Finding small stabilizers for unstable graphs

Abstract. In an instance of the classical, cooperative *matching game* introduced by Shapley and Shubik [Int. J. Game Theory '71] we are given an undirected graph $G = (V, E)$, and we define the value $\nu(S)$ of each subset $S \subseteq V$ as the cardinality of a maximum matching in the subgraph $G[S]$ induced by S. The *core* of such a game contains all *fair* allocations of $\nu(V)$ among the players of V, and is well-known to be non-empty iff graph G is *stable*. G is stable if its inessential vertices (those that are exposed by at least one maximum matching) form a stable set. In this paper we study the following natural edge-deletion question: given a graph $G = (V, E)$, can we find a minimum-cardinality *stabilizer*? I.e., can we find a set F of edges whose removal from G yields a stable graph? We show that this problem is vertex-cover hard. We then prove that there is a minimum-cardinality stabilizer that avoids some maximum-matching of G. We employ this insight to give efficient approximation algorithms for sparse graphs, and for regular graphs.

1 Introduction

Given an undirected graph $G = (V, E)$, a subset of edges $M \subseteq E$ is a *matching* if every node $v \in V$ is incident to at most one edge in M. Dually, a subset of vertices $U \subseteq V$ is called *vertex cover* if every edge has at least one endpoint in U. The corresponding optimization problems of finding a matching and vertex cover of largest and smallest size, respectively, have a rich history in the field of Combinatorial Optimization. Relaxing canonical integer programming formulations for these problems yields the following primal-dual pair of linear programs.

$$\nu_f(G) := \max\{\mathbf{1}^T x : x(\delta(v)) \leq 1 \; \forall v \in V, \; x \geq 0\} \tag{P}$$

where $\delta(v)$ denotes the set of edges incident to v, and

$$\tau_f(G) := \min\{\mathbf{1}^T y : y_u + y_v \geq 1 \; \forall uv \in E, \; y \geq 0\} \tag{D}$$

We will henceforth refer to feasible solutions of (P) and (D) as *fractional* matchings and vertex covers, respectively. An application of duality theory easily yields

$$\nu(G) \leq \nu_f(G) = \tau_f(G) \leq \tau(G)$$

where $\nu(G)$ and $\tau(G)$ denote the size of maximum matching and minimum vertex cover respectively.

In this paper, we study *stable* graphs – undirected graphs G for which $\nu(G) = \tau_f(G)$. The class of such graphs properly subsumes the well-studied class of *König-Egerváry* (KEG) graphs (e.g., see [20, 14, 11, 15, 12]). Stable graphs arise quite naturally in the study of cooperative *matching games* introduced by Shapley and Shubik in their seminal paper [19]. An instance of this game is associated with an undirected graph $G = (V, E)$, and its *core* consists of all *stable* allocations of $\nu(G)$ among the vertices in V in which no coalition of vertices has an incentive to deviate; i.e.

$$\text{core(G)} := \{y \in \mathbb{R}^V_+ : \sum_{v \in S} y_v \geq \nu(G[S]) \\ \sum_{v \in V} y_v = \nu(G)\},$$

where $G[S]$ is the subgraph of G induced by vertex set S. It is well-known (e.g., see [5]) that core(G) is non-empty iff G is stable.

Matching games in turn are closely related to *network bargaining*, a natural, recent network generalization of Nash's famous bargaining solution [16] to networks due to Kleinberg and Tardos [9]. Here, we are given an undirected graph $G = (V, E)$ whose vertices correspond to players, and whose edges correspond to potential, unit-value deals between the incident players. Each player is allowed to engage in at most one deal with one of its neighbours. Hence, a permissible outcome is naturally associated with a matching M among the vertices of G, as well as an allocation $y \in \mathbb{R}^V_+$ of $|M|$ among M's endpoints. Kleinberg and Tardos define an allocation to be *stable* if $y_u + y_v \geq 1$ for all $uv \in E$. The authors further define *outside option* α_u as

$$\alpha_u := \max\{1 - y_v \, : \, uv \in \delta(u) \setminus M\},$$

for each vertex $u \in V$, and say that an outcome (M, y) is *balanced* if for every edge $uv \in M$, the surplus $1 - \alpha_u - \alpha_v$ is split evenly among u and v. The main result in [9] is that an instance of network bargaining has a stable outcome iff it has a balanced one. One now realizes (see also [3]) that a stable outcome exists iff the core of the underlying matching game instance is non-empty and hence iff G is stable.

In this paper, we focus on *unstable* instances of the matching game, where the core is empty. Our motivating goal is to establish strategies for *stabilizing* such instances in the *least intrusive way*. Specifically, we would like to alter the input graph in as few places as possible, and we would like to maintain the value of the grand coalition $\nu(V)$ in the process. We accomplish this by addressing the following natural *edge-deletion* problem: find the smallest set $F \subseteq E$ of edges such that the subgraph $G[E \setminus F]$ induced by the edges not in F is stable.

The problem of removing vertices or edges from a graph in order to attain a certain graph property is natural, and hence it is not surprising that there is a vast amount of literature in Combinatorial Optimization that deals with its various facets. Much of the work on deletion problems addresses *monotone* graph properties (e.g., see [22, 1]) that are invariant under edge- or vertex removal. Crucially, graph stability is not a monotone property as one easily verifies: the triangle is not stable, and adding a single pendant edge to one of its nodes yields a stable graph.

1.1 Our results

Our first result (Section 2) shows a structural property of all minimum cardinality stabilizers. We show that for a graph G and a minimum stabilizer F, $\nu(G \setminus F) = \nu(G)$. Hence, one can stabilize an unstable instance of the matching game *without* affecting the value of the grand coalition! Our result is algorithmic: given any stabilizer F, we can efficiently find a maximum matching M in G and a stabilizer F' such that $|F'| \leq |F|$ and $M \cap F' = \emptyset$. The last equality implies that M is still a maximum matching in $G \setminus F'$.

The structural property mentioned before implies that there is always a maximum matching M in G that is still a maximum matching in $G \setminus F$. In Section 3, we therefore investigate the seemingly easier *M-stabilizer* question where we want to find a minimum stabilizer of G that is disjoint from a given maximum matching M. In network bargaining terms, this corresponds to the problem of stabilizing a *given* set of deals between players. We show that finding a minimum M-stabilizer is NP-hard by reduction from vertex cover, and that no efficient algorithm achieving $(2-\varepsilon)$-approximation is possible assuming the Unique Games Conjecture, for any $\varepsilon > 0$. We match this inapproximability result by developing an LP based 2-approximation algorithm for computing a minimum M-stabilizer.

We then focus in Section 4 on the algorithmic question of finding a minimum stabilizer for a graph G. We extend the inapproximability result mentioned before for the M-stabilizer to hold for this case. From an approximation point of view, we develop a $O(\omega)$-approximation algorithm where ω is the sparsity of the graph. We do not know whether a constant factor approximation algorithm can be developed for arbitrary graphs. However, we give a 2-approximation algorithm for regular graphs. In network bargaining model, this corresponds to the case where every player has the same number of potential deals to make. The analysis of both our algorithms uses famous classical results about matchings (such as the structure of basic solutions of (P) and (D) and the Edmonds-Gallai decomposition).

