On the diameter of half-integral polytopes

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Abstract

We prove a new bound on the diameter of lattice polytopes, that improves on a 20 year old bound by Kleinschmidt and Onn. This result implies that the diameter of a d-dimensional half-integral polytope is at most $\lfloor \frac{3}{2} d \rfloor$. For this family of polytopes we also show that the bound is tight.

1 Introduction

The 1-*skeleton* of a polytope P is the graph whose nodes are the vertices of P, and that has an edge joining two nodes if and only if the corresponding vertices of P are adjacent on P. Given vertices u, v of P, the *distance* $\delta^P(u, v)$ between u and v is the length of a shortest path connecting u and v on the 1-skeleton of P. We may write $\delta(u, v)$ instead of $\delta^P(u, v)$ when the polytope we are referring to is clear from the context. The *diameter* $\delta(P)$ of P is the smallest natural number that bounds the distance between any pair of vertices of P. We refer to [7] for basic notions and terminology on polytopes and graphs.

A *lattice polytope* is a nonempty polytope whose vertices are integral. For $k \in \mathbb{N}$, a $(0, k)$-*polytope* $P \subseteq \mathbb{R}^n$ is a lattice polytope contained in $[0, k]^n$. Naddef [4] showed that the diameter of a d-dimensional $(0, 1)$-polytope is at most d, and such bound is tight for the hypercube $[0, 1]^d$. Kleinschmidt and Onn [3] extended such result by proving that the diameter of a d-dimensional $(0, k)$-polytope cannot exceed kd.

We now state our main results.

Theorem 1. *For $k \geq 2$, the diameter of a d-dimensional $(0, k)$-polytope is at most $\lfloor \frac{2k-1}{2} d \rfloor$.*

In the special case where $k = 2$, we prove that the bound of Theorem 1 is tight, i.e. for any natural number d there exists a d-dimensional $(0, 2)$-polytope attaining the bound.

Theorem 2. *The diameter of a d-dimensional $(0, 2)$-polytope is at most $\lfloor \frac{3}{2} d \rfloor$, and the bound is tight.*

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A polytope P is called *half-integral* if every vertex has only 0, 1 or 1/2 components, or equivalently if $2P = \{2x : x \in P\}$ is a $(0,2)$-polytope. As P and $2P$ have the same 1-skeleton and the same dimension, Theorem 2 implies that the diameter of a half-integral polytope is at most $\lfloor \frac{3}{2} d \rfloor$, and the bound is tight.

In Section 2 we present some lemmas that will be crucial to prove our main results, in Section 3 we give the proof of Theorem 1, and in Section 4 we prove Theorem 2.

2 Tools

We say that two vertices u, v of a polytope P are *neighbors* if $\delta(u, v) = 1$.

Lemma 1. *Let $P \subseteq \mathbb{R}^n$ be a lattice polytope, let u be a vertex of P, and let $c \in \mathbb{Z}^n$. If there exists a vector $v \in P$ such that $cv < cu$, then there exists a neighbor u^1 of u with $cu^1 \leq cu - 1$.*

Proof. It is well known that there exists a path of vertices $u^0, u^1, \dots, u^t$ on the 1-skeleton of P with $u^0 = u$, and $cu^t \leq cv$, where the vertices along the path attain strictly decreasing values under c, i.e. $cu^i < cu^{i-1}$ for $i = 1, \dots, t$ (see e.g. [2]). Then u^1 is a neighbor of u with $cu^1 < cu$. By the integrality of c, u, u^1, we have $cu^1 \leq cu - 1$. $\square$

Lemma 2. *Let $P \subseteq \mathbb{R}^n$ be a lattice polytope, and let u be a vertex of P. Let $c \in \mathbb{Z}^n$, $\gamma = \min\{cx : x \in P\}$, and $F = \{x \in P : cx = \gamma\}$. Then there exists a vertex u' of F such that $\delta(u, u') \leq cu - \gamma$.*

