COLORFUL LINEAR PROGRAMMING, NASH EQUILIBRIUM, AND PIVOTS

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Abstract. Let $\mathbf{S}_1, \ldots, \mathbf{S}_k$ be k sets of points in $\mathbb{Q}^d$. The *colorful linear programming problem*, defined by Bárány and Onn (*Mathematics of Operations Research*, **22** (1997), 550–567), aims at deciding whether there exists a $T \subseteq \bigcup_{i=1}^k \mathbf{S}_i$ such that $|T \cap \mathbf{S}_i| \leq 1$ for $i = 1, \ldots, k$ and $\mathbf{0} \in \mathrm{conv}(T)$. They proved in their paper that this problem is NP-complete when $k = d$. They leave as an open question the complexity status of the problem when $k = d + 1$. Contrary to the case $k = d$, this latter case still makes sense when the points are in a generic position.

We solve the question by proving that this case is also NP-complete. The proof is inspired by the proof of the NP-completeness of the linear complementarity problem and uses some relationships between colorful linear programming and complementarity problems that we explicit in this paper. We also show that if P=NP, then there is an easy polynomial-time algorithm computing Nash equilibrium in bimatrix games using any polynomial-time algorithm solving the case with $k = d + 1$ and $|\mathbf{S}_i| \leq 2$ for $i = 1, \ldots, d + 1$ as a subroutine. On our track, we found a new way to prove that a complementarity problem belongs to the PPAD class with the help of Sperner's lemma. We also show that we can adapt algorithms proposed by Bárány and Onn for computing a feasible solution T in a special case and get what can be interpreted as a "Phase I" simplex method, without any projection or distance computation.

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1. Introduction

A set of points is said to be *positively dependent* if it is nonempty and contains $\mathbf{0}$ in its convex hull. Given a configuration of k sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_k$ in $\mathbb{R}^d$, a set $T \subseteq \bigcup_{i=1}^k \mathbf{S}_i$ such that $|T \cap \mathbf{S}_i| \leq 1$ for $i = 1, \ldots, k$ is said to be *colorful*. The *colorful linear programming problem*, called Colorful Linear Programming and defined by Bárány and Onn [4], is the following.

Colorful Linear Programming
Input. A configuration of k sets of points $\mathbf{S}_1, \ldots, \mathbf{S}_k$ in $\mathbb{Q}^d$.
Task. Decide whether there exists a positively dependent colorful set for this configuration of points.

The usual linear programming is the special case of this problem when $\mathbf{S}_1 = \cdots = \mathbf{S}_k$.

In their paper, they proved among several other results that the colorful linear programming problem is NP-complete (Theorem 5.1 of their paper) using a reduction of the partition problem to the case when $k = d$. In a comment of their theorem, they note that it "would be interesting to determine the complexity of deciding the existence of a positively dependent colorful set when $k = d + 1$ but the $\mathbf{S}_i$ are not necessarily positively dependent". This question is motivated by the colorful Carathéodory theorem, found by Bárány [2], whose statement is: *When each of the* $\mathbf{S}_i$ *is positively dependent and* $k = d + 1$*, there exists a positively dependent colorful set.* If $k = d + 1$ and each $\mathbf{S}_i$ is positively dependent, the colorful Carathéodory theorem implies that the answer to Colorful Linear Programming is always 'yes'. Not requiring each $\mathbf{S}_i$ to be positively dependent but keeping the condition $k = d+1$ is a way to get a true decision problem while slightly relaxing the condition of the colorful Carathéodory theorem. Another motivation is the fact that, contrary to the case $k = d$, the case $k = d + 1$ still makes sense when the points are in a generic position. We answer the question.

Theorem 1. Colorful Linear Programming *with* k *sets of points* $\mathbf{S}_1, \ldots, \mathbf{S}_k$ *in* $\mathbb{Q}^d$ *is* NP-*complete when* $k = d + 1$*, even if each* $\mathbf{S}_i$ *has cardinality at most* 2*.*

Polynomially checkable sufficient conditions ensuring the existence of a positively dependent colorful set exist: the condition of the colorful Carathéodory theorem is one of them. More general polynomially checkable sufficient conditions when $k = d+1$ are given in [1, 10, 19]. However, Theorem 1 implies that there is no polynomially checkable conditions that are simultaneously sufficient and necessary for a positively dependent colorful set to exist when $k = d + 1$, unless P=NP.

A problem similar to Colorful Linear Programming was proposed by Meunier and Deza [19] as a byproduct of an existence theorem, the Octahedron lemma [3, 8], which by some features has a common flavor with the colorful Carathéodory theorem. The Octahedron lemma states that if each $\mathbf{S}_i$ of the configuration is of size 2 and if the points are in a generic position, the number of positively dependent colorful sets is even. The problem we call Finding Another Colorful Simplex is the following.

