equations.

Theorem 6.1 *Suppose that **Assumption 2** holds. Then $(\mu;\mu_0)$ is a valid equation if and only if there exists some $\bar{\lambda} \in \mathbb{R}^m$ such that*

$$A^*\bar{\lambda} = \mu \text{ and } \langle b, \bar{\lambda}\rangle = \mu_0$$

for all $b \in \mathcal{B}$.

Proof.

($\Leftarrow$) It's easy to see that the condition in Theorem 6.1 is sufficient. Suppose there exists $\bar{\lambda} \in \mathbb{R}^m$ such that

$$A^*\bar{\lambda} = \mu \text{ and } \langle b, \bar{\lambda}\rangle = \mu_0$$

for all $b \in \mathcal{B}$. Then for any $z \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ we have

$$\langle \mu, z\rangle = \langle A^*\bar{\lambda}, z\rangle = \langle \bar{\lambda}, Az\rangle = \langle \bar{\lambda}, b\rangle = \mu_0$$

where the last equation follows from the fact that $z \in \mathcal{S}(A,\mathcal{K},\mathcal{B})$ and hence $b \in \mathcal{B}$, proving that $(\mu;\mu_0)$ is a valid equation.

($\Rightarrow$) To prove the necessity of the condition, suppose that $(\mu;\mu_0)$ is a valid equation. Let $\hat{b}$ and $\hat{z}$ be as described in the statement of Theorem 6.1, and consider

$$\inf_z \{\langle \mu, z\rangle : \ Az = \hat{b}, \ z \in \mathcal{K}\}.$$

Since there exists $\hat{z} \in \text{int}(\mathcal{K})$ satisfying $A\hat{z} = \hat{b}$, this problem is strictly feasible, moreover the solution set of this problem is contained in $\mathcal{S}(A,\mathcal{K},\mathcal{B})$ and thus $(\mu;\mu_0)$ being a v.e., implies that its optimum value is equal to μ_0. Using strong conic duality we get

$$\mu_0 = \sup_\lambda \{\langle \hat{b}, \lambda\rangle : \ A^*\lambda \preceq_{\mathcal{K}^*} \mu\},$$

which implies the existence of an optimal solution $\bar{\lambda}$ satisfying

$$A^*\bar{\lambda} \preceq_{\mathcal{K}^*} \mu \text{ and } \langle \hat{b}, \bar{\lambda}\rangle = \mu_0$$

Note that any feasible solution to the primal problem is optimal including the strictly feasible solution $\hat{z}$. Therefore using the complementary slackness of conic duality we have

$$\langle \hat{z}, \mu - A^*\bar{\lambda}\rangle = 0.$$

Since $\hat{z}$ is in int$(\mathcal{K})$ this is possible if and only if $A^*\bar{\lambda} = \mu$. So now we have established that there exists $\bar{\lambda}$ satisfying $A^*\bar{\lambda} = \mu$ and $\langle \hat{b}, \bar{\lambda}\rangle = \mu_0$. , Now consider any $b \in \mathcal{B}$, then we have

$$\mu_0 = \langle \mu, z_b\rangle \;\geq\; \inf_z\{\langle \mu, z\rangle : \; Az = b, \; z \in \mathcal{K}\} \;\geq\; \sup_\lambda\{\langle \hat{b}, \lambda\rangle : \; A^*\lambda \preceq_{\mathcal{K}^*} \mu\} \;\geq \langle \hat{b}, \bar{\lambda}\rangle. \tag{9}$$

Moreover

$$-\mu_0 = \langle -\mu, z_b\rangle \;\geq\; \inf_z\{\langle -\mu, z\rangle : \; Az = b, \; z \in \mathcal{K}\} \;\geq\; \sup_\lambda\{\langle \hat{b}, \lambda\rangle : \; A^*\lambda \preceq_{\mathcal{K}^*} -\mu\} \;\geq \langle \hat{b}, -\bar{\lambda}\rangle \tag{10}$$

where the second inequality follows from weak duality and the last inequality follows from the fact that $-\bar{\lambda}$ is a feasible solution to the dual. By combining (9) and (10), we get $\mu_0 = \langle \hat{b}, \bar{\lambda}\rangle$, which completes the proof.