Proof. We prove the statement by induction on the integer value $cu - \gamma \geq 0$. The statement is trivial for $cu - \gamma = 0$. Assume $cu - \gamma \geq 1$. Because F is nonempty, there exists a vector $v \in F$ such that $cu - cv = cu - \gamma \geq 1$. Thus, by Lemma 1, there exists a neighbor u^1 of u with $cu^1 \leq cu - 1$. As $cu^1 - \gamma \leq cu - \gamma - 1$, by the induction hypothesis there exists a vertex u' of F such that $\delta(u^1, u') \leq cu^1 - \gamma$. Therefore $\delta(u, u') \leq \delta(u, u^1) + \delta(u^1, u') \leq 1 + cu^1 - \gamma \leq cu - \gamma$. $\square$

Lemma 3. *Let $P \subseteq \mathbb{R}^n$ be a lattice polytope, and let u, v be vertices of P. Let $c \in \mathbb{Z}^n$, $\gamma = \min\{cx : x \in P\}$, and $F = \{x \in P : cx = \gamma\}$. Then $\delta(u, v) \leq \delta(F) + cu + cv - 2\gamma$.*

Proof. By Lemma 2 there exist vertices u', v' of F such that $\delta(u, u') \leq cu - \gamma$ and $\delta(v, v') \leq cv - \gamma$. As $u', v' \in F$, we have $\delta(u', v') \leq \delta(F)$, hence $\delta(u, v) \leq \delta(u, u') + \delta(u', v') + \delta(v', v) \leq cu - \gamma + \delta(F) + cv - \gamma$. $\square$

Given a vector $v \in \mathbb{R}^n$ we denote by v_i its i-th component, for $i = 1, \dots, n$. We denote by e^i, for $i = 1, \dots, n$, the i-th vector of the standard basis of $\mathbb{R}^n$. Moreover, given a scalar k, we denote by k^n the n-dimensional vector with all entries equal to k.

Given two vertices u and v of a $(0, k)$-polytope P in $\mathbb{R}^n$, by applying Lemma 3 with some $c \in \mathbb{Z}^n$, we can derive an upper bound on $\delta(u, v)$ that depends on c. In the particular case where $c = e^i$, for $i \in \{1, \dots, n\}$, as $\min\{x_i : x \in P\} \geq 0$, Lemma 3 immediately implies $\delta(u, v) \leq \delta(F) + u_i + v_i$, where F is the face of P that minimizes x_i. Similarly, if Lemma 3 is applied with $c = -e^i$, for $i \in$

$\{1, \ldots, n\}$, as $\min\{-x_i : x \in P\} \geq -k$, it follows $\delta(u, v) \leq \delta(F) - (u_i + v_i) + 2k$, where F is the face of P that maximizes x_i. Note that, for $i = 1, \ldots, n$, $u_i + v_i \geq k$ if and only if $-(u_i + v_i) + 2k \leq k$. Thus, the bound $\delta(u, v) \leq \delta(F) + k$ always holds, for some face F of P of dimension at most $n-1$. Moreover, if $u+v \neq k^n$, then there exists an index $i \in \{1, \ldots, n\}$ such that either $u_i + v_i \leq k - 1$ or $u_i + v_i \geq k + 1$. By applying Lemma 3 with $c = e^i$ or $c = -e^i$, it follows that $\delta(u, v) \leq \delta(F) + k - 1$, for some face F of P of dimension at most $n - 1$.

In order to bound the diameter of a non full-dimensional polytope P, we will use a theorem by Naddef and Pulleyblank [5]. The statement below is slightly modified with respect to Theorem 3.3. in [5], based on the details given in the first lines of its proof. The *projection of P onto the i-coordinate hyperplane* is the polytope

$$\{\bar{x} \in \mathbb{R}^{n-1} : \exists\, x \in P \text{ with } x_j = \bar{x}_j \text{ for } j = 1, \ldots, i-1,\ x_j = \bar{x}_{j-1} \text{ for } j = i+1, \ldots, n\}.$$

That is, we simply drop the i-th coordinate from all vectors in P.