1.2 Related work

Our work is closely related to that of Mishra et al. [15] on vertex- and edge-removal problems for the König-Egerváry graph property. Just like stability, KEG is not a monotone property. Mishra et al. showed that it is NP-hard to approximate the corresponding edge-deletion problem to within 2.88. Assuming the Unique Games Conjecture [7], no constant-factor approximation may exist for the problem. We note that the reductions used in [15] will likely not be helpful for proving hardness for the above stabilizer problem as the graphs constructed are stable. On the positive side, the authors show that, for a given graph $G = (V, E)$ one can efficiently find a KEG (and hence stable) subgraph with at least $3|E|/5$.

The recent paper by Könemann et al. [10] addressed the related, NP-hard problem of finding a minimum-size *blocking set* in an input graph $G = (V, E)$. Here one wants to find a set of edges $F \subseteq E$ such that $G[E \setminus F]$ has a fractional vertex cover of size no larger than $\nu(G)$. Note that the resulting graph might not be stable as $\nu(G[E \setminus F])$ may well be smaller then $\nu(G)$. Therefore, the size of a minimum blocking set is incomparable to that of a minimum stabilizer.

1.3 Preliminaries

Given an undirected graph G and a matching M in G, a path is called M-*alternating* if it alternates edges from M and those from $E \setminus M$. An odd cycle of length $2k+1$ in which exactly k edges are in M is called an M-*blossom*. An M-*flower* is an even M-alternating path from an exposed vertex to a vertex u such that there exists a blossom through u. For a subset of vertices $S \subseteq V$, we use $E(S)$ to denote the set of edges in the graph induced by S and $G[S]$ to denote the subgraph induced by S. A graph $G = (V, E)$ is called *factor-critical* if for all $v \in V$, $G[V \setminus \{v\}]$ has a perfect matching; i.e. a matching that does not expose any vertex. A vertex v is called *inessential* for G if there exists a maximum matching M that exposes v, and *essential* otherwise. In this paper, we will also use the following characterization of stable graphs.

Theorem 1 ([9]). *The following are equivalent: (i) G is stable, (ii) The set of inessential vertices of G form a stable set, (iii) G contains no M-flower for any maximum matching M. Moreover, if G is not stable, then G containsan M-flower for every maximum matching M.*

Given a graph G, the Edmonds-Gallai decomposition is a partition of its vertex set into three parts $B(G), C(G), D(G)$, where $B(G)$ is the set of inessential vertices, the set $C(G)$ consists of the neighbours of $B(G)$ and $D(G) = V \setminus (B(G) \cup C(G))$. We list several standard but useful propoerties. For a proof see, e.g., [18].

Theorem 2. *Given a graph G, the Edmonds-Gallai decomposition of the graph $B(G), C(G), D(G)$ can be computed in polynomial time. Further, we have the following properties.*

1. *Each component of $G[B(G)]$ is factor-critical.*
2. *Every maximum matching M in G exposes at most one vertex in each component K of $G[B(G)]$.*

The following propositions are immediate consequences of the Edmonds-Gallai decomposition theorem. We include a proof in the appendix.

Proposition 1. *Let U be a non-trivial factor-critical component in $G[B(G)]$. Then, $\nu(G - G[U]) < \nu(G)$.*

Proposition 2. *Let M be a maximum matching in G that also matches the maximum possible number of isolated vertices in $G[B(G)]$. Let k be the number of non-trivial factor-critical components with at least one vertex exposed by M. Then $k = 2(\nu_f(G) - \nu(G))$.*

We say that a graph $G = (V, E)$ is ω-sparse if $\forall S \subseteq V$ we have $|E(S)| \leq \omega |S|$. We say that a graph is *regular* if all nodes have the same degree.

2 Maximum matchings and minimum stabilizers

Theorem 3. *For every minimum stabilizer F, we have $\nu(G \setminus F) = \nu(G)$.*

Proof. Let F be a minimum stabilizer. Find a maximum matching M in G such that $|M \cap F|$ is minimum. Suppose $|M \cap F| \neq \emptyset$.

Consider $G' := G \setminus (F \setminus M)$, the graph obtained by removing all the edges of $F \setminus M$ from G. Clearly M is still a maximum matching in G'. However, since F is minimum, G' is not stable. By Theorem 1, this implies that there exists an M-flower in G' starting at an M-exposed node w.

Suppose the M-flower contains an edge $uv \in F$. Then, $uv \in M$, since all other edges from F have been removed in G'. Therefore, we can find an even M-alternating path P from w to either u or v. Switching along the edges of this path, we obtain another maximum matching $M' = M \Delta P$ in G with $|F \cap M'| < |F \cap M|$, a contradiction.

It follows that the M-flower does not contain any edge from F, and therefore the M-flower also exists in $G \setminus F$. However, since $G \setminus F$ is stable, this implies that $M \setminus F$ is not a maximum matching in $G \setminus F$. Apply Edmonds' maximum matching algorithm on the graph $G \setminus F$ initialized with the matching $M \setminus F$, and construct an $M \setminus F$-alternating tree starting with the exposed node w. There are two possibilities: either we find an augmenting path P or a frustrated tree rooted at w. In the first case, the path P starts with w and ends with a $M \setminus F$-exposed node, say w'. However, such a path cannot exist in G because M is a maximum matching, and therefore w' must have been incident to an edge $f \in M \cap F$. Also, note that the path P is in $G \setminus F$. Hence, $P + f$ is an even M-alternating path in G containing exactly one edge in $M \cap F$. Switching along the edges of this path, we obtain another maximum matching $M' = M \Delta P$ in G with $|F \cap M'| < |F \cap M|$, a contradiction.

The only remaining possibility is that we find a frustrated tree T rooted at w. Let $G[T] = (V_T, E_T)$ be the graph induced by all nodes in the frustrated tree T (after expanding pseudonodes). In this case, $M \cap E_T$ is a maximum matching in $G[T]$, and the M-flower is contained in E_T. However, if we continue Edmonds' algorithm, it would remove the vertices of the frustrated tree, and continue running in the resulting subgraph to find a maximum matching. Therefore it ends by computing a maximum matching M^* in $G \setminus F$ with $M^* \cap E_T = M \cap E_T$. Therefore, we have a M^*-flower in $G \setminus F$, again a contradiction. □

We remark here that the above proof is algorithmic, therefore given a stabilizer F in polynomial time we can find a maximum matching M in G and another stabilizer $F' \subseteq F$ with $|F'| \leq |F|$ such that $M \cap F' = \emptyset$. The first step of computing a maximum matching M in G with minimum intersection with F can be done by assigning a cost of one to the edges in F, zero to the rest of the edges, and computing a min-cost matching in G of cardinality $\nu(G)$.

We next prove a lower bound on the cardinality of a stabilizer.

Theorem 4. *For every minimum stabilizer F, we have $|F| \geq 2(\nu_f(G) - \nu(G))$.*

Proof. Let $B(G), C(G), D(G)$ denote the Edmonds-Gallai decomposition and let M be a maximum matching in G that also matches the maximum possible number of isolated vertices in $G[B(G)]$. Let $U_1, \ldots, U_k$ denote the components in $G[B(G)]$ with at least one vertex exposed by M. Let F be a minimum stabilizer and $H = G \setminus F$. For each component $U_1, \ldots, U_k$, at least one vertex $v_i \in U_i$ becomes essential in H. Suppose not, then all nodes of some U_i are inessential in H. This implies that F contains all edges in $G[U_i]$. Thus, by Proposition 1, we have that $\nu(H) < \nu(G)$, a contradiction to Theorem 3.

