

On the Unique-Lifting Property

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Joint Work with Gennadiy Averkov

18th Aussois Combinatorial Optimization Workshop

Cut Generating Functions for Mixed-Integer Linear Programs

- ▶ Solve problem as Linear Program and obtain optimal simplex tableau.
- ▶ If some basic variables are non-integral, “apply” **Cut Generating Functions** to one or more rows of the tableau obtain **Cutting Planes**.
- ▶ Add these cutting planes and re-iterate (usually combined with some enumeration scheme).

General Framework for Cut Generating Functions

$$x = f + \sum_{j=1}^k r^j s_j + \sum_{j=1}^{\ell} p^j y_j$$

$$x \in \mathbb{Z}^q$$

$$s \in \mathbb{R}_+^k$$

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such that the inequality

$$\sum_{j=1}^k \psi_f(r^j) s_j + \sum_{j=1}^{\ell} \pi_f(p^j) y_j \geq 1$$

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Want **minimal valid pairs** to remove redundancies.

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- Let $B \in \mathbb{R}^q$ be a maximal lattice-free convex polytope with $f \in \text{int}(B)$: $B = \{x \in \mathbb{R}^q : a_i(x - f) \leq 1, i \in I\}$. Define the function

$$\phi_{f,B}(r) = \max_{i \in I} a_i r, \quad \forall r \in \mathbb{R}^q.$$

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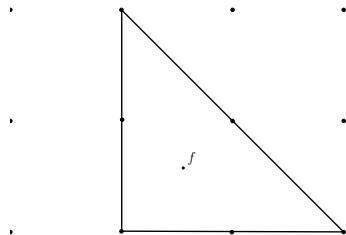
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- For some special f, B 's, minimal liftings are unique. **Gives us formulas for minimal valid pairs.**

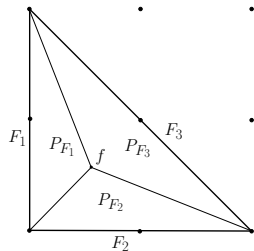
Recognizing pairs f, B with unique minimal liftings

1. Given f, B



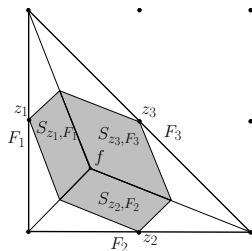
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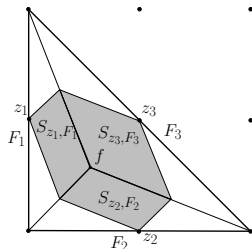
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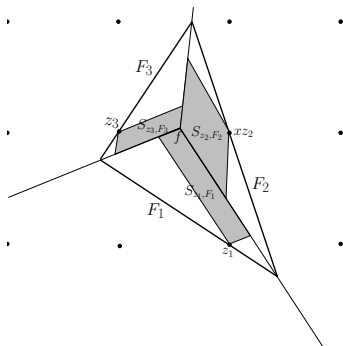
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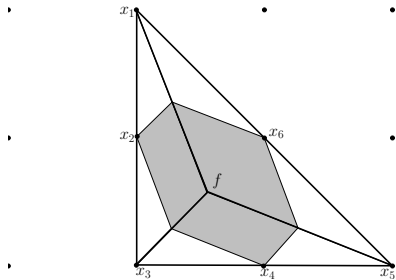
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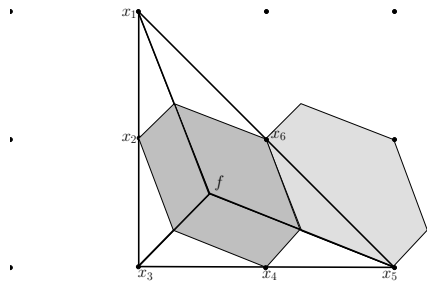
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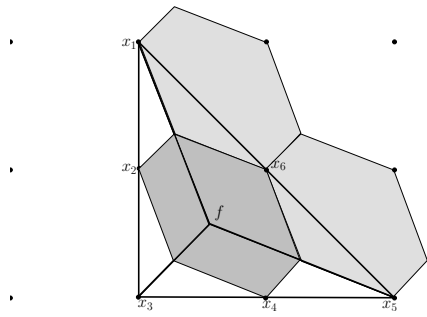
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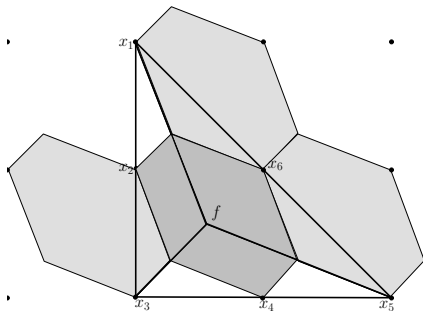
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Main Credit for sparking this line of research:
Santanu Dey and Laurence Wolsey 2009.