Finding another colorful simplex
Input. A configuration of $d + 1$ pairs of points $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ in $\mathbb{Q}^d$ and a positively dependent colorful set in this configuration.
Task. Find another positively dependent colorful set.

Another positively dependent colorful set exists for sure. Indeed, by a slight perturbation, we can assume that all points are in a generic position. If there were only one positively dependent colorful set, there would also be only one positively dependent colorful set in the perturbed configuration, which violates the evenness property stated by the Octahedron lemma. In their paper, Meunier and Deza question the complexity status of this problem. We solve the question by proving that it is actually a generalization of the problem of computing a Nash equilibrium in a bimatrix game. The problem of computing a Nash equilibrium in a bimatrix game – called Bimatrix – is known as a "difficult" problem with the powerful machinery of complexity theory. Many problems for which the solution is known to exist belong to a complexity class called PPAD, defined by Papadimitriou in 1992 [20], which contains complete problems for this class: the PPAD-complete problems. Since Bimatrix is PPAD-complete [5], Finding Another Colorful Simplex is PPAD-complete as well.

Proposition 1. Finding Another Colorful Simplex *is* PPAD*-complete.*

Moreover, we show that any algorithm solving Colorful Linear Programming can be used to solve Finding Another Colorful Simplex, which shows that any NP-complete problem is at least as hard as any PPAD-complete problem. It has already been noted that P=NP would imply P=PPAD, see [21]. We give here a concrete example of this implication. We do not know whether such an example was already known.

Theorem 1 and Proposition 1 are our two main results. In addition to them, other results are also obtained.

For instance, we propose an adaptation of two pivoting algorithms by Bárány and Onn for finding a positively dependent colorful set under the condition of the colorful Carathéodory theorem. The first of their algorithms is directly inspired by the proof of the colorful Carathéodory theorem [2] and requires a distance computation at each iteration. The second one is an adaptation of the first and consists roughly in replacing the distance computation by a projection on a segment. These algorithms are not polynomial and the question whether there is a polynomial algorithm finding a positively dependent colorful set under the condition of the colorful Carathéodory theorem is an important open question [4, 8]. We show in Section 3 that the distance computation or the projection can be replaced in the aforementioned algorithms by a classical reduced cost consideration. We get in this way an algorithm similar to the "Phase I" simplex method. Numerical performances of this approach will be investigated in future work.

We also discuss links between colorful linear programming and linear complementarity problems (Section 4).

2. Finding Another Colorful Simplex and Nash equilibria

The main message of this section is that Bimatrix can be reduced in polynomial time to Finding Another Colorful Simplex. We first explain why Finding Another Colorful Simplex is a PPAD problem. We end the section with a discussion on the links with Colorful Linear Programming.

2.1. Finding Another Colorful Simplex is in PPAD.

In [19], it was noted that Finding Another Colorful Simplex is in PPA. The class PPA, also defined by Papadimitriou in 1994 [20], contains the class PPAD. Recall that a *search problem* is like a decision problem but a certificate is sought in addition to the 'yes' or 'no' answer. The class of search problems whose decision counterpart has always a 'yes' answer is called TFNP. A subclass of TFNP is PPA, which contains the problems that can be polynomially reduced to the problem of finding another degree 1 vertex in a graph whose vertices have all degree at most 2 and in which a degree 1 vertex is already given. The graph is supposed to be implicitly described by the neighbor function, which, given a vertex, gives in polynomial time its neighbors. The PPAD class is the subclass of PPA for which the implicit graph is oriented and such that each vertex has an outdegree at most 1 and an indegree at most 1. The problem becomes then: given an *unbalanced* vertex, that is a vertex v such that $\deg^+(v)+\deg^-(v)=1$, find another unbalanced vertex. See [20] for further precisions.

We prove in this subsection that Finding Another Colorful Simplex is in PPAD. We proceed by showing that the existence of the other positively dependent colorful set is a consequence of Sperner's lemma [23]. Our method for proving that Finding Another Colorful Simplex belongs to PPAD is adaptable for other complementarity problems, among them Bimatrix. We believe that our method is new. It avoids the use, as in [5, 13, 20, 25], of *oriented primoids* or *oriented duoids* defined by Todd [24].

One of the multiple versions of Sperner's lemma is the following theorem, proposed by Scarf [22], which involves a triangulation of a sphere, whose vertices are labeled. A simplex whose vertices get pairwise distinct labels is said to be *fully-labeled*.