Pick a maximum matching N in H. Then, N will cover all these vertices $v_1, \ldots, v_k$. Since $G[U_i]$ is factor-critical and M matches all but one vertex in U_i using edges in $G[U_i]$, we may assume without loss of generality, that M misses all these vertices. The graph $M \Delta N$ is a disjoint union of even cycles and even paths since $|M| = |N| = \nu(G)$. Consider the k disjoint paths starting at the vertices $v_1, \ldots, v_k$ in $M \Delta N$. We observe that at least one of the M edges in each of these paths should belong to F, otherwise we can obtain a maximum matching in H that exposes the starting vertex v_i, thus contradicting $v_i \notin B(H)$. Hence $|F| \geq k$. The result follows by Proposition 2.

□

3 Minimum M-stabilizers

Theorem 3 shows that for any minimum stabilizer F, there exists a maximum matching M in G that is disjoint from F. This observation gives rise to the following natural question: For a chosen maximum matching M, what is the cardinality of a minimum stabilizer F_M that is disjoint from M?

In the network bargaining setting, this question asks how to stabilize a given set of deals by forbidding a minimum number of other deals. In other words, can we ensure a stable outcome considering the given set of deals with the least modification to the underlying network?

In this section, we address the *M-stabilizer problem*: given a maximum matching M of G, find a minimum cardinality edge set $F_M \subseteq E$ with the property that F_M is a stabilizer for G, and in addition $F_M \cap M = \emptyset$. Biró et. al. [4] considered the weighted *M-stabilizer problem*, where edge weights are given and the task is to find a minimum weight set of edges disjoint from M whose removal yields a stable graph. They state that it is NP-hard even in the unit weight case but omit the proof.

The next two theorems characterize the hardness and the approximability of the M-stabilizer problem. Theorem 5 complements the results of Biró et. al. [4] by providing a stronger inapproximability result for the M-stabilizer problem. It is based on an approximation preserving reduction from the Vertex Cover Problem. [1]

Theorem 5. *The M-stabilizer problem is NP-hard. Furthermore, assuming the Unique Game Conjecture is true, it is NP-hard to compute a $(2-\varepsilon)$-approximate solution for any $\varepsilon > 0$.*

[1] We refer the reader to the appendix for all missing proofs.

We match this inapproximability result by presenting a 2-approximation algorithm. The main idea is to show that a suitable linear programming relaxation for the problem has a half-integral optimal solution.

Theorem 6. *There is a 2-approximation algorithm for the M-stabilizer problem.*

We remark that this result generalizes easily to the weighted case as well.

Corollary 1. *There is a 2-approximation algorithm for the weighted M-stabilizer problem.*

4 Finding minimum stabilizers

In this section, we return to the problem of finding minimum stabilizers. Our first result is a hardness result obtained by extending the proof of Theorem 5.

Theorem 7. *The stabilizer problem is NP-hard. Furthermore, assuming Unique Game Conjecture, it is NP-hard to compute a $(2-\varepsilon)$-approximate solution for any $\varepsilon > 0$.*

We now focus on the main algorithmic results of this paper, namely on approximation algorithms for the minimum stabilizer problem.

The results in the previous section suggest that if we know the maximum matching of G that "survives" in the stable subgraph, then we can find a 2-approximate minimum stabilizer for G. However, as shown by Example **??** in the appendix, not every maximum matching survives. In fact, the example shows a planar factor-critical graph, where, for two different maximum matchings M and M', the cardinality of F_M and $F_{M'}$ differ by a factor of $\Omega(|V|)$.

In Section 4.1, we design an approximation algorithm whose approximation factor depends on the *sparsity* of the graph. In Section 4.2, we design a 2-approximate algorithm for regular graphs.

4.1 An $O(\omega)$-approximation algorithm for sparse graphs

Theorem 8. *There is a $O(\omega)$-approximation algorithm for the minimum stabilizer problem that runs in time polynomial in the number of vertices.*

Before proving Theorem 8, we state and prove the following lemma that is the main ingredient for our algorithm.

Lemma 1. *Let G be a graph with $\nu_f(G) > \nu(G)$. There exists an algorithm to find a set of edges L with $|L| = O(\omega)$, such that*

(i) $\nu(G \setminus L) = \nu(G)$,
(ii) $\nu_f(G \setminus L) \leq \nu_f(G) - \frac{1}{2}$.

Moreover, the algorithm runs in time that is polynomial in the number of vertices.

In other words, Lemma 1 shows that we can find a small subset of edges to remove from G without decreasing the size of the maximum matching but reducing the size of the minimum fractional vertex cover. The proof of Lemma 1 will use the following two classical results on the structure of fractional and integral matchings.

Theorem 9. *[**?**] Every basic feasible solution to (P) has components equal to* $0, 1$ *or* $\frac{1}{2}$*. Moreover, the edges with half integral components induce node disjoint cycles.*

Theorem 10. *[2, 21] Let* $\hat{x}$ *be a fractional maximum matching in a graph* G *having half integral fractional components for a minimum number of odd cycles* $C_1, \ldots, C_q$*. Let* $\bar{M} := \{e \in E : \hat{x}_e = 1\}$ *and* M_i *be a maximum matching in* C_i*. Then* $M = \bar{M} \cup M_1 \cup \ldots, \cup M_q$ *is a maximum matching in* G*. Moreover, such* $\hat{x}$ *and* M *can be found in time polynomial in the number of vertices.*

We are now ready to prove Lemma 1.

Proof (Proof of Lemma 1). Consider $\hat{x}$ and M as in Theorem 10 for the graph G. By duality theory, there exists a fractional vertex cover y with $\mathbf{1}^T y = \mathbf{1}^T \hat{x}$ satisfying complementary slackness conditions with $\hat{x}$. Moreover, we can always find such a vector y with half integral components [**?**].

We will algorithmically find a vertex u with the following properties:

(a) $y_u = \frac{1}{2}$,
(b) $L_u := \{uw : y_w = \frac{1}{2}\}$ satisfies $\nu(G \setminus L_u) = \nu(G)$ and $|L_u| = O(\omega)$.

First, let us argue that $(a) + (b)$ implies the result. Assume we can find such a vertex u. Consider the vector y' defined as $y'_v = y_v$ for all $v \neq u$ and $y'_u = 0$ otherwise. Then y' is a fractional vertex cover for $G \setminus L_u$ (vertex u cannot be adjacent to nodes with y-value zero because y is a fractional vertex cover for G).

Now let us prove that a vertex u satisfying $(a) + (b)$ can be found efficiently. Consider an arbitrary cycle in $\hat{x}$, e.g. C_1. Since $\hat{x}_e > 0$ for every edge $e = uv$ in C_1, it follows that the vertex cover constraint is tight (i.e., $y_u + y_v = 1$ holds) for all edges in C_1, and therefore $y_v = \frac{1}{2}$ for all vertices in C_1.

Set $H := C_1$, and mark all nodes in C_1. Note that C_1 is an odd cycle, therefore if we isolate any marked node in H we do not decrease the size of a maximum integral matching in the resulting graph. Repeat the following process, which will maintain the following invariants for the graph H: (i) Every node in H has y-value $\frac{1}{2}$, and (ii) Removing any subset of edges incident to one marked node of H does not decrease the size of a maximum matching.