Theorem 3 ([5])**.** *Let $P \subseteq \mathbb{R}^n$ be a polytope contained in the hyperplane $ax = \beta$ with $a_i \neq 0$ for $i \in \{1, \ldots, n\}$, and let $\bar{P}$ be the projection of P onto the i-coordinate hyperplane. Then P and $\bar{P}$ have the same* 1*-skeleton.*

Theorem 3 implies the following.

Lemma 4. *Let P be a d-dimensional $(0, k)$-polytope in $\mathbb{R}^n$. Then there exists a full-dimensional $(0, k)$-polytope in $\mathbb{R}^d$ with the same* 1*-skeleton as P.*

Proof. For every fixed $d \in \mathbb{N}$, the proof is by induction on n. The base case is when $n = d$. Then P is full-dimensional and the statement is trivially true.

Let now $n \geq d + 1$, i.e. P is not full-dimensional. Then there exists a hyperplane $ax = \beta$ with $a_i \neq 0$ for $i \in \{1, \ldots, n\}$, that contains P. Let $\bar{P}$ be the projection of P onto the i-coordinate hyperplane. By Theorem 3, P and $\bar{P}$ have the same 1-skeleton. Moreover $\bar{P}$ is a $(0, k)$-polytope in $\mathbb{R}^{n-1}$, and has the same dimension of P (see for example [6]). Thus by the induction hypothesis there exists a full-dimensional $(0, k)$-polytope in $\mathbb{R}^d$ with the same 1-skeleton as $\bar{P}$. □

For $d, k \in \mathbb{N}$, we define δ_k^d to be the maximum possible diameter of a $(0, k)$-polytope of dimension at most d, i.e.

$$\delta_k^d = \max\{\delta(P) : P \text{ is a } (0, k)\text{-polytope of dimension at most } d\}.$$

Note that the maximum in the definition of δ_k^d always exists. In fact, it follows from Lemma 4 that the number of vertices of a d-dimensional $(0, k)$-polytope is at most k^d, thus also its diameter is upper bounded by k^d, which is a number independent on the dimension of the ambient space of P. Moreover, for fixed k, the value δ_k^d is clearly non-decreasing in d.

In the following lemma, that is an immediate consequence of Lemma 3, we give a bound on the diameter of a $(0, k)$-polytope P contained in $\mathbb{R}^n$ that does not depend on a particular face of P, but only on δ_k^{n-1}.

Lemma 5. *Let P be a $(0, k)$-polytope in $\mathbb{R}^n$, and suppose that there exists $i \in \{1, \ldots, n\}$ such that $x_i \in [l, h]$, with $0 \leq l \leq h \leq k$ for every $x \in P$. Then $\delta(P) \leq \delta_k^{n-1} + (h - l)$.*

Proof. By performing the change of variable $\tilde{x}_i = x_i - l$, we can assume that $0 \leq x_i \leq h - l$ for every $x \in P$. Let u, v be vertices of P. Then $u_i + v_i \leq h - l$, or $u_i + v_i \geq h - l$. If $u_i + v_i \leq h - l$ let $c = e^i$, otherwise let $c = -e^i$. Let $\gamma = \min\{cx : x \in P\}$, and $F = \{x \in P : cx = \gamma\}$. Note that $\gamma \geq 0$ if $c = e^i$ and $\gamma \geq -(h-l)$ if $c = -e^i$. Moreover, F is a $(0,k)$-polytope of dimension at most $n-1$, thus $\delta(F) \leq \delta_k^{n-1}$. Then, by Lemma 3, we have that $\delta(u, v) \leq \delta(F) + cu + cv - 2\gamma \leq \delta_k^{n-1} + h - l$. □

We are now ready to give the proof of Theorem 1.