Theorem 2 (Sperner's lemma)**.** *Let T be a triangulation of an n-dimensional sphere $\mathcal{S}^n$ and let V be its vertex set. Assume that the elements of V are labeled according to a map $\lambda: V\rightarrow E$, where E is some finite set. If E is of cardinality $n+1$, then there is an even number of fully-labeled n-simplices.*

We state now the main proposition of this subsection.

Proposition 2. Finding Another Colorful Simplex *is in* PPAD.

Proof. By a perturbation argument, we can assume the points to be in a generic position, see [17] for instance for a description of such a polynomial-time computable perturbation. The proof consists then in proving the existence of another positively dependent colorful set via a polynomial reduction to Sperner's lemma.

Let X and X' be two disjoint copies of $\bigcup_{i=1}^{d+1}\mathbf{S}_i$. We define two simplicial complexes K and K'. The simplicial complex K has X as vertex set and its simplices are the colorful sets. K' is defined on X' as follows

$$\mathsf{K}'=\{\sigma\subset X':\ X'\setminus\sigma \text{ is positively dependent}\}.$$

Note that any superset of a positively dependent set is a positively dependent set. It ensures that K' is a simplicial complex. The points being in a generic position, the dimension of K' is $2(d+1)-(d+1)-1=d$.

The *join* of K and K', denoted $\mathsf{K}*\mathsf{K}'$, is the simplicial complex $\{\sigma\cup\sigma':\ \sigma\in\mathsf{K},\sigma'\in\mathsf{K}'\}$. Note that K is homeomorphic to the boundary of the $(d+1)$-dimensional crosspolytope and as such is a triangulation of a d-dimensional sphere $\mathcal{S}^d$. The simplicial complex K' is a triangulation of $\mathcal{S}^d$ as well. It can be seen using Gale transform and Corollary 5.6.3 (iii) of [15]. Therefore, $\mathsf{K}*\mathsf{K}'$ is a triangulation of the sphere $\mathcal{S}^{2d+1}$, see p.76 of [16] for a proof that the join of two spheres is again a sphere.

Now, for v a vertex of $\mathsf{K}*\mathsf{K}'$, define $\lambda(v)$ to be the point in $\bigcup_{i=1}^{d+1}\mathbf{S}_i$ of which v is the copy. The important fact here is that a fully-labeled $(2d+1)$-simplex of the form $\sigma\cup\sigma'$ with $\sigma\in\mathsf{K},\sigma'\in\mathsf{K}'$ is such that $\lambda(\sigma)$ is a positively dependent colorful set, and conversely any positively dependent colorful set provides exactly one such fully-labeled $(2d+1)$-simplex.

Applying Theorem 2 (Sperner's lemma) with $\mathsf{T}=\mathsf{K}*\mathsf{K}'$, $n=2d+1$, and $E=\bigcup_{i=1}^{d+1}\mathbf{S}_i$ shows that there is an even number of fully-labeled simplices in $\mathsf{K}*\mathsf{K}'$, and hence, an even number of positively dependent colorful sets. Since there is a proof of Sperner's lemma using an oriented path-following argument [18, 22] and since the triangulation here can easily be encoded by a Turing machine computing the neighbors of any simplex in the triangulation in polynomial time, it is routine to conclude. $\square$

The idea of introducing the join $\mathsf{K} * \mathsf{K}'$ was inspired by the proof of the topological Helly theorem by Kalai and Meshulam [11]. In their paper, K is seen as a partition matroid.

2.2. **Reduction of** Bimatrix**.** We start by recalling some basic things about Nash equilibria in bimatrix games.

A bimatrix game involves two $m \times n$ matrices with real coefficients $A = (a_{ij})$ and $B = (b_{ij})$. There are two players. The first player chooses a probability distribution on $\{1, \ldots, m\}$, the second a probability distribution on $\{1, \ldots, n\}$. Once these probability distributions have been chosen, a pair $(\bar{i}, \bar{j})$ is drawn at random according to these distributions. The first player gets a payoff equal to $a_{(\bar{i},\bar{j})}$ and the second a payoff equal to $b_{(\bar{i},\bar{j})}$. A Nash equilibrium is the choice of the distributions in such a way that if a player changes his distribution, he will not get in average a strictly better payoff.