∎

In addition to the above characterization, we can relate each v.e. with its corresponding set D given by (3) as follows:

Corollary 6.1 *Suppose that **Assumption 2** holds.* $(\mu; \eta_0)$ *is a v.e. if there exists* λ_μ *satisfying* $D := \{\lambda : A^*\lambda \preceq_{\mathcal{K}^*} \mu\} = \lambda_\mu + \{\lambda : \; A^*\lambda \preceq_{\mathcal{K}^*} 0\}$ *and* $\eta_0 = \inf_{b\in\mathcal{B}} \sigma_D(b) = \sup_{b\in\mathcal{B}} \sigma_D(b)$. *Also if* $(\mu; \eta_0)$ *satisfies these conditions, then it is a valid* $\mathcal{K}$*-sublinear inequality.*

Proof. Suppose $(\mu; \eta_0)$ is a v.e., then by Theorem 6.1, there exists $\bar{\lambda} =: \lambda_\mu$ such that $\mu = A^*\bar{\lambda}$ and $\mu_0 = \langle b, \bar{\lambda}\rangle$ for all $b \in \mathcal{B}$. Thus, we have

$$D = \{\lambda : \; A^*\lambda \preceq_{\mathcal{K}^*} A^*\bar{\lambda}\} = \{\bar{\lambda} + \lambda : \; A^*\lambda \preceq_{\mathcal{K}^*} 0\},$$

and

$$\inf_{b\in\mathcal{B}} \sigma_D(b) = \inf_{b\in\mathcal{B}} \sup_\lambda\{\langle b, \bar{\lambda} + \lambda\rangle : \; A^*\lambda \preceq_{\mathcal{K}^*} 0\} = \inf_{b\in\mathcal{B}} \left[\underbrace{\langle b, \bar{\lambda}\rangle}_{=\eta_0} + \underbrace{\sup_\lambda\{\langle b, \lambda\rangle : \; A^*\lambda \preceq_{\mathcal{K}^*} 0\}}_{\in\{0,+\infty\}}\right] = \eta_0,$$

where the last equation follows from the fact that $\eta_0 \in \mathbb{R}$. Similarly, we can show that $\eta_0 = \sup_{b\in\mathcal{B}} \sigma_D(b)$.

To see the reverse, suppose $(\mu; \eta_0)$ is such that there exists λ_μ satisfying

$$D = \{\lambda : A^*\lambda \preceq_{\mathcal{K}^*} \mu\} = \lambda_\mu + \{\lambda : \; A^*\lambda \preceq_{\mathcal{K}^*} 0\} = \{\lambda : \; A^*\lambda \preceq_{\mathcal{K}^*} A^*\lambda_\mu\},$$

and $\eta_0 = \inf_{b\in\mathcal{B}} \sigma_D(b)$. Then, by Proposition 5.4, it is valid and by Proposition 5.5 $(\mu; \eta_0) \in C_a(A, \mathcal{K}, \mathcal{B})$. Therefore by Theorem 5.1, we have

$$\langle \mu, z\rangle \geq \sigma_D(Az) = \langle A^* \lambda_\mu, z\rangle$$

for all $z \in \mathcal{K}$. ∎

When $\mathcal{K} = \mathbb{R}^n_+$ (or any cone where each pair of its extreme rays is orthogonal), we note that the statement of Corollary 6.1 gives a complete characterization of v.e.

7 Conclusions and Further Research

Gomory's corner polyhedron serves as a major tool in the MILP framework, as most cutting planes used in MILP can be viewed in this context. Therefore the generalization of Gomory's corner polyhedron from MILPs to MICPs offered by $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ suggests its potential to serve as a fundamental equivalent relaxation for MICPs. In particular, by studying the structure of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$, we can design better cutting planes for MICPs. In the MILP literature, a natural first step in studying the corner polyhedron has been investigating the associated semi-infinite relaxation. Such an alternative seems to be meaningful only when the associated cone is the nonnegative orthant. To the best of our knowledge, the extensions of other well-known regular cones such as the Lorentz cone and the cone of positive semidefinite matrices to the infinite dimensional cases are not well defined. Moreover in the practical cutting plane procedures to solve MILPs (and also MICPs), one is indeed faced with a problem in a finite dimensional space. Therefore we chose to follow a different path and focus on a finite dimensional case defined by a given particular instance.