1. If there is a marked vertex in H with $|L_u| < 4\omega$, then u satisfies properties (a) and (b). STOP.
2. Otherwise, consider an arbitrary marked node in H that is adjacent to a node $w \notin H$ with $y_w = \frac{1}{2}$. Such a w must be matched in M, otherwise we do have an M-augmenting path, contradicting the maximality of M in G.

3. Let z be the node matched to w by M. By complementary slackness, $y_z = \frac{1}{2}$. Add w and z to H and mark z. Go to 1.

It only remains to verify that we can always perform Step 2, i.e. if all marked nodes u have $|L_u| > 4\omega$, then there exists a marked node in H that is adjacent to a node $w \notin H$ with $y_w = \frac{1}{2}$. Suppose not. Consider the subgraph $G[H]$ induced by the nodes in H. This subgraph has the property that the degree of every marked node u in $G[H]$ is $|L_u| > 4\omega$. However, the number of marked nodes is more than half the total number of nodes in $G[H]$. This contradicts the ω-sparsity of $V(H)$ in G.

The running time of the above process being polynomial is straightforward. □

With this Lemma at hand, we are now ready to prove our main theorem.

Proof (Proof of Theorem 8).

Let G be an unstable graph. We use the following algorithm:

INITIALIZE $G' = G$.
FOR $i = 1, \ldots, 2(\nu_f(G) - \nu(G))$:

(i) Let L be the set of edges returned by the algorithm in Lemma 1 when its input is the current graph G'.
(ii) Set $G' \leftarrow G' \setminus L$.
(iii) If G' is stable, STOP.

We will now prove that (i) whenever the above algorithm stops, the current graph G' is stable, and (ii) the total number of edges removed during the complete execution of the algorithm is $O(\omega)|F^*|$, where F^* is a minimum stabilizer. Clearly (a) + (b) implies the result.

First, let us argue about stability. If the algorithm stops in step (iii) for some iteration $i < 2(\nu_f(G) - \nu(G))$, this is clear. So we may assume that the algorithm stops after performing all $2(\nu_f(G) - \nu(G))$ iterations. The graph G' output at this point has $\nu_f(G') \leq \nu_f(G) - \frac{1}{2}(2(\nu_f(G) - \nu(G))) = \nu(G) = \nu(G')$. This is because, by Lemma 1, in each iteration the size of a minimum fractional vertex cover decreases by at least $\frac{1}{2}$ while the size of the maximum matching is maintained. Hence, by definition of stability, G' is stable.

In each iteration we remove $O(\omega)$ edges and the total number of iterations is at most $2(\nu_f(G) - \nu(G))$. The bound on the approximation factor follows from Theorem 4. The running time bound also follows since the number of applications of the algorithm in Lemma 1 is at most $2(\nu_f(G) - \nu(G)) \leq |F^*| \leq |E|$ times. □

4.2 A 2-approximation algorithm for regular graphs

In this section, we give a 2-approximation algorithm for solving the minimum stabilizer problem in regular graphs. We begin by describing an algorithm to stabilize arbitrary graphs, whose approximation factor guarantee for arbitrary graphs is of the order of the maximum degree among all nodes of G (and in fact, the analysis is tight). We then prove the approximation factor of the algorithm reduces to 2 for regular graphs.

Our algorithm is based on Edmonds-Gallai decomposition. Recall the Edmonds-Gallai decomposition $B(G), C(G), D(G)$. The following proposition is a consequence of the structural property shown in Theorem 3.

Proposition 3. *For every minimum stabilizer F, we have $B(G \setminus F) \subseteq B(G)$.*

Proof. Suppose we have a vertex $u \in C(G) \cup D(G)$ such that $u \in B(G \setminus F)$. Let M_u be a maximum matching in $G \setminus F$ that misses u. Since $|M_u| = \nu(G)$ by Theorem 3, we have that u is inessential in G, contradicting $u \in C(G) \cup D(G)$. □

Consider the following algorithm.

Algorithm 1.

1. Construct the Edmonds-Gallai decomposition of G.
   Let $B := B(G)$ denote the inessential vertices. Let S denote the isolated vertices (or *singletons*) in $G[B]$.
2. Find a maximum matching M in G that also matches the maximum possible number of vertices in S.
3. Pick a vertex u for each non-singleton factor-critical component in $G[B]$ with at least one vertex exposed by M. Add all edges adjacent to u in G to the set F.

Lemma 2. *The set F output by Algorithm 1 is a stabilizer. Moreover, the algorithm can be implemented in polynomial time.*

Proof. Suppose there exists an edge $uv \in G \setminus F$ such that $u, v \in B(G \setminus F)$. First observe that M is a maximum matching in $G \setminus F$. Since M is a maximum matching, it follows that M should match either u or v. Without loss of generality, suppose M matches u. Let M_u be a maximum matching in $G \setminus F$ that exposes u. Then there exists a path P in $M_u \Delta M$ starting at u. Say, $P = e_1, e_2, \ldots, e_t$, where $e_1 \in M$. If the path P is an odd length path, then P is an M_u-augmenting path, a contradiction to M_u being a maximum matching in $G \setminus F$. So, the path P is of even length. Suppose P ends at a vertex a. Then, a is exposed by M and hence $a \in B(G \setminus F) \subseteq B(G)$. This also implies that the path is of length at least two. Let $a_0 = u, a_1, a_2, \ldots, a_{t-1} = b, a_t = a$ be the vertices occurring in the path in that order.

Suppose a is a singleton vertex in $G[B(G)]$. Since M matches the maximum possible number of vertices in S, it follows that a_{t-2} is also a singleton vertex in $G[B(G)]$. Repeating this argument, we obtain that $a_0 = u$ is a singleton vertex in $G[B(G)]$, contradicting the existence of the edge $uv \in G[B(G)]$.

So, we may assume that a belongs to a factor-critical component K of $G[B(G)]$. Since a is exposed by M, by Theorem **??**, this is the only exposed vertex in the non-trivial factor-critical K. Hence, the algorithm must have isolated a. Therefore the edge $e_t = a_{t-1}a_t$ cannot exist in $G \setminus F$, a contradiction since $e_t \in M_u \subseteq G \setminus F$.

The running time of the algorithm is polynomial since Edmonds-Gallai decomposition can be computed in polynomial time and a maximum matching that maximizes the number of matched vertices in S can also be computed in polynomial time [17]. □

We now derive the approximation factor of Algorithm 1 for regular graphs.

Theorem 11. *There exists a polynomial time algorithm to find a* 2*-approximate stabilizer in regular graphs.*

Proof. We use Algorithm 1. By Lemma 2, the set F is a stabilizer and the algorithm runs in polynomial time. Consider a d-regular graph G, i.e. a regular graph where every node has degree d. Let k denote the number of non-singleton factor-critical components in $G[B(G)]$ with at least one vertex exposed by M. It is clear that the size of F output by the algorithm is exactly kd. The following claim shows that every stabilizer in G is of size at least $kd/2$ completing the proof.

Claim. Every stabilizer in G is of size at least $kd/2$.