Let Δ^k be the set of vectors $\boldsymbol{x} \in \mathbb{R}_+^k$ such that $\sum_{i=1}^k x_i = 1$. Formally, a *Nash equilibrium* is a pair $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ with $\boldsymbol{y}^* \in \Delta^m$ and $\boldsymbol{z}^* \in \Delta^n$ such that

$$\boldsymbol{y}'^T A \boldsymbol{z}^* \leq \boldsymbol{y}^{*T} A \boldsymbol{z}^* \text{ for all } \boldsymbol{y}' \in \Delta^m \quad \text{and} \quad \boldsymbol{y}^{*T} B \boldsymbol{z}' \leq \boldsymbol{y}^{*T} B \boldsymbol{z}^* \text{ for all } \boldsymbol{z}' \in \Delta^n. \tag{1}$$

It is well-known that if the matrices have rational coefficients, there is a Nash equilibrium with small rational coefficients. Bimatrix is the following problem: given A and B with rational coefficients, find a Nash equilibrium. Papadimitriou showed in 1994 the following theorem [20].

Theorem 3. Bimatrix *is in* PPAD*.*

Later, Chen, Deng, and Teng [5] proved its PPAD-completeness.

Theorem 4. Bimatrix *is* PPAD-*complete.*

A combinatorial approach to these equilibria consists in studying the *complementary* solutions of the two systems

$$[A, I_m]\boldsymbol{x} = (1, \ldots, 1)^T \text{ and } \boldsymbol{x} \in \mathbb{R}_+^{n+m} \tag{2}$$

and

$$[I_n, B^T]\boldsymbol{x} = (1, \ldots, 1)^T \text{ and } \boldsymbol{x} \in \mathbb{R}_+^{n+m}. \tag{3}$$

By *complementary solutions*, we mean a solution $\boldsymbol{x}_A$ of (2) and a solution $\boldsymbol{x}_B$ of (3) such that $\boldsymbol{x}_A \cdot \boldsymbol{x}_B = 0$. Indeed, complementary solutions with $\mathrm{supp}(\boldsymbol{x}_A) \neq \{n+1, \ldots, n+m\}$ or $\mathrm{supp}(\boldsymbol{x}_B) \neq \{1, \ldots, n\}$ give a Nash equilibrium. This point of view goes back to Lemke and Howson [14]. A complete proof within this framework can be found in Remark 6.1 of [18].

We use Theorem 4 to derive the difficulty of Finding Another Colorful Simplex.

Proof of Proposition 1. We prove that the equivalent version of Finding Another Colorful Simplex with cones, given in Section 4.2, is PPAD-complete.

First note that we can assume that all coefficients of A and B are positive. Indeed, adding a same constant to all entries of the matrices does not change the game.

Consider the following $(m+n) \times (2(m+n))$ matrix,

$$M = \begin{pmatrix} A & I_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_n & B^T \end{pmatrix}.$$

We denote by M_i the ith column of M. Note that the vector $\boldsymbol{u} = (1, \ldots, 1) \in \mathbb{R}^{2(n+m)}$ is in the conic hull of $T = \{M_{n+1}, \ldots, M_{n+m}, M_{n+m+1}, \ldots, M_{2n+m}\}$. Indeed, the corresponding submatrix is the identity matrix.

Let $\mathbf{S}_i$ be the pair $\{M_i, M_{m+n+i}\}$ for $i = 1, \ldots, m+n$. Since all coefficients of A and B are positive, $\mathbf{0}$ is not in the convex hull of the columns of M and $\boldsymbol{u}$. A polynomial time algorithm solving Finding Another Colorful Simplex with T as input set would find another colorful set T' such that $\boldsymbol{u} \in \operatorname{cone}(T')$. The decomposition of $\boldsymbol{u}$ on the points in T' gives a vector $\boldsymbol{x}$ such that $M\boldsymbol{x} = \boldsymbol{u}$, $x_i x_{m+n+i} = 0$ for $i = 1, \ldots, m+n$, and $\operatorname{supp}(\boldsymbol{x}) \neq \{n+1, \ldots, 2n+m\}$. Such an $\boldsymbol{x}$ can be written $(\boldsymbol{x}_A, \boldsymbol{x}_B)$ with $\boldsymbol{x}_A, \boldsymbol{x}_B \in \mathbb{R}_+^{m+n}$ satisfying $\boldsymbol{x}_A \cdot \boldsymbol{x}_B = 0$ and such that $\operatorname{supp}(\boldsymbol{x}_A) \neq \{n+1, \ldots, n+m\}$ or $\operatorname{supp}(\boldsymbol{x}_B) \neq \{1, \ldots, n\}$. In other words, it would find a Nash equilibrium. Proposition 2 and Theorem 4 imply therefore that Finding Another Colorful Simplex is PPAD-complete. □

This proof shows that Finding Another Colorful Simplex is more general than computing complementary solutions of Equations (2) and (3). In [19], a pivoting algorithm for solving Finding Another Colorful Simplex is proposed. It reduces to the classical pivoting algorithm due to Lemke and Howson [14] used for computing such complementary solutions.