We introduced the class of $\mathcal{K}$-minimal valid inequalities in the MICP context. We have seen that this class contains a small yet essential set of irredundant v.i. In particular, under a mild assumption, the class of $\mathcal{K}$-minimal v.i. together with the trivial cone-implied inequalities are sufficient to describe the closed convex hull of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$. This indicates that $\mathcal{K}$-minimal v.i. are of great interest, and an efficient cutting plane procedure for solving MICPs should aim at separating v.i. from this class. Nevertheless the definition of $\mathcal{K}$-minimality reveals little about the structure of $\mathcal{K}$-minimal v.i. To address this issue, we showed that the class of $\mathcal{K}$-minimal v.i. is contained in a slightly larger class of so-called $\mathcal{K}$-sublinear v.i. We established a close connection between $\mathcal{K}$-sublinear v.i. for MICPs and convex sets with certain structure. This led to practical ways of showing that an inequality is $\mathcal{K}$-sublinear and $\mathcal{K}$-minimal. In all cases, our results generalized the earlier results from MILP setup, i.e., when $\mathcal{K} = \mathbb{R}^n_+$, indicating that $\mathcal{K}$-minimal inequalities for MILPs are generated by sublinear (positively homogeneous, subadditive and convex) functions that are also piecewise linear.

In this work, we have shed some light on the structure of $\mathcal{K}$-minimal and $\mathcal{K}$-sublinear valid inequalities for solution sets of the form $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ involving a regular cone $\mathcal{K}$. But many questions remain open when we start considering regular cones other than $\mathbb{R}^n_+$. In particular, we find the following questions of interest:

- *[Finiteness of the $\mathcal{K}$-minimal conic inequalities]* When $\mathcal{K} = \mathbb{R}^n_+$ and $\mathcal{B}$ is finite, Johnson [45] proved that the cone of $\mathcal{K}$-minimal inequalities is finitely generated, i.e., G_C is finite. Note that G_L is always finite. For general regular cones, e.g., $\mathcal{L}^n, \mathcal{S}^n_+$, expecting the convex hull description of $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ to be given by finitely many linear inequalities is too much, and against the inherit nonlinear nature of these cones. Even for $\mathcal{L}^3$, there are examples showing that this is not possible, i.e., there exists $\mathcal{S}(A, \mathcal{K}, \mathcal{B})$ that requires infinitely many $\mathcal{K}$-minimal and also extremal linear inequalities (see Example 5.2). On the other hand, in that example, it is clear that the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$ only involves two linear and one conic inequality involving $\mathcal{L}^3$. While the $\mathcal{K}$-minimality notion is seemingly defined for linear v.i., we can immediately extend it to a conic inequality by saying that a conic v.i. is $\mathcal{K}$-minimal if the associated (possibly infinite) set of all linear inequalities are $\mathcal{K}$-minimal. We believe that instead of focusing on the finiteness of linear inequalities describing $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$, it is more natural and relevant to focus on the finiteness of conic inequalities (of the same type of $\mathcal{K}$) describing $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$. Therefore we wonder what can be said in terms of the number of $\mathcal{K}$-minimal conic inequalities required in the description of $\overline{\text{conv}}(\mathcal{S}(A, \mathcal{K}, \mathcal{B}))$. Is it a finite number when $\mathcal{B}$ is finite? Is it finite regardless of the size of $\mathcal{B}$? Or can we at least identify the cases where it is finite?

- *[Relations with other valid inequalities for MICPs]* We showed that conic MIR inequalities introduced in [6] can be interpreted in this framework. It would be nice to understand the relation of our framework and other recently developed valid inequalities for MICPs based on split or disjunctive arguments as in [3, 12, 31, 51].