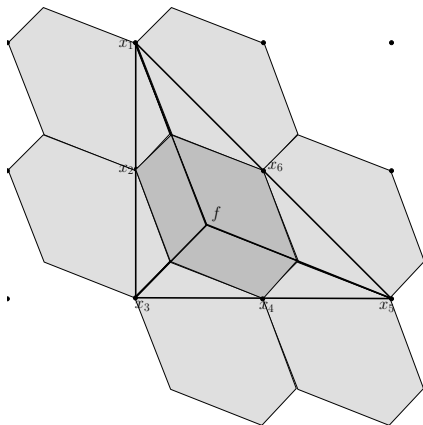




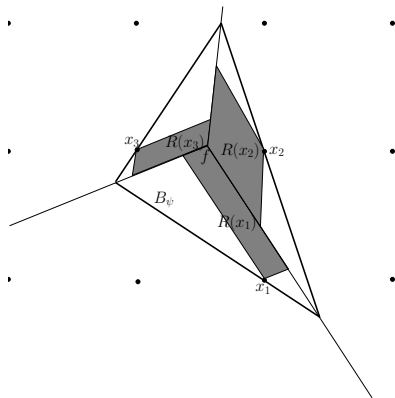




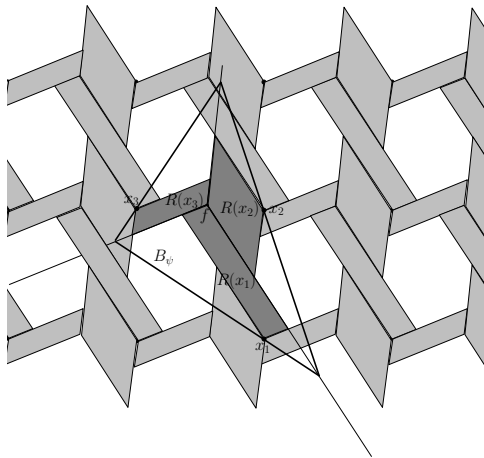
$$R_\psi + \mathbb{Z}^q = \mathbb{R}^q$$



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MORAL :

1. If the pair f, B has the **unique-lifting property**, then we get closed form formulas for a minimal valid pair.
2. The question of deciding if f, B has the **unique-lifting property** is equivalent to deciding if $R(f, B) + \mathbb{Z}^q = \mathbb{R}^q$.

Potentially connects with a lot of research on coverings and tilings by star-shaped bodies, extensively studied in **Geometry of Numbers**.

Invariance of the Unique-lifting property

For a fixed maximal lattice-free convex polytope B , $R(f, B)$ (in fact ψ_f itself) depends on the position of f in the interior. So, *a priori*, the same lattice-free set B might have the unique-lifting property when paired with one f_1 , and have the multiple-lifting property when paired with a different f_2 .

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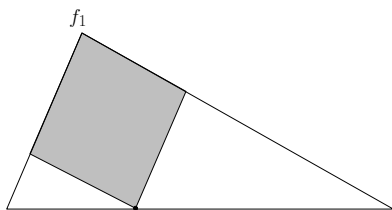
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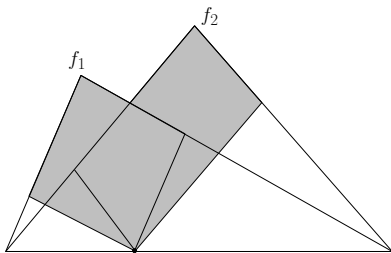
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Proof Ingredient 2 : Take care of intersections due to lattice translations: We find a closed form integral expression for $\text{vol}_{\mathbb{T}^q}(R_\psi/\mathbb{Z}^q)$.