2.3. **From** Colorful Linear Programming **to** Bimatrix**.** Let $\mathcal{A}$ be an algorithm solving Colorful Linear Programming. In this subsection, we describe an algorithm solving Finding Another Colorful Simplex, and therefore Bimatrix because of the reduction described in Section 2.2, by calling exactly $d+1$ times $\mathcal{A}$. We are probably not aware of the good references, but we were not able to find another such problem in the literature giving a concrete illustration of the fact that NP-complete problems are harder than PPAD problems.

We describe now the algorithm for Finding Another Colorful Simplex. The input is given by the $d+1$ pairs of points $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ and the positively dependent colorful set T. The algorithm selects successively a point in each of the $\mathbf{S}_i$'s. Each iteration consists in testing with the help of $\mathcal{A}$ which point of $\mathbf{S}_i$ is in a positively dependent colorful set compatible with the already selected points and in selecting such a point, with the priority given to $\mathbf{S}_i \setminus T$. A typical iteration is

> Define $\mathbf{S}'_i := \mathbf{S}_i \setminus T$; apply $\mathcal{A}$ to $\mathbf{S}'_1, \ldots, \mathbf{S}'_i, \mathbf{S}_{i+1}, \ldots, \mathbf{S}_{d+1}$; if the answer is 'no', define instead $\mathbf{S}'_i := \mathbf{S}_i \cap T$.

At the end, the algorithm outputs $\bigcup_{i=1}^{d+1} \mathbf{S}'_i$.

Since we know that there is another positively dependent colorful set, the answer will be 'yes' for at least one i. The returned colorful simplex is therefore a positively dependent colorful set distinct from T. This algorithm returns another positively dependent colorful set after calling $d+1$ times $\mathcal{A}$.

3. Bárány-Onn algorithms and the simplex method

Recall that the colorful Carathéodory theorem states that when each of the $\mathbf{S}_i$ is positively dependent and $k = d+1$, there exists a positively dependent colorful set. The pivoting algorithms proposed by Bárány and Onn, for finding a positively dependent colorful set under these conditions go roughly as follows. The input is the sets $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ of points in $\mathbb{Q}^d$, each of cardinality $d+1$ and positively dependent. All points are moreover assumed to be in generic position.

Bárány-Onn algorithms

- Choose a first colorful set T_1 of size $d+1$ and let $i := 0$.

- Repeat:
  - Let $i := i + 1$.
  - If $\mathbf{0} \in \mathrm{conv}(T_i)$, stop an output T_i.
  - Otherwise, find $F \subseteq T_i$ of cardinality d such that $\mathrm{aff}(F)$ separates $T_i \setminus F$ from $\mathbf{0}$; choose in the half-space containing $\mathbf{0}$ a point $\boldsymbol{t}$ of the same color as $T_i \setminus F$; define $T_{i+1} := F \cup \{\boldsymbol{t}\}$.

Since each $\mathrm{conv}(\mathbf{S}_i)$ contains $\mathbf{0}$, there is always a point of each color in the half-space delimited by $\mathrm{aff}(F)$ and containing $\mathbf{0}$. It explains that a point $\boldsymbol{t}$ as in the algorithm can always be found as long as the algorithm has not terminated.

The technical step is the way of finding the subset F and requires a distance computation or a projection [4], or the computation of the intersection of a fixed ray and $\mathrm{conv}(T_i)$ [19]. Deza and al. [7] proceed to an extensive computational study of algorithms solving this problem, with many computational experiments. In addition to some heuristics, "multi-update" versions are also proposed, but they do not avoid this kind of operations.

We propose to modify the approach as follows. We add a dummy point $\boldsymbol{v}$ and define the following optimization problem.

$$\begin{array}{ll} \min & z \\ \text{s.t.} & A\boldsymbol{\lambda} + z\bar{\boldsymbol{v}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\ & \boldsymbol{\lambda} \geq \mathbf{0},\ z \geq 0, \end{array}$$

where $\bar{\boldsymbol{v}} = (\boldsymbol{v}, 1)$ and where A is the $(d+1) \times (d+1)^2$ matrix whose columns are the points of $\bigcup_{i=1}^{d+1} \mathbf{S}_i$ with an additional 1 on the $(d+1)$th row. This optimization problem simply looks for an expression of $\mathbf{0}$ as a convex combination of the points in $\{\boldsymbol{v}\} \cup \bigcup_{i=1}^{d+1} \mathbf{S}_i$ with a minimal weight on $\boldsymbol{v}$. Especially, if $\mathbf{0} \in \mathrm{conv}(\bigcup_i \mathbf{S}_i)$, the optimal value is 0. The idea consists in seeking an optimal basis, with the terminology of the linear programming, which in addition is required to be colorful. The colorful Carathéodory theorem ensures that such a basis exists.