References

[1] K. Abhishek and J. T. Linderoth. An outer-approximation-based solver for nonlinear mixed integer programs. *Lecture Notes in Computer Science*, 2081:293–303, 2001.

[2] F. Alizadeh. Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM J. Optim.*, 5:13–51, 1993.

[3] K. Andersen and A. N. Jensen. Intersection cuts for mixed integer conic quadratic sets. In Goemans and Correa [35], pages 37–48.

[4] K. Andersen, Q. Louveaux, R. Weismantel, and L. A. Wolsey. Inequalities from two rows of a simplex tableau. In M. Fischetti and D. P. Williamson, editors, *IPCO*, volume 4513 of *Lecture Notes in Computer Science*, pages 1–15. Springer, 2007.

[5] A. Atamtürk, G. Berenguer, and M. Shen. A conic integer programming approach to stochastic joint location-inventory problems. *Oper. Res.*, 60(2):366–381, 2012.

[6] A. Atamtürk and V. Narayanan. Conic mixed-integer rounding cuts. *Math. Program.*, 122(1):1–20, 2010.

[7] A. Atamtürk and V. Narayanan. Lifting for conic mixed-integer programming. *Math. Program.*, 126(2):351–363, 2011.

[8] E. Balas. Integer programming and convex analysis: Intersection cuts from outer polars. *Math. Program.*, 2:330–382, 1972.

[9] E. Balas. Disjunctive programming. *Ann. Discrete Math.*, 5:3–51, 1979.

[10] E. Balas, S. Ceria, and G. Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Math. Program.*, 58:295–324, 1993.

[11] E. Balas and A. Qualizza. Intersection cuts from multiple rows: A disjunctive programming approach. Technical report, 2012. `http://arxiv.org/abs/1206.1630`.

[12] P. Belotti, J. C. Goez, I. Polik, T. K. Ralphs, and T. Terlaky. On families of quadratic surfaces having fixed intersections with two hyperplanes. Technical report, 2012. `http://www.optimization-online.org/DB_FILE/2012/06/3494.pdf`.

[13] A. Ben-Tal and A. S. Nemirovski. Robust convex optimization. *Math. Oper. Res.*, 23(4):769–805, 1998.

[14] A. Ben-Tal and A. S. Nemirovski. *Lectures on modern convex optimization: analysis, algorithms, and engineering applications*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2001.

[15] D. Bienstock and A. Michalka. Strong formulations for convex functions over nonconvex sets. Technical report, 2011. `http://www.optimization-online.org/DB\_FILE/2011/12/3278.pdf`.

[16] D. Bienstock and A. Michalka. Cutting-planes for optimization of convex functions over nonconvex sets. Technical report, 2013. `http://www.optimization-online.org/DB\_FILE/2013/05/3881.pdf`.

[17] A. Billionnet, E. Sourour, and P. Marie-Christine. Improving the performance of standard solvers for quadratic 0-1 programs by a tight convex reformulation: The QCR method. *Discrete Appl. Math.*, 157(6):1185–1197, Mar. 2009.

[18] P. Bonami. Lift-and-project cuts for mixed integer convex programs. In O. Günlük and G. J. Woeginger, editors, *IPCO*, volume 6655 of *Lecture Notes in Computer Science*, pages 52–64. Springer, 2011.

[19] P. Bonami, L. T. Biegler, A. R. Conn, G. Cornuéjols, I. E. Grossmann, C. D. Laird, J. Lee, A. Lodi, F. Margot, N. Sawaya, and A. Wächter. An algorithmic framework for convex mixed integer nonlinear programs. *Discret. Optim.*, 5(2):186–204, May 2008.

[20] V. Borozan and G. Cornuéjols. Minimal valid inequalities for integer constraints. *Math. Oper. Res.*, 34(3):538–546, 2009.

[21] C. Buchheim, A. Caprara, and A. Lodi. An effective branch-and-bound algorithm for convex quadratic integer programming. *Math. Program.*, 135(1-2):369–395, 2012.