Proof of the claim. Let S_u denote the vertices in S that are exposed by M. We first observe that the size $\nu(G)$ of the maximum matching in G is $(|V| - k - |S_u|)/2$. Consider the following primal and dual linear programs.

$$\begin{aligned} \min \sum_{e \in E} z_e & \qquad (\mathcal{P}) \\ y_u + y_v + z_{uv} \geq 1 \ \forall \ uv \in E & \\ \sum_{u \in V} y_u = \nu(G) & \\ y, z \geq 0 & \end{aligned}$$

$$\begin{aligned} \max \sum_{e \in E} \alpha_e - \gamma\nu(G) & \qquad (\mathcal{D}) \\ \alpha(\delta(u)) \leq \gamma \ \forall \ u \in V & \\ 0 \leq \alpha \leq 1 & \end{aligned}$$

By setting z to be the indicator vector of the minimum stabilizer, we can obtain y such that (y, z) is a feasible solution to the primal program. This is because, if z is the indicator vector of a stabilizer in G, then by definition there exists a fractional vertex cover y in $G \setminus \text{Support}(z)$ with size equal to $\nu(G \setminus \text{Support}(z))$. We also know by Theorem 3 that for every minimum stabilizer F, $\nu(G \setminus F) = \nu(G)$.

Thus, the primal program is a relaxation of the minimum stabilizer problem. Consequently, the objective value of any feasible solution to the dual program is a lower bound on the size of a minimum stabilizer. We will provide a dual feasible solution with objective value at least $kd/2$.

Consider the dual solution $(\gamma = d, \ \alpha_e = 1 \ \forall \ e \in E)$. Since the graph is d-regular we have that $\alpha(\delta(u)) = d$. Thus, all dual constraints are satisfied and hence, it is a dual feasible solution. The objective value is

$$\sum_{e \in E} \alpha_e - \gamma\nu(G) = \frac{d|V|}{2} - d\left(\frac{|V| - k - |S_u|}{2}\right) = d\left(\frac{k + |S_u|}{2}\right) \geq \frac{kd}{2}.$$

□

We conclude the paper with a remark about the linear program ($\mathcal{P}$). If we add the integrality constraints on the z variables, we obtain an integer program (IP) and it follows by our result that the integrality gap of the resulting IP is at most 2 for d-regular graphs. Koenemann et. al. [10] proved a $\Theta(n)$-bound on the integrality gap of the IP for general graphs. However, the resulting IP is *not* a formulation for our minimum stabilizer problem, since the integral optimum solution of the IP could be $\Omega(n)$ away from the size of a minimum stabilizer for arbitrary graphs (not necessarily regular). In order to obtain a formulation for our minimum stabilizer problem, we could introduce additional variables and impose the existence of a matching in $G \setminus \text{Support}(z)$ of size $\nu(G)$. However, we show a $\Omega(n)$ bound on the integrality gap of this formulation. Details can be found in the appendix.

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5 Appendix

5.1 Edmonds Gallai Decomposition

Proof (Proof of Proposition 1). Let $H' = G - G[U]$ and $N = M \cap H'$. Clearly, $|N| < |M| = \nu(G)$. We will show that N is a maximum matching in H'. Suppose not. Then there exists an N-augmenting path in H'. The path necessarily has to go through a vertex $u \in U$ since otherwise, we will have an M-augmenting path contradicting the maximality of M. Since N exposes all vertices in U, we may assume that the path starts at $u \in U$. Let the path be $u_0 = u, u_1, \ldots, u_{2t+1}$. Then, $u_1 \in C(G)$. By induction, we can show that every vertex in $C(G)$ that occurs in the path is at odd distance from u_0 in the path. Hence, the path contains vertices only from $B(G) \cup C(G)$.

If $u_{2t+1} \in C(G)$, then either u_{2t+1} is unmatched by M, contradicting that u_{2t+1} is an essential vertex or u_{2t+1} is matched by M to a vertex in U. The latter is impossible since $|M \cap \delta(U)| \leq 1$ by the property of Edmonds-Gallai decomposition and hence $M \cap \delta(U) = \emptyset$. If $u_{2t+1} \in B(G)$, then we have a component $U' \in G[B(G)]$ with an exposed vertex $u_{2t+1} \in U'$. Since the path is non-empty, there exists a matching edge $e = ab$ in the path, where $a \in C(G)$ and $b \in U'$. Thus, by the property of Edmonds-Gallai decomposition, U' cannot have any M-exposed vertex, a contradiction. □

Proof (Proof of Proposition 2). An odd cycle C in G is said to be separated by a maximum matching N if $N \cap \delta(C) = \emptyset$. We define the following quantities.

$$\begin{aligned}
\sigma(G, N) &:= \text{Maximum number of disjoint odd cycles separated by N} \\
\sigma(G) &:= \max_{\text{max matchings N in G}} \sigma(G, N) \\
\gamma(G, x) &:= \text{Number of odd cycles in the support of x} \\
\gamma(G) &:= \min_{\text{max fractional matchings x in G}} \gamma(G, x)
\end{aligned}$$

By a result of Balas [2], we know that $2(\nu_f(G) - \nu(G)) = \sigma(G) = \gamma(G)$. Further, it is straightforward that $\sigma(G) \geq \sigma(G, M) \geq k$. We will show that $\gamma(G) \leq k$ and the result follows. For this, we will derive a maximum fractional matching x with $\gamma(G, x) = k$. Consider x obtained from M as follows: for each factor-critical component $U_i \in G[B(G)]$ that has a vertex exposed by M, pick a fractional perfect matching in U_i with exactly one odd cycle in the support; set $x(e) = 1/2$ for edges in the odd cycle, and $x(e) = 1$ for matching edges; for every edge $e \notin \cup_{i=1}^{k} U_i$, set $x(e) = 1$ if $e \in M$ and 0 otherwise. It is clear that $\gamma(G, x) = k$. By Theorem 4 of [17], it follows that x is a maximum fractional matching in G. □

5.2 Proofs for the M-Stabilizer problem

Proof (of Theorem 5). We give a reduction from the vertex cover problem. Let $G = (V, E)$ be a vertex cover instance. The approach is to extend the graph by connecting every vertex of V by a two-edge-path to a super-source v_0. We observe that every edge in G induces a cycle of length 5 including v_0 in the new graph. We choose the maximum matching M such that each of the 5-cycle forms an M-flower with the exposed vertex v_0. Two of these M-flowers are edge-disjoint if and only if the two corresponding edges in G are vertex-disjoint. We show that in order to ensure that there are no M-flowers, the stabilizer should remove as many edges in the new graph as the number of vertices needed to cover all edges in G.

Formally, we construct the new graph $G' = (V', E')$ as follows:

$$V' := \{v_0\} \cup \{v', v'' \colon v \in V\} \quad \text{and}$$

$$E' = \{v_0 v', v'' v' \colon v \in V\} \cup \{u'' v'' \colon uv \in E\}.$$

We set the matching $M := \{v'' v' \colon v \in V\}$. We observe that G' is factor-critical and M is a maximum matching exposing v_0. See figure 1 for an example.

(a) Vertex cover instance (b) M-stabilizer instance

Fig. 1: The M-stabilizer instance constructed from a vertex cover instance

Every edge $uv \in E$ in G corresponds to an M-flower

$$\{v_0 v', v' v'', u'' v'', u' u'', v_0 u'\}$$

in G'. We further observe that by the choice of M, all M-flowers in G' are necessarily of this kind.