Now, choose a first *transversal* F_1, which is a colorful set of cardinality d. Choose the dummy point $\boldsymbol{v}$ so that $F_1 \cup \{\boldsymbol{v}\}$ contains $\mathbf{0}$ in the interior of its convex hull. Note that $F_1 \cup \{\boldsymbol{v}\}$ is a feasible basis. The algorithm proceeds with simplex pivots, going from feasible colorful basis to feasible colorful basis, until an optimal colorful basis is found. We start with $i := 0$. We repeat then

- Let $i := i + 1$.
- Choose a point $\boldsymbol{t}$ of the missing color in F_i with negative reduced cost. The reduced costs are computed according to the current basis $F_i \cup \{\boldsymbol{v}\}$.
- Proceed to a simplex pivot operation with $\boldsymbol{t}$ entering the current basis.
- If $\boldsymbol{v}$ leaves the basis, stop and output $F_i \cup \{\boldsymbol{t}\}$ (it is an optimal colorful basis).
- Otherwise, define F_{i+1} to be the new basis minus $\boldsymbol{v}$.

This algorithm eventually finds a positively dependent colorful set because of the following lemma. The remaining arguments are exactly the same as above: as long as a positively dependent colorful set has not been found, there is a point of the missing color in the half-space delimited by $\mathrm{aff}(F_i)$ and containing $\mathbf{0}$.

Lemma 1. *The points in the half-space delimited by* $\operatorname{aff}(F_i)$ *and containing* $\mathbf{0}$ *are precisely the points with a negative reduced cost.*

Proof. Let $F_i = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d\}$ and let $\boldsymbol{t}$ be any other point in $\left(\bigcup_{j=1}^{d+1} \mathbf{S}_j\right) \setminus F_i$. Consider $x_1, \ldots, x_d, r, s \in \mathbb{R}$ such that

$$rt + s\boldsymbol{v} + \sum_{i=1}^{d} x_i \boldsymbol{u}_i = \mathbf{0}, \tag{4}$$

with $r > 0$ and $r + s + \sum_{i=1}^{d} x_i = 0$. We have $s \neq 0$ by genericity assumption. The reduced cost of $\boldsymbol{t}$ is exactly s/r. Therefore, proving the lemma amounts to prove that s is negative exactly when $\boldsymbol{t}$ is in the half-space delimited by $\operatorname{aff}(F_i)$ and containing $\mathbf{0}$.

To see this, note that Equation (4) can be rewritten

$$r(\boldsymbol{t} - \boldsymbol{u}_1) + s(\boldsymbol{v} - \boldsymbol{u}_1) + \sum_{i=2}^{d} x_i(\boldsymbol{u}_i - \boldsymbol{u}_1) = \mathbf{0}. \tag{5}$$

Now, take the unit vector $\boldsymbol{n}$ orthogonal to $\operatorname{aff}(F_i)$ and take the scalar product of Equation (5) and $\boldsymbol{n}$. It gives

$$r\boldsymbol{n} \cdot (\boldsymbol{t} - \boldsymbol{u}_1) + s\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{u}_1) = 0$$

and the conclusion follows since $\boldsymbol{v}$ and $\mathbf{0}$ are in the same half-space delimited by $\operatorname{aff}(F_i)$. □

This approach is reminiscent of the "Phase I" simplex method, which computes a first feasible basis by solving an auxiliary linear program whose optimal value is 0 on such a basis.

4. Linear complementarity problem and colorful linear programming

4.1. **Almost a generalization.** We have mentioned the existence of a link between colorful linear programming and linear complementarity problem. We make the things more precise in this section and show that colorful linear programming generalizes linear complementarity problems provided that some condition is satisfied. The *linear complementarity problem* can be formalized as follows.

Linear Complementarity Problem
Input. A matrix $M \in \mathbb{Q}^{d \times d}$ and a vector $\boldsymbol{q} \in \mathbb{Q}^d$.
Task. Decide whether there exist two vectors $\boldsymbol{w}$ and $\boldsymbol{z}$ in $\mathbb{Q}_+^d$ such that $\boldsymbol{w} \cdot \boldsymbol{z} = 0$ and $\boldsymbol{w} - M\boldsymbol{z} = \boldsymbol{q}$.