3 Proof of Theorem 1

Let P be a d-dimensional $(0,k)$-polytope in $\mathbb{R}^n$, for $n, d, k \in \mathbb{N}$ with $n \geq d \geq 1$, $k \geq 2$. By Lemma 4, there exists a full-dimensional $(0,k)$-polytope in $\mathbb{R}^d$ with the same 1-skeleton as P. Hence, from now on, we assume that P is full-dimensional, i.e. $n = d$.

The proof is by induction on d. The base cases are $d = 0$ and $d = 1$. The diameter of a 0-dimensional polytope is clearly zero, and the diameter of a 1-dimensional polytope is at most one, thus also bounded by $\lfloor \frac{2k-1}{2} \rfloor = k - 1$ since $k \geq 2$.

We now assume $d \geq 2$. Let u, v be vertices of P. By the induction hypothesis we assume that Theorem 1 is true for $(0,k)$-polytopes of dimension at most $d-1$. In particular, $\delta_k^{d-1} \leq \lfloor \frac{2k-1}{2}(d-1) \rfloor$, and $\delta_k^{d-2} \leq \lfloor \frac{2k-1}{2}(d-2) \rfloor$. Thus, in order to prove the inductive step, we just need to show one of the following two inequalities:

$$\delta(u, v) \leq \delta_k^{d-1} + k - 1, \tag{1}$$

$$\delta(u, v) \leq \delta_k^{d-2} + 2k - 1. \tag{2}$$

If $u + v \neq k^d$, there exists an index $i \in \{1, \dots, d\}$ such that $u_i + v_i \leq k - 1$ or $u_i + v_i \geq k + 1$ and, by Lemma 3 applied with $c = e^i$ or $c = -e^i$, respectively, $\delta(u, v) \leq \delta(F) + k - 1$, where F is the face of P that minimizes cx. As F is a $(0,k)$-polytope of dimension at most $d - 1$, we have $\delta(F) \leq \delta_k^{d-1}$, therefore $\delta(u, v) \leq \delta_k^{d-1} + k - 1$, i.e. (1) is satisfied. Therefore we now assume that $u + v = k^d$.

If there exists a facet G of the hypercube $[0, k]^d$ with $P \cap G = \emptyset$, then let $i \in \{1, \dots, d\}$ be such that $l \leq x_i \leq h$, with $l > 0$ or $h < k$. By Lemma 5, $\delta(u, v) \leq \delta_k^{d-1} + k - 1$, i.e. (1) is satisfied. Therefore we now assume that $P \cap G$ is nonempty for every facet G of the hypercube $[0, k]^d$.

If u has one component u_i, $i \in \{1, \dots, d\}$, with $1 \leq u_i \leq k - 1$, then we show that (2) is satisfied. Since the sets $P \cap \{x \in \mathbb{R}^d : x_i = 0\}$ and $P \cap \{x \in \mathbb{R}^d : x_i = k\}$ are nonempty, by Lemma 1 (with $c = \pm e^i$), u has a neighbor s with $s_i \leq u_i - 1$, and a neighbor t with $t_i \geq u_i + 1$. If $s_j = t_j = u_j$ for all $j \in \{1, \dots, d\}$, $j \neq i$, then by setting $\lambda = \frac{t_i - u_i}{t_i - s_i}$ we have $\lambda s + (1 - \lambda)t = u$, contradicting the fact that u is a vertex of P. Thus, there exists an index $j \in \{1, \dots, d\}$ with $j \neq i$ such that either $s_j \neq u_j$ or $t_j \neq u_j$. Therefore there exists a neighbor w of u such that for distinct indices $i, j \in \{1, \dots, d\}$, $w_i \neq u_i$ and $w_j \neq u_j$.

We assume wlog $w_i < u_i$ (if not, we can perform the change of variable $\tilde{x}_i = k - x_i$). Analogously, we assume wlog $w_j < u_j$. As $u + v = k^n$, we have $w_i + w_j + v_i + v_j \leq 2k - 2$. Let $\gamma = \min\{x_i + x_j : x \in P\}$ and $F = \{x \in P : x_i + x_j = \gamma\}$. By Lemma 3 (with $c = e^i + e^j$), $\delta(w, v) \leq \delta(F) + w_i + w_j + v_i + v_j - 2\gamma \leq \delta(F) + 2k - 2 - 2\gamma$.