We will show that the size of the minimum vertex cover in G is equal to the size of the minimum M-stabilizer in G'. This implies that the reduction is approximation preserving and the inapproximability results for the vertex cover problem [6, 8] carry over to the problem of finding a minimum M-stabilizer.

It remains to show that the sizes are equal. We will show this in two steps. First, we show that a minimum vertex cover in G can be used to obtain a M-stabilizer in G' of the same cardinality. Second, we show that there exists a minimum M-stabilizer in G' that only consists of edges of the type $v_0 v'$ for some node $v \in V$. The second part allows us to construct a vertex cover from such a minimum M-stabilizer F as follows: For every edge $v_0 v'$ in the M-stabilizer F, we take the corresponding node v into the vertex cover. By construction, the resulting set of vertices is a vertex cover, since an uncovered edge in G would induce an M-flower in $G' \setminus F$ thus contradicting the stability of $G' \setminus F$

(Theorem 1). We further note that the cardinality of the resulting vertex cover is the same as that of the minimum M-stabilizer. Hence, the size of the minimum M-stabilizer in G' is equal to the size of the minimum vertex cover in G. We now prove the two claims formally.

Claim. Let W be a minimum vertex cover in G. Then the set $F' = \{v_0v' : v \in W\}$ is a M-stabilizer in G'.

Proof. We observe that $G \setminus F'$ has no M-flowers since any M-flower corresponding to an edge $uv \in E$ in G contains at least one edge from F'. This is because W is a vertex cover. Thus, $G \setminus F'$ is stable by Theorem 1 and hence F' is a stabilizer. □

Claim. There exists a minimum M-stabilizer in G' that only consists of edges of the type v_0v' for some node $v \in V$

Proof. Suppose that F' is an M-stabilizer that contains an edge $e \in E'$ that is not of the type v_0v' for some node $v \in V$. We observe that $e = u''v''$ for some edge $uv \in E$. Then we can replace e by v_0u' in F', since e only intersects with the M-flower corresponding to the edge $uv \in E$. □

This concludes the proof of Theorem 5. □

Proof (of Theorem 6). We obtain a 2-approximation algorithm for finding a minimum M-stabilizer for a given maximum matching in graph by showing that a suitable linear program has a half-integral optimum solution. It is first proven that the formulation has an integral optimum solution for bipartite graphs. Then we construct a suitable new bipartite graph G' from our original instance G that allows us to derive a half-integral optimum solution for G. A final rounding step yields an M-stabilizer for G that is at most twice as large as the minimum M-stabilizer.

Let $V(M) \subseteq V$ denote the set of vertices that are incident to an edge in the given matching M. We introduce a variable x_v for every vertex $v \in V(M)$. We consider the following covering linear program:

$$\begin{aligned} \min \quad & \sum_{e \in E \setminus M} z_e && (\bar{P}) \\ \text{s.t.} \quad & x_u + x_v = 1 \quad \forall uv \in M \\ & x_u + x_v + z_e \geq 1 \quad \forall e = uv \in E \text{ and } u, v \in V(M) \\ & x_v + z_e \geq 1 \quad \forall e = uv \in E \text{ and } v \in V(M), u \notin V(M) \\ & x, z \geq 0 \end{aligned}$$

The first set of constraints enforces that $|M| = \sum_{v \in V(M)} x_v$. The following two sets of constraints imply that every edge not in M is covered by the corresponding z-variable or the x-variables of its end points. We note that at least one of the end points of any edge is in $V(M)$, since M is a maximal matching (even maximum). We observe that if a feasible solution (x, z) of $(\bar{P})$ satisfies $z \in \{0,1\}^{E \setminus M}$, then $F := \{e \in E : z_e = 1\}$ is an M-stabilizer. This is because we have a fractional vertex cover x in $G \setminus F$ of the same size as the maximum matching in $G \setminus F$.

Claim. For a bipartite graph $G = (V, E)$ and a maximal matching M in G, the linear program $(\bar{P})$ has an integral optimum solution (x^*, z^*).

Proof. Let A denote the coefficient matrix of the constraints in $(\bar{P})$ for G. Then the matrix A has the form

$$A = \begin{bmatrix} A' \; I \end{bmatrix}$$

where A' is a $|E| \times |V(M)|$ submatrix of the edge-vertex incidence matrix A_G of G and I is a $|E| \times |E \setminus M|$ submatrix of the $|E| \times |E|$ identity matrix $I_{|E|}$. Now, we observe that the matrix $\begin{bmatrix} A_G \; I_{|E|} \end{bmatrix}$ is totally unimodular since G is bipartite and thus A_G is totally unimodular. Since A is a (column-indexed) submatrix of $\begin{bmatrix} A_G \; I_{|E|} \end{bmatrix}$, we conclude that A is totally unimodular as well. This implies that there is an integral optimum solution (x^*, z^*) of $(\bar{P})$. □

Observe that the claim implies that we can find a minimum M'-stabilizer in a bipartite graph G' by solving the linear program $(\bar{P})$. We will use this to find an M-stabilizer in G that is at most twice as large as the optimum by constructing a bipartite graph as follows. Let $G' = (V', E')$ denote the new bipartite graph with $V' := V_1 \cup V_2$ for $V_i := \{v_i \colon v \in V\}$ and $E' := \{u_1v_2, u_2v_1 \colon uv \in E\}$.

We set $M' := \{u_1v_2, u_2v_1 \colon uv \in M\}$. We note that M' is a maximal matching in G', but may not necessarily be maximum (M-flowers in G correspond to M'-augmenting paths in G'). Let $(\bar{P}')$, $(\bar{P})$ denote the corresponding linear programs for G' and G, respectively.

We show first that the minimum M-stabilizer F in G induces an M'-stabilizer F' in G' of size $|F'| = 2|F|$. To see this, let x denote the fractional vertex cover of $G \setminus F$ with size $\sum_{v \in V(M)} x_v = |M|$. Such a fractional vertex cover exists exists because $G \setminus F$ is stable. We set $y'_{u_i} := x_u$ for all $u \in V, i = 1, 2$, and $z'_{u_1v_2} = z'_{u_2v_1} = 1$ for all $uv \in F$. Now, (y', z') is a feasible solution of $(\bar{P}')$ of cost $2|F|$ and since z' is integral, $F' := \{u_1v_2, u_2v_1 \colon uv \in F\}$ is an M'-stabilizer.

Next, we show that the optimum integral solution of $(\bar{P}')$ allows us to find an half-integral solution of $(\bar{P})$. Let (x', z') be the optimal integral solution of $(\bar{P}')$. Then $x_u := (1/2)(x'_{u_1} + x'_{u_2})$ and $z_{uv} = \max\{z'_{u_1v_2}, z'_{u_2v_1}\}$ defines a feasible solution for $(\bar{P})$: For $uv \in M$, we get $x_u + x_v = (x'_{u_1} + x'_{v_2} + x'_{u_2} + x'_{v_1})/2 = 1$ and for $uv \in E \setminus M$ with $u, v \in V(M)$, we get

$$\begin{aligned} x_u + x_v &= \frac{x'_{u_1} + x'_{v_2} + x'_{u_2} + x'_{v_1}}{2} \geq \frac{1 - z'_{u_1v_2} + 1 - z'_{u_2v_1}}{2} \\ &= 1 - \frac{z'_{u_1v_2} + z'_{u_2v_1}}{2} \geq 1 - \max\{z'_{u_1v_2}, z'_{u_2v_1}\} = 1 - z_{u,v}. \end{aligned}$$

The case $uv \in E \setminus M$ with $u \in V(M)$ and $v \notin V(M)$ follows in an analoguous manner.