We associate to this problem the following input of Colorful Linear Programming:

$$\mathbf{S}_i = \{\boldsymbol{e}_i, -M_i\} \text{ for } i = 1, \ldots, d \quad \text{and} \quad \mathbf{S}_{d+1} = \{-\boldsymbol{q}\},$$

where M_i is the ith column of M. As it can be readily checked, solving $\operatorname{LCP}(\boldsymbol{q}, M)$ amounts to solve Colorful Linear Programming with this input when M satisfies the following condition: there does not exist $\boldsymbol{z}'$ in $\mathbb{Q}_+^d \setminus \{\mathbf{0}\}$ such that $\boldsymbol{z}'^T M \boldsymbol{z}' = 0$ and $M\boldsymbol{z}' \geq \mathbf{0}$. We do not know whether this latter condition is polynomially checkable or not. Note however that this condition is satisfied for positive definite matrices and matrices with all coefficients being positive.

The proof of Theorem 1 given in Section 5 is inspired by this point of view and by the proof of the NP-completeness of Linear Complementarity Problem given in [6].

4.2. **The conic version of** Colorful Linear Programming**.** A more general version of Colorful Linear Programming can be defined with conic hulls instead of convex hulls.

Colorful Linear Programming (conic version)
Input. A configuration of k sets of points $\mathbf{S}_1,\ldots,\mathbf{S}_k$ in $\mathbb{Q}^{d+1}$ and an additional point $\boldsymbol{p}$ in $\mathbb{Q}^{d+1}$.
Task. Decide whether there exists a colorful set T such that $\boldsymbol{p}\in\operatorname{cone}(T)$.

By an easy geometric argument, this version in dimension $d+1$ coincides with Colorful linear programming in dimension d when $\operatorname{conv}(\{\boldsymbol{p}\}\cup\bigcup_{i=1}^k\mathbf{S}_i)$ does not contain $\mathbf{0}$. The same remark holds for the problem Finding Another Colorful Simplex. The following problem is an equivalent formulation.

Finding another colorful simplex (conic version)
Input. A configuration of $d+1$ pairs of points $\mathbf{S}_1,\ldots,\mathbf{S}_{d+1}$ in $\mathbb{Q}^{d+1}$, an additional point $\boldsymbol{p}$ in $\mathbb{Q}^{d+1}$ such that $\operatorname{conv}(\{\boldsymbol{p}\}\cup\bigcup_{i=1}^{d+1}\mathbf{S}_i)$ does not contain $\mathbf{0}$, and a colorful set T such that $\boldsymbol{p}\in\operatorname{cone}(T)$.
Task. Find another colorful set T' such that $\boldsymbol{p}\in\operatorname{cone}(T')$.

The conic version of Colorful Linear Programming fully contains Linear Complementarity Problem, without the restriction outlined in Section 4.1, by working in dimension d and defining the input

$$\mathbf{S}_i=\{\boldsymbol{e}_i,-M_i\}\text{ for }i=1,\ldots,d\quad\text{and}\quad\boldsymbol{p}=\boldsymbol{q}.$$

5. Proof of the NP-completeness in the case $k=d+1$

The proof of Theorem 1 uses a reduction of the subset sum problem, denoted Subset Sum. Given $n+1$ integers $a_1,\ldots,a_n,b$, Subset Sum aims at deciding whether there exists $I\subseteq\{1,\ldots,n\}$ such that $\sum_{i\in I}a_i=b$. Subset Sum is NP-complete [9, 12].

Proof of Theorem 1. Take an input $a_1,\ldots,a_n,b$ of Subset Sum. Define $d=n+2$. For $i=1,\ldots,d$, let $\boldsymbol{e}_i$ be the point in $\mathbb{R}^d$ such that $\boldsymbol{e}_i=(0,\ldots,0,1,0,\ldots,0)$ with a '1' at position i and a '0' elsewhere (it is the unit vector). For $i=1,\ldots,d-2$, let $\boldsymbol{f}_i$ be the point in $\mathbb{R}^d$ such that $\boldsymbol{f}_i=(0,\ldots,0,1,0,\ldots,0,-1,1)$ with a '-1' at position $d-1$, a '1' at positions i and d, and a '0' elsewhere. Finally, we define the point $\boldsymbol{g}\in\mathbb{R}^d$ to be $\boldsymbol{g}=(-a_1,\ldots,-a_n,b,-b)$.