[22] S. Burer and A. Saxena. The MILP road to MIQCP. In *Mixed Integer Nonlinear Programming*, pages 373–405. Springer, 2012.

[23] M. T. Çezik and G. Iyengar. Cuts for mixed 0-1 conic programming. *Math. Program.*, 104(1):179–202, Sept. 2005.

[24] M. Conforti, G. Cornuéjols, A. Daniilidis, C. Lemaréchal, and J. Malick. Cut-generating functions. In Goemans and Correa [35], pages 123–132.

[25] M. Conforti, G. Cornuéjols, and G. Zambelli. Equivalence between intersection cuts and the corner polyhedron. *Oper. Res. Lett.*, 38:153–155, 2010.

[26] M. Conforti, G. Cornuéjols, and G. Zambelli. Corner polyhedron and intersection cuts. *Surveys in Operations Research and Management Science*, 16:105–120, 2011.

[27] G. Cornuéjols. Valid inequalities for mixed integer linear programs. *Math. Program.*, 112(1):3–44, July 2007.

[28] G. Cornuéjols and F. Margot. On the facets of mixed integer programs with two integer variables and two constraints. *Math. Program.*, 120:429–456, 2009.

[29] G. Cornuéjols, L. Wolsey, and S. Yıldız. Sufficiency of cut-generating functions. Technical report, 2013. http://integer.tepper.cmu.edu/webpub/draftCGF.pdf.

[30] CPLEX. IBM ILOG, 12.5. http://www-01.ibm.com/software/commerce/optimization/cplex-optimizer/.

[31] D. Dadush, S. S. Dey, and J. P. Vielma. The split closure of a strictly convex body. *Oper. Res. Lett.*, 39(2):121–126, Mar. 2011.

[32] S. Drewes. *Mixed integer second order cone programming*. PhD thesis, Technische Universität, 2009.

[33] S. Drewes and S. Pokutta. Cutting-planes for weakly-coupled 0/1 second order cone programs. *Electronic Notes in Discrete Mathematics*, 36:735–742, 2010.

[34] M. X. Goemans. Semidefinite programming in combinatorial optimization. *Math. Program.*, 79:143–161, 1997.

[35] M. X. Goemans and J. R. Correa, editors. *Integer Programming and Combinatorial Optimization - 16th International Conference, IPCO 2013, Valparaíso, Chile, March 18-20, 2013. Proceedings*, volume 7801 of *Lecture Notes in Computer Science*. Springer, 2013.

[36] M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *J. ACM*, 42(6):1115–1145, Nov. 1995.

[37] R. E. Gomory. Some polyhedra related to combinatorial problems. *Lin. Alg. Appl.*, 2(4):451–558, 1969.

[38] R. E. Gomory and E. L. Johnson. Some continuous functions related to corner polyhedra. *Math. Program.*, 3:23–85, 1972.

[39] R. E. Gomory and E. L. Johnson. Some continuous functions related to corner polyhedra, II. *Math. Program.*, 3:359–389, 1972.

[40] Gurobi Optimization. Reference Manual, 5.5. http://www.gurobi.com/pdf/reference-manual.pdf.

[41] R. Hildebrand. An LMI description for the cone of Lorentz-positive maps I. *Linear and Multilinear Algebra*, 55(6):551–573, 2007.

[42] R. Hildebrand. An LMI description for the cone of Lorentz-positive maps II. *Linear and Multilinear Algebra*, 59(7):719–731, 2011.

[43] J.-B. Hiriart-Urruty and C. Lemaréchal. *Fundamentals of Convex Analysis*. Springer Verlag, 2001.

[44] E. L. Johnson. On the group problem for mixed integer programming. *Math. Program.*, 2:137–179, 1974.

[45] E. L. Johnson. Characterization of facets for multiple right-hand side choice linear programs. *Mathematical Programming Study*, 14:137–179, 1981.