We now show that $\delta(F) \leq \delta_k^{n-2} + \gamma$. Let $\bar{F}$ be the projection of F onto the j-coordinate hyperplane. $\bar{F}$ is a $(0, k)$-polytope in $\mathbb{R}^{n-1}$ and, by Lemma 4, $\bar{F}$ has the same 1-skeleton of F. Note that, for any $x \in F$, $x_i = \gamma - x_j$ and $x_j \geq 0$ imply $x_i \leq \gamma$. Therefore, $x_i \leq \gamma$ for any $x \in \bar{F}$. Then, by Lemma 5, $\delta(\bar{F}) \leq \delta_k^{n-2} + \gamma$, thus $\delta(F) \leq \delta_k^{n-2} + \gamma$.

This implies $\delta(w, v) \leq \delta_k^{n-2} + 2k - 2 - \gamma$ and, since $\gamma \geq 0$ and $\delta(u, w) = 1$, finally $\delta(u, v) \leq \delta(u, w) + \delta(w, v) \leq \delta_k^{n-2} + 2k - 1$, i.e. (2) is satisfied.

Thus we now assume that $u \in \{0, k\}^d$. By possibly performing the change of variable $\tilde{x}_1 = k - x_1$, we can further assume wlog $u_1 = k$, and $v_1 = 0$.

Let F be the face of P defined by $F = \{x \in P : x_1 = 0\}$. F is a $(0, k)$-polytope of dimension at most $n - 1$, thus $\delta(F) \leq \delta_k^{n-1}$. By Lemma 2 (with $c = e^1$), there exists a vertex u' of F such that $\delta(u, u') \leq k$. Observe that both u' and v lie in F and therefore $\delta(u', v) \leq \delta_k^{d-1}$.

If $u' = (0, u_2, \ldots, u_d)$, then u and u' are adjacent vertices of the hypercube $[0, k]^d$, implying that $\mathrm{conv}\{u, u'\}$ is an edge of $[0, k]^d$. As P is convex and it is contained in $[0, k]^d$, it follows that $\mathrm{conv}\{u, u'\}$ is also an edge of P. Therefore, $\delta(u, u') = 1$ and consequently $\delta(u, v) \leq \delta_k^{d-1} + 1$. As $k \geq 2$, it follows $\delta(u, v) \leq \delta_k^{d-1} + k - 1$, i.e. (1) is satisfied.

Thus we now assume $u' \neq (0, u_2, \ldots, u_d)$. Then, there exists an index $i \in \{2, \ldots, d\}$ such that $u'_i + v_i \leq k - 1$ or $u'_i + v_i \geq k + 1$. We assume wlog $u'_i + v_i \leq k - 1$ (if not, we can perform the change of variable $\tilde{x}_i = k - x_i$). Let $\gamma = \min\{x_i : x \in F\}$, $F' = \{x \in F : x_i = \gamma\}$. F' is a $(0, k)$-polytope, and it has dimension at most $d-2$ because it is contained in the intersection of two linearly independent hyperplanes in $\mathbb{R}^d$. It follows that $\delta(F') \leq \delta_k^{d-2}$. Then, by applying Lemma 3 to polytope F and vertices u' and v, we have $\delta(u', v) \leq \delta(F') + u'_i + v_i \leq \delta_k^{d-2} + k - 1$. This implies $\delta(u, v) \leq \delta(u, u') + \delta(u', v) \leq \delta_k^{d-2} + 2k - 1$, i.e. (2) is satisfied. $\square$

Remark 4. *The result of Theorem 1 can be further improved by refining the bound in low dimensional cases.*

For $d = 1$, Theorem 1 yields the bound $\delta_k^1 \leq k - 1$. As $\delta_k^1 = 1$, the upper bound of $\lfloor \frac{2k-1}{2} d \rfloor$ on δ_k^d can be improved by subtracting the term $(k - 1) - 1$. In other words, for $k \geq 2$, $n \geq 1$, the bound $\delta_k^d \leq \lfloor \frac{2k-1}{2} d \rfloor - k + 2$ holds.