As the cost of (x', z') is at most $2|F|$, the cost of the solution (x, z) of $(\bar{P})$ that we constructed is also bounded by $2|F|$. However, z is integral and thus defines an M-stabilizer in G of size at most twice the size of the minimum M-stabilizer. □

M-stabilizers are far from minimum stabilizer

Proposition 4. *There exist a graph $G = (V, E)$ and two different maximum matchings M, M' in G such that the size of a minimum M-stabilizer and a minimum M'-stabilizer differ by a factor of $\Omega(|V|)$.*

Proof. Let G' denote the graph constructed from a complete graph $G = (V, E)$ with $|V| = n$ vertices by adding a supersource v_0 that is connected to every vertex of G by a path of length 2 (as in the proof of theorem 5). We observe that the maximum matching $M := \{v'v'' : v \in V\}$ only allows for an M-stabilizer of size $n - 1$. Now let M' denote the maximum matching obtained from M by replacing one edge $u'u''$ by v_0u' for some $u \in V$. Then the single edge $u'u''$ forms an M'-stabilizer. As a consequence the minimum stabilizer in G' is of size one. We refer to Figure 2 for an example constructed from a complete graph on 5 vertices. □

(a) M-stabilizer of size 4 (b) M'-Stabilizer of size 1

Fig. 2: Example for a large M-stabilizer and a small M'-stabilizer (dashed edges)

5.3 Hardness of the Stabilizer problem

Proof (of Theorem 7). As in the proof of Theorem 5, we give a reduction from the vertex cover problem. The construction is very similar, except that we introduce a gadget graph H instead of a two-edge-path in order to enforce that a minimum stabilizer selects edges incident to the super-source.

Let $G = (V, E)$ be a vertex cover instance. We construct a new graph G' as in the proof of theorem 5, but where the edges of the type vv'' are replaced by a gadget graph H that connects v' and v'' to a $K_{n,n}$, the complete bipartite graph on n vertices, where $n = |V|$ is the number of nodes in the vertex cover instance. See Figure 3a for an illustration of H for $n = 4$ and Figure 3b for the instance G' (the vertex cover instance G is a cycle of length four). In the rest of the proof, for every vertex $v \in V$, we will refer to H_v as the gadget graph H between the vertices v' and v'' in G'.

Claim. Let W be a vertex cover in G. Then, $F' = \{v_0v' : v \in W\}$ is a stabilizer in G' and moreover, $|F'| = |W|$.

(a) gadget graph H_v (b) Stabilizer instance

Fig. 3: Example for the reduction from vertex cover to the stabilizer problem

Proof. By Theorem 1, it is sufficient to show that F' is an M-stabilizer for any chosen maximum matching M in G'. We choose M to be the maximum matching that leaves v_0 exposed and has a perfect matching in each gadget graph H_v. Suppose for contradiction that there is an M-flower in $G' \setminus F'$. Since v_0 is the only exposed vertex, every M-flower has to contain it. By the choice of M and the construction of the gadget graph H, this M-flower has to contain an edge of the type $u''v''$ for an edge $uv \in E$ and thus also the edges v_0u' and v_0v'. This is a contradiction to W being a vertex cover. □

By the above claim, we can also conclude that a minimum stabilizer in G' is of size strictly smaller than n.

Claim. There exists a minimum stabilizer in G' that only consists of edges of the type v_0v' for some nodes $v \in V$.

Proof. Let F be a minimum stabilizer in G' that does not satisfy this property. To prove the claim, we will show that we can replace every edge in F of a different type by an edge of the type v_0v' for some node $v \in V$.

Theorem 3 yields a maximum matching M' in G' with $F \cap M' = \emptyset$. We observe that by construction of G', it is factor-critical and hence, every maximum matching in G' contains at most one edge of the type $u''v''$ for an edge $uv \in E$. Thus we are left with only two kinds of maximum matchings in G':

1. M' does not contain any edge of the type $u''v''$ for an edge $uv \in E$.
   If M' leaves v_0 exposed, we observe that every M'-flower in G' corresponds to an odd cycle of the form $v_0, v', \ldots, v'', u'', \ldots, u', v_0$ (with some path of length 3 through the gadgets H_u, H_v to connect v' to v'' and u'' to u', respectively). This implies that an edge $u''v'' \in F$ can be replaced by v_0u' without violating the stabilizing property. Similarly, an edge in F that belongs to a gadget H_v for some $v \in V$ can also be replaced by v_0v', since every flower containing this edge must also contain v_0v'.

   If $v_0u' \in M'$, then the exposed vertex is either u'' or adjacent to u' in H_u. Then there exists an even alternating path P from the exposed vertex to v_0 of length two or four. We change M' along P to a new maximum matching $N := M' \Delta P$. Now, if P contains an edge $f \in F$, we can exchange f with v_0u' in F, and the resulting set is an N-stabilizer.

2. M' contains an edge $u''v''$ and thus w.l.o.g. also v_0u'.

   We distinguish two cases:

   (a) The node v' is exposed.

   We observe that the edge v_0v' is then necessarily present in the minimum stabilizer F, since otherwise the number of M'-flowers (with edge-disjoint paths through H_u and H_v) is n and all these can be avoided in $G' \setminus F$ only by removing at least n edges.

   We further observe that any edge $p''w''$ in F can be replaced by v_0p' as in the previous case, since they only belong to M'-blossoms with base v_0 (reached via an even alternating path from v' through v'', u'' and u'). A similar argument applies to the edges of F within a gadget H_w, since every flower has to contain the edge v_0w' as well.

   Finally, we claim that a minimum stabilizer cannot contain edges from the gadgets H_v or H_u, because otherwise the stabilizer would have size at least n. Since there are n edge-disjoint paths through the gadget from v' to v'', a stabilizer must contain either at least n edges from the gadget to block all flowers using a path from v' to v'' or none of those edges. The same argument holds for the u'-u''-paths as well.

   (b) The exposed vertex is a node t adjacent to v'' in the gadget H_v.

   As in the previous case, a minimum stabilizer cannot contain edges from the gadgets H_v or H_u, since otherwise the stabilizer would have size at least n. Observe further that any edge $p''w''$ in F can be replaced by v_0p' as before, since they only belong to M'-blossoms with base v_0 (reached via an even alternating path from t through v'', u'' and u').