[46] M. R. Kılınç, J. Linderoth, and J. Luedtke. Effective separation of disjunctive cuts for convex mixed integer nonlinear programs. Technical report, 2010. http://www.optimization-online.org/DB_FILE/2010/11/2808.pdf.

[47] M. Kojima and L. Tunçel. Cones of matrices and successive convex relaxations of nonconvex sets. *SIAM J. Optim.*, 10(3):750–778, July 1999.

[48] J. B. Lasserre. An explicit exact SDP relaxation for nonlinear 0-1 programs. *Lecture Notes in Computer Science*, 2081:293–303, 2001.

[49] M. Lobo, M. Fazel, and S. Boyd. Portfolio optimization with linear and fixed transaction costs. *Ann. of Oper. Res.*, 152(1):341–365, 2007.

[50] L. Lovasz and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. *Siam J. Optim.*, 1:166–190, 1991.

[51] S. Modaresi, M. R. Kılınç, and J. P. Vielma. Intersection cuts for nonlinear integer programming: Convexification techniques for structured sets. Technical report, 2013. http://arxiv.org/abs/1302.2556.

[52] MOSEK. The MOSEK optimization tools manual, 6.0. http://docs.mosek.com/6.0/tools.pdf.

[53] G. L. Nemhauser and L. A. Wolsey. *Integer and Combinatorial Optimization*. John Wiley and Sons, New York , NY, USA, 1988.

[54] A. Qualizza, P. Belotti, and F. Margot. Linear programming relaxations of quadratically constrained quadratic programs. In *Mixed Integer Nonlinear Programming*, pages 407–426. Springer, 2012.

[55] R. T. Rockafellar. *Fundamentals of Convex Analysis*. Princeton University Press, 1970.

[56] A. Saxena, P. Bonami, and J. Lee. Disjunctive cuts for non-convex mixed integer quadratically constrained programs. In A. Lodi, A. Panconesi, and G. Rinaldi, editors, *IPCO*, volume 5035 of *Lecture Notes in Computer Science*, pages 17–33. Springer, 2008.

[57] A. Saxena, P. Bonami, and J. Lee. Convex relaxations of non-convex mixed integer quadratically constrained programs: Extended formulations. *Math. Program.*, 124(1-2):383–411, 2010.

[58] A. Saxena, P. Bonami, and J. Lee. Convex relaxations of non-convex mixed integer quadratically constrained programs: Projected formulations. *Math. Program.*, 130(2):359–413, 2011.

[59] H. D. Sherali and W. P. Adams. *A reformulation-linearization technique for solving discrete and continuous nonconvex problems*. Springer, 1998.

[60] H. D. Sherali and C. Shetti. Optimization with disjunctive constraints. *Lectures on Econ. Math. Systems*, 181, 1980.

[61] H. D. Sherali and C. H. Tunçbilek. A reformulation-convexification approach for solving nonconvex quadratic programming problems. *J. Global Optim.*, 7:1–31, 1995.

[62] R. A. Stubbs and S. Mehrotra. A branch-and-cut method for 0-1 mixed convex programming. *Math. Program.*, 86:515–532, 1999.

[63] R. A. Stubbs and S. Mehrotra. Generating convex polynomial inequalities for mixed 0-1 programs. *J. Global Optim.*, 24(3):311–332, Nov. 2002.

[64] M. Tawarmalani and N. V. Sahinidis. *Convexification and global optimization in continuous and mixed-integer nonlinear programming: theory, algorithms, software, and applications*. Springer, 2002.

[65] M. Tawarmalani and N. V. Sahinidis. Global optimization of mixed-integer nonlinear programs: A theoretical and computational study. *Math. Program.*, 99(3):563–591, Apr. 2004.

[66] M. Tawarmalani and N. V. Sahinidis. A polyhedral branch-and-cut approach to global optimization. *Math. Program.*, 103(2):225–249, June 2005.

[67] J. P. Vielma, S. Ahmed, and G. L. Nemhauser. A lifted linear programming branch-and-bound algorithm for mixed-integer conic quadratic programs. *INFORMS J. on Computing*, 20(3):438–450, 2008.