For $d = 2$, Theorem 1 yields the bound $\delta_k^2 \leq 2k - 1$. It is known that any 2-dimensional $(0, k)$-polytope has at most $5.5 \cdot k^{2/3}$ vertices [1], which immediately implies $\delta_k^2 \leq 5.5 \cdot k^{2/3}$. Therefore, the bound of Theorem 1 can be further refined by subtracting the term $(2k - 1) - 5.5 \cdot k^{2/3}$, i.e. for $k, n \geq 2$, the bound $\delta_k^d \leq \lfloor \frac{2k-1}{2} d \rfloor - (2k - 1) + 5.5 \cdot k^{2/3}$ holds. Note that this bound, for large k dominates the bound of $\lfloor \frac{2k-1}{2} d \rfloor - k + 2$.

4 Tightness

This section is devoted to proving Theorem 2. We first introduce some well-known results concerning the cartesian product of polytopes. For the sake of completeness, we next provide short proofs of such results. Let $P^1 \subseteq \mathbb{R}^n$ and $P^2 \subseteq \mathbb{R}^m$ be two polytopes of dimension d^1 and d^2, respectively. We define the *cartesian product* of P^1 and P^2 as $P^1 \times P^2 = \{(x,y) \in \mathbb{R}^{n+m} : x \in P^1, y \in P^2\}$. Observe that $P^1 \times P^2$ has dimension equal to $d^1 + d^2$. We say that an inequality is *valid* for P if it is satisfied by all points in P.

Lemma 6. *Let P^1 and P^2 be two polytopes, and let $P = P^1 \times P^2$. Then F is a nonempty face of P if and only if $F = F^1 \times F^2$, where F^1 and F^2 are nonempty faces of P^1 and P^2, respectively.*

Proof. Let $F^1 = \{x \in P^1 : a^1x = \beta^1\}$ and $F^2 = \{y \in P^2 : a^2y = \beta^2\}$, where $a^1x \leq \beta^1$ and $a^2y \leq \beta^2$ are valid for P^1 and P^2, respectively. Then $a^1x + a^2y \leq \beta^1 + \beta^2$ is valid for P and $F = \{(x,y) \in P : a^1x + a^2y = \beta^1 + \beta^2\}$ is a nonempty face of P. It can be verified that, in fact, $F = F^1 \times F^2$.

Conversely, suppose that $F = \{(x,y) \in P^1 \times P^2 : a^1x + a^2y = \beta\}$ is a nonempty face of P, where $a^1x + a^2y \leq \beta$ is valid for P, and define $\beta^1 = \max\{a^1x : x \in P^1\}$ and $\beta^2 = \max\{a^2y : y \in P^2\}$. Then $a^1x \leq \beta^1$ and $a^2y \leq \beta^2$ are valid for P^1 and P^2, respectively, and $F^1 = \{x \in P^1 : a^1x = \beta^1\}$, $F^2 = \{y \in P^2 : a^2y = \beta^2\}$ are the associated nonempty faces of P^1 and P^2. It can be verified that, in fact, $\beta = \beta^1 + \beta^2$ and that this implies $F = F^1 \times F^2$. $\square$

Lemma 7. *Let P^1 and P^2 be two polytopes and let $P = P^1 \times P^2$. Then $\delta(P) = \delta(P^1) + \delta(P^2)$.*