Operations that preserve that Unique-lifting property

Pyramid Construction. Let $B \subseteq \mathbb{R}^q$ be a maximal lattice-free polytope. Consider B as embedded in \mathbb{R}^{q+1} , i.e., $B \subseteq \mathbb{R}^q \times \{0\} \subseteq \mathbb{R}^{q+1}$. Let $v \in \mathbb{R}^{q+1} \setminus (\mathbb{R}^q \times \{0\})$. Let $C(B, v)$ be the cone formed with $B - v$ as base. We define

$$\text{Pyr}(B, v) = (C(B, v) + v) \cap \{x \in \mathbb{R}^{q+1} : x_{q+1} \geq -1\}.$$

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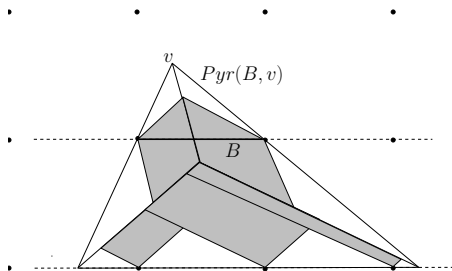
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Let B be a maximal lattice-free polytope and $v \in \mathbb{R}^{q+1}$ be such that $\text{Pyr}(B, v)$ is a maximal lattice-free polytope. Suppose further that the base of $\text{Pyr}(B, v)$ contains an integer translate of B . If B is a body with unique lifting, then $\text{Pyr}(B, v)$ is a body with unique lifting.

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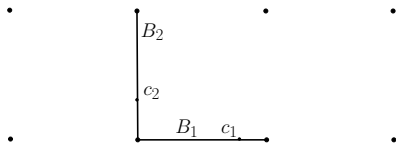
$$B_1 \oplus B_2 := \text{conv}\left(\left(\left\{\frac{B_1 - c_1}{1 - \mu} \times \{o_2\}\right\} \cup \left(\{o_1\} \times \frac{B_2 - c_2}{\mu}\right)\right) + (c_1, c_2)\right).$$

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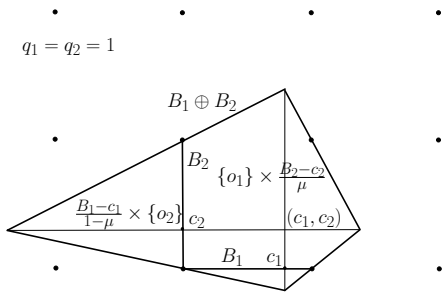
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Let $a = (a_1, \dots, a_q)$ be an q -tuple of real numbers such that $\frac{1}{a_1} + \dots + \frac{1}{a_q} = 1$. Then $S(a) := \text{conv}\{0, a_1 e^1, a_2 e^2, \dots, a_q e^q\} \subseteq \mathbb{R}^q$ is a maximal lattice-free simplex.

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Let P be a maximal lattice-free pyramid in \mathbb{R}^q such that every facet of P contains exactly one integer point in its relative interior. P has the unique-lifting property if and only if P is an affine unimodular transformation of $\text{conv}\{0, qe^1, \dots, qe^q\}$.

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4. Combinatorial geometry of zonotopes $\Rightarrow S_{z,F}(f)$ is a parallelotope. This implies P is a simplex.

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PROOF:

1. Let f be the apex of P and consider $S_{z,F}(f)$.
2. Venkov-Alexandrov-McMullen theorem $\Rightarrow S_{z,F}(f)$ is centrally symmetric with centrally symmetric facets.
3. McMullen's characterization of zonotopes $\Rightarrow S_{z,F}(f)$ is a zonotope whose every belt is of length 4.
4. Combinatorial geometry of zonotopes $\Rightarrow S_{z,F}(f)$ is a parallelotope. This implies P is a simplex.
5. Minkowski's Convex Body theorem + Minkowski-Hajós theorem $\Rightarrow P$ is an affine unimodular transformation of $\text{conv}\{0, qe^1, \dots, qe^q\}$.

Conclusions

- Maximal Lattice-free polytopes with the unique-lifting property give closed form formulas for minimal cut generating functions.
- Detecting the unique-lifting property can be converted into a geometric question, i.e., covering by lattice translates. Can leverage a lot of research done in Geometry of Numbers and Discrete Geometry.
- Characterize maximal lattice-free polytopes with the unique-lifting property.
- Invariance of Unique-lifting property for S -free sets.

THANK YOU !

Questions/Comments ?