This proves our claim. □

We set $W := \{v \colon v_0v' \in F\}$ for the minimum stabilizer F with the above property. We now claim that W is a vertex cover in G. For contradiction, suppose that an edge uv was not covered by W. This implies that neither v_0v' nor v_0u' is in F. Then there exists a cycle $v_0, v', \ldots, v'', u'', \ldots, u', v_0$ (with some path of length three through the gadgets H_u, H_v) in $G' \setminus F$. We observe that the matching M defined in the beginning of the proof is a maximum matching in $G' \setminus F$ and thus this cycle forms an M-flower in $G' \setminus F$, thus contradicting that F is a stabilizer in G'. □

5.4 Integrality Gap

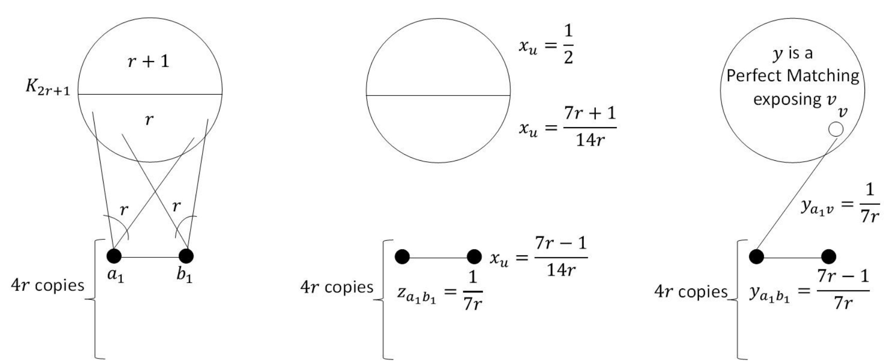
We show that the integrality gap of the following linear program on factor critical graphs is $\Omega(|V|)$.

$$
\begin{aligned}
\min \sum_{e \in E} z_e \\
x_u + x_v + z_{uv} \geq 1 \ \forall \ uv \in E \\
\sum_{u \in V} x_u = \nu(G) \\
y(\delta(v)) \leq 1 \ \forall \ v \in V \\
y(E[S]) \leq \frac{|S|-1}{2} \ \forall \ S \subseteq V, \ |S| \text{ odd} \\
\sum_{e \in E} y_e = \nu(G) \\
y_e + z_e \leq 1 \ \forall \ e \in E \\
x, y, z \geq 0
\end{aligned}
$$

The gap instance and a feasible solution is shown in figure (i) below. The instance consists of an odd clique on $2r+1$ vertices Q. Fix a subset $S \subseteq Q$ of r vertices. Next we add $4r$ new vertices $\{a_i, b_i : i \in [4r]\}$ and edges $\{a_i b_i : i \in [4r]\} \cup \{a_i u, b_i u : u \in S, i \in [4r]\}$. The number of vertices in the graph is $|V| = 10r + 1$.

Claim. The graph G is factor critical.

Proof. By a characterization of factor-critical graphs due to Lovász [13], it is sufficient to give an ear construction of the graph using only odd ears. Here is one possible ear construction of G: first construct K_{2r+1} using odd ears. Repeat for $i = 1, \ldots, 4r$: add an odd ear containing the i'th edge—namely, the odd ear $(ua_i, a_i b_i, b_i u)$ where $u \in S$. Next add edges $a_i v$ followed by $b_i v$ for all vertices $v \in S$. □



(i) Integrality gap instance G (ii) (x, z) values (iii) y values

Next we construct a feasible solution (x, y, z) to the LP for this instance. Fix a vertex $v \in S$ in the clique and let M be a perfect matching in the clique that exposes v.

$$y_e = \begin{cases} 1 & \text{if } e \in M \\ \frac{7r-1}{7r} & \text{if } e = a_i b_i \ i \in [4r] \\ \frac{1}{7r} & \text{if } e = a_i v,\ i \in [4r] \\ 0 & \text{otherwise,} \end{cases}$$

$$x_u = \begin{cases} \frac{1}{2} & \text{if } u \in Q \setminus S \\ \frac{7r+1}{14r} & \text{if } u \in S \\ \frac{7r-1}{14r} & \text{if } u = a_i \text{ or } b_i,\ i \in [4r] \\ 0 & \text{otherwise,} \end{cases}$$

$$z_e = \begin{cases} \frac{1}{7r} & \text{if } e = a_i b_i,\ i \in [4r] \\ 0 & \text{otherwise.} \end{cases}$$

Claim. The solution (x, y, z) is feasible and has objective value $4/7$.

Proof. We show feasibility of the solution by verifying that y satisfies all odd-set constraints. The rest of the constraints can be verified easily. In order to show that y satisfies all matching constraints, we will express it as a convex combination of $4r + 1$ integral matchings. Take $M_0 = M \cup \{a_i b_i : i \in [4r]\}$. Now, for each $i = 1, \ldots, 4r$, take $M_i = M \cup \{a_j b_j : j \in [4r], j \neq i\} \cup \{a_i v\}$. It is immediately seen that

$$y = \frac{3}{7}\chi_{M_0} + \sum_{i=1}^{4r} \frac{1}{7r}\chi_{M_i}$$

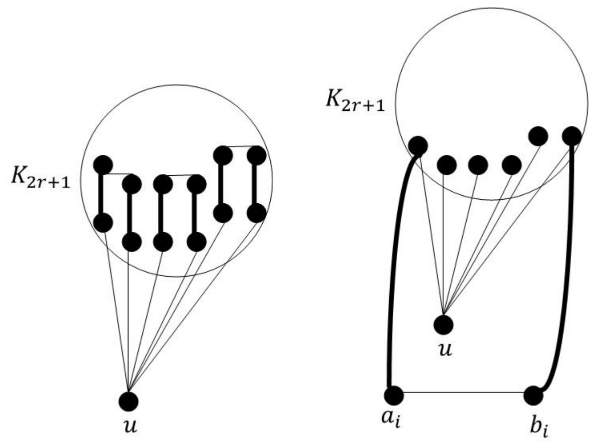
where χ_M denotes the indicator vector of M. The objective value of the linear program is easy to verify. ☐

In order to exhibit the integrality gap, it remains to show that the minimum stabilizer in this graph is of size $\Omega(r)$.

Claim. The minimum stabilizer for G is of size $\Omega(r)$.

Proof. We know that there exists a maximum matching M^* in G and a minimum stabilizer F that is disjoint from M^*. Let $H = G \setminus F$. Since the graph G is factor-critical, we know that the matching M^* exposes exactly one vertex u. We have two cases.

1. Suppose this vertex u is in the odd clique. Then, the number of edge-disjoint triangles through u containing exactly one edge from M^* is at least r. Since H is stable, H cannot contain any of these triangles. Hence $|F| \geq r$.
2. Suppose this vertex $u \in \{a_1, \ldots, a_{4r}, b_1, \ldots, b_{4r}\}$. Consider the neighbors S of u in G. If t among these neighbors are matched inside the odd clique by M^*, then we can pair up these matching edges and find $t/2$ disjoint 5-cycles



(a) Disjoint 5-cycles inside (b) Disjoint 5-cycles outside

through u in G each containing exactly two edges from M^* (See figure (a)). Since H is stable, H cannot contain any of these 5-cycles. Thus, the stabilizer has to remove at least $t/2$ edges.

(a) If $t \geq r/2$, then the stabilizer has to remove at least $t/2 \geq r/4$ edges.

(b) If $t < r/2$, then M^* matches $r - t \geq r/2$ vertices to vertices outside outside the clique. If one of the $r-t$ vertices is matched to some a_i, then the vertex b_i is either matched to a vertex inside the clique or $b_i = u$ (see figure (b)). Thus, we can once again identify $(r - t - 1)/2 \geq (r - 2)/4$ disjoint 5-cycles through u in G each containing exactly two edges from M^*. Since H is stable, H cannot contain any of these 5-cycles. Thus, the stabilizer has to remove at least $(r - 2)/4 \geq r/8$ edges.

Thus, we have a lower bound of $\Omega(r) = \Omega(|V|)$ on the size of the optimum stabilizer. □