We define the input of the colorful linear programming problem as follows

$$\mathbf{S}_i=\{\boldsymbol{e}_i,\boldsymbol{f}_i\}\text{ for }i=1,\ldots,d-2,\quad\mathbf{S}_{d-1}=\{\boldsymbol{e}_{d-1}\},\quad\mathbf{S}_d=\{\boldsymbol{e}_d\},\quad\text{and}\quad\mathbf{S}_{d+1}=\{\boldsymbol{g}\}.$$

This reduction being obviously polynomial, it remains to check that a 'yes' answer to Colorful Linear Programming with this input is equivalent to a 'yes' answer to Subset Sum.

Suppose first that the answer is 'yes' to Colorful Linear Programming. Then there exists a positively dependent set $T\subseteq\bigcup_{i=1}^{d+1}\mathbf{S}_i$ such that $|T\cap\mathbf{S}_i|\leq 1$ for $i=1,\ldots,d+1$. Since T is positively dependent, there exists a nonzero vector $\boldsymbol{\lambda}\in\mathbb{R}_+^{d+1}$ such that

$$\sum_{i=1}^{d+1}\lambda_i\boldsymbol{p}_i=\mathbf{0},\tag{6}$$

where $\boldsymbol{p}_i$ is in $T\cap\mathbf{S}_i$ for $i=1,\ldots,d+1$.

We define I as the set of indices $i \in \{1, \ldots, n\}$ such that $T \cap \mathbf{S}_i = \{\boldsymbol{f}_i\}$. The last two coordinates in Equation (6) lead to

$$-\sum_{i\in I} \lambda_i + \lambda_{d-1} + \lambda_{d+1} b = 0 \quad \text{and} \quad \sum_{i\in I} \lambda_i + \lambda_d - \lambda_{d+1} b = 0.$$

Since all λ_i are nonnegative, summing these two equalities shows that $\lambda_{d-1} = \lambda_d = 0$ and thus

$$\sum_{i\in I} \lambda_i = \lambda_{d+1} b. \tag{7}$$

Reading the first $d-2$ rows of Equation (6) leads to $\lambda_i - \lambda_{d+1} a_i = 0$ for $i = 1, \ldots, d-2$. Combining these equalities with Equation (7) gives $\sum_{i\in I} \lambda_{d+1} a_i = \lambda_{d+1} b$. The number λ_{d+1} is nonzero otherwise $\boldsymbol{\lambda} = \mathbf{0}$. Thus, the last equality reads $\sum_{i\in I} a_i = b$.

Conversely, suppose that the answer is 'yes' to Subset Sum. We have then a subset I of $\{1, \ldots, n\}$ such that $\sum_{i\in I} a_i = b$. For each $i \in \{1, \ldots, n\}$, define $\boldsymbol{p}_i$ to be $\boldsymbol{f}_i$ if i is in I and to be $\boldsymbol{e}_i$ otherwise. We have $\sum_{i=1}^n a_i \boldsymbol{p}_i + \boldsymbol{g} = \mathbf{0}$. The set $T = \bigcup_{i=1}^n \{\boldsymbol{p_i}\} \cup \{\boldsymbol{g}\}$ is a positively dependent colorful set. □

The positively dependent colorful set T used in the proof is such that $\mathbf{0}$ is in the convex hull of $d-1$ points: the weights λ_{d-1} and λ_d are zero. We may ask whether the generalization of Theorem 1 with $\mathbf{0}$ required not to be in the convex hull of less than $d+1$ points of $\bigcup_{i=1}^{d+1} \mathbf{S}_i$ remains true. The answer is yes. To see this, we slightly modify the proof, with the same $\mathbf{S}_i$'s, but instead of deciding whether $\mathbf{0}$ is in the convex hull of some positively dependent colorful set, we decide whether $\boldsymbol{q}$ is in the convex hull of some positively dependent colorful set, where

$$\boldsymbol{q} = \left(\frac{1}{n^2(1+A)}, \ldots, \frac{1}{n^2(1+A)}, \frac{1}{n(1+A)}, \frac{1}{n(1+A)} \right),$$

with $A = \sum_{i=1}^n a_i$. Proving that this generalization of Theorem 1 is true with $\boldsymbol{q}$ in place of $\mathbf{0}$ shows that the generalization is true with $\mathbf{0}$. Indeed, the former is equivalent to the latter with the points of the $\mathbf{S}_i$'s translated by $-\boldsymbol{q}$.

The slight modification of the proof goes as follows. We assume without loss of generality that $n \geq 4$ and $A \geq 1$. In the proof, if the answer is 'yes' to Colorful Linear Programming, instead of proving directly that $\sum_{i\in I} a_i = b$, we prove that $\left| \sum_{i\in I} a_i - b \right| < 1$. Since the a_i and b are integers, the inequality implies that the answer is 'yes' to Subset Sum. The reverse implication is proved along the same lines.

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