Proof. We first show that, for any two vertices $w = (w^1, w^2)$ and $z = (z^1, z^2)$ of P, it holds $\delta^P(w,z) = \delta^{P^1}(w^1, z^1) + \delta^{P^2}(w^2, z^2)$. We just need to show that $u = (u^1, u^2)$ and $v = (v^1, v^2)$ are adjacent vertices in P if and only if either u^1 and v^1 are adjacent vertices in P^1 and $u^2 = v^2$ is a vertex of P^2, or u^2 and v^2 are adjacent vertices in P^2 and $u^1 = v^1$ is a vertex of P^1. First, Lemma 6 implies that u and v are vertices of P if and only if u^1, v^1 are vertices of P^1 and u^2, v^2 are vertices of P^2. Recall that u and v are neighbors on P if and only if they lie on a common 1-dimensional face of P. Then, by Lemma 6, $\text{conv}\{u, v\}$ is an edge of P if and only if either $u^1 = v^1$ and $\text{conv}\{u^2, v^2\}$ is an edge of P^2, or $u^2 = v^2$ and $\text{conv}\{u^1, v^1\}$ is an edge of P^1.

Let $w = (w^1, w^2)$ and $z = (z^1, z^2)$ be vertices of P such that $\delta^P(w,z) = \delta(P)$. We then have $\delta^P(w,z) = \delta^{P^1}(w^1, z^1) + \delta^{P^2}(w^2, z^2)$, thus $\delta(P) \leq \delta(P^1) + \delta(P^2)$. Suppose by contradiction that $\delta(P) < \delta(P^1) + \delta(P^2)$ and let $\bar{w}^i, \bar{z}^i$ be vertices of P^i such that $\delta(P^i) = \delta^{P^i}(\bar{w}^i, \bar{z}^i)$, $i \in \{1, 2\}$. Then, $\bar{w} = (\bar{w}^1, \bar{w}^2)$ and $\bar{z} = (\bar{z}^1, \bar{z}^2)$ are vertices of P such that $\delta^P(\bar{w}, \bar{z}) = \delta^{P^1}(\bar{w}^1, \bar{z}^1) + \delta^{P^2}(\bar{w}^2, \bar{z}^2) = \delta(P^1) + \delta(P^2) > \delta(P)$, a contradiction. $\square$

We are now ready to prove Theorem 2.

Proof of Theorem 2. Theorem 1 (with $k = 2$), immediately implies that the diameter of a d-dimensional $(0,2)$-polytope is at most $\left\lfloor \frac{3}{2}d \right\rfloor$. For any natural number d, we construct a d-dimensional $(0,2)$-polytope whose diameter equals $\left\lfloor \frac{3}{2}d \right\rfloor$.

Let $H^0 = \{0\}$, $H^1 = [0,2]$ and denote by H^2 the hexagon defined by

$$H^2 = \mathrm{conv}\{(0,0),(1,0),(0,1),(2,1),(1,2),(2,2)\}.$$

For $d \in \mathbb{N}$, $d \geq 3$ define

$$H^d = \begin{cases} \underbrace{H^2 \times \cdots \times H^2}_{\lfloor d/2 \rfloor \text{ times}} \times H^1 & d \text{ odd} \\ \underbrace{H^2 \times \cdots \times H^2}_{d/2 \text{ times}} & d \text{ even.} \end{cases}$$

Observe that, by construction, H^d is a d-dimensional $(0,2)$-polytope in $[0,2]^d$. We show that $\delta(H^d) = \left\lfloor \frac{3}{2}d \right\rfloor$ for any $d \in \mathbb{N}$. The cases $d = 0,1,2$ trivially verify the claim.

For odd $d \geq 3$, Lemma 7 directly implies $\delta(H^d) = \left\lfloor \frac{d}{2} \right\rfloor \cdot \delta(H^2) + \delta(H^1) = \left\lfloor \frac{d}{2} \right\rfloor \cdot 3 + 1 = \left\lfloor \frac{3}{2}d \right\rfloor$, where the last equality is implied by the fact that d is odd. For even $d \geq 4$, Lemma 7 directly implies $\delta(H^d) = \frac{d}{2} \cdot \delta(H^2) = \frac{d}{2} \cdot 3 = \left\lfloor \frac{3}{2}d \right\rfloor$, where the last equality is implied by the fact that d is even. $\square$

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