On the Unique-Lifting Property

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Joint Work with Gennadiy Averkov

18th Aussois Combinatorial Optimization Workshop

Cut Generating Functions for Mixed-Integer Linear Programs

- Solve problem as Linear Program and obtain optimal simplex tableau.
- If some basic variables are non-integral, "apply" Cut Generating Functions to one or more rows of the tableau obtain Cutting Planes.
- Add these cutting planes and re-iterate (usually combined with some enumeration scheme).

General Framework for Cut Generating Functions

$$x = f + \sum_{j=1}^{k} r^j s_j + \sum_{j=1}^{\ell} p^j y_j$$

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such that the inequality

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Want minimal valid pairs to remove redundancies.

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$$\phi_{f,B}(r) = \max_{i \in I} a_i r, \quad \forall r \in \mathbb{R}^q.$$

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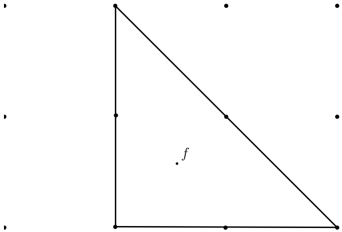
- For some special f, B's, minimal liftings are unique. Gives us formulas for minimal valid pairs.

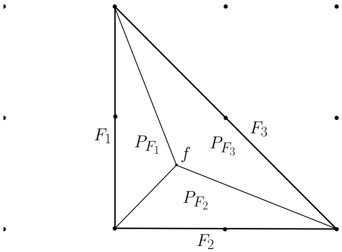
Recognizing pairs f, B with unique minimal liftings

1. Given f, B



Recognizing pairs f, B with unique minimal liftings

1. Given f, B.
2. For every facet F, construct $P_F := \operatorname{conv}(F \cup \{f\})$.



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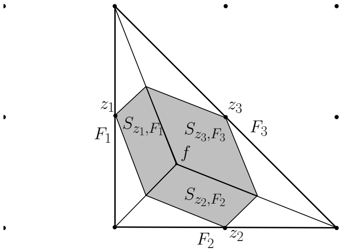

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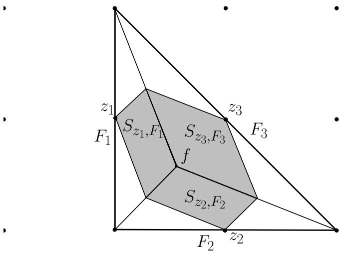

$$R(f,B) = \bigcup_{\text{Facets } F} \; \bigcup_{z \in \mathbb{Z}^q \cap \textbf{relint}(F)} S_{z,F}(f)$$

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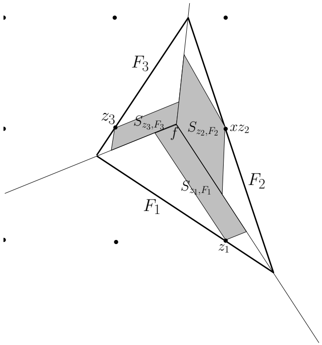

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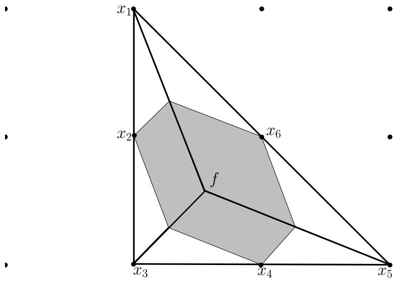
Main Credit for sparking this line of research:
Santanu Dey and Laurence Wolsey 2009.



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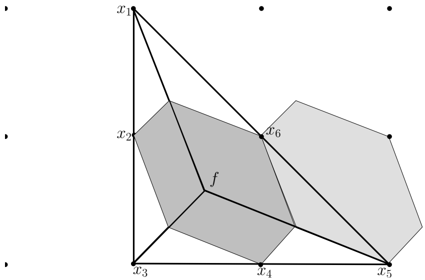

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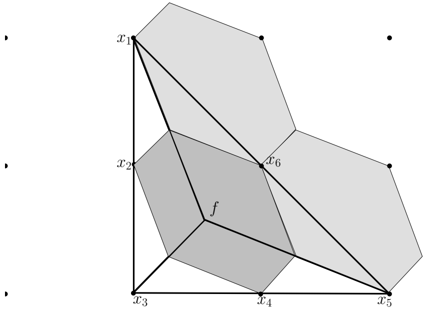
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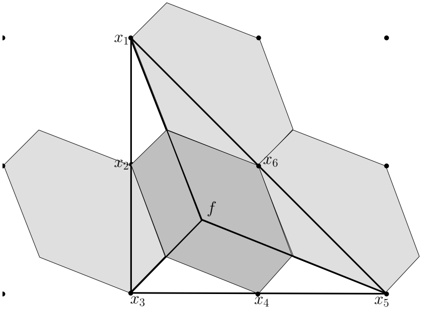
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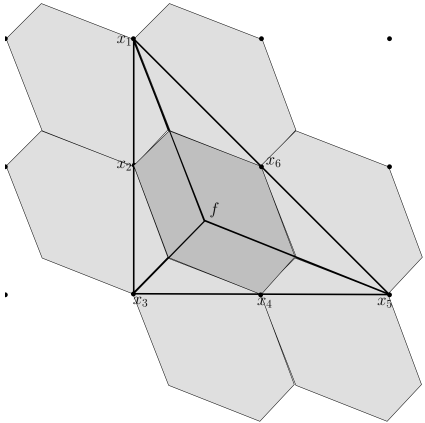

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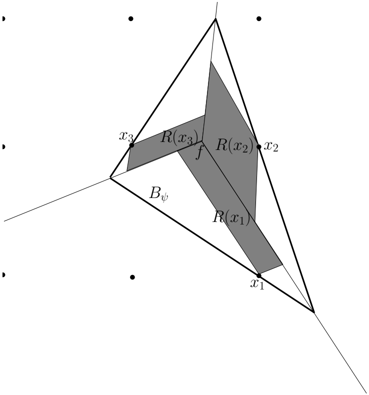

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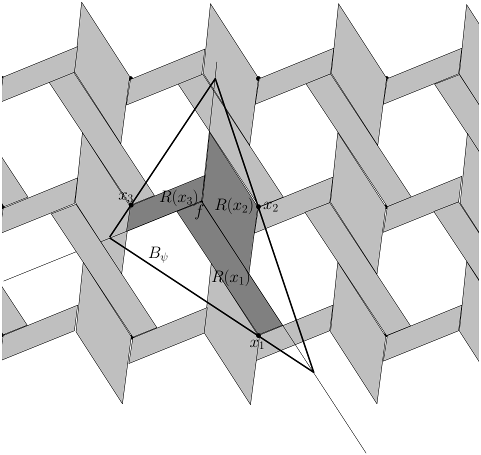

MORAL :

1. If the pair f, B has the unique-lifting property, then we get closed form formulas for a minimal valid pair.
2. The question of deciding if f, B has the unique-lifting property is equivalent to deciding if $R(f, B) + \mathbb{Z}^q = \mathbb{R}^q$.

Potentially connects with a lot of research on coverings and tilings by star-shaped bodies, extensively studied in Geometry of Numbers.

Invariance of the Unique-lifting property

For a fixed maximal lattice-free convex polytope B, $R(f, B)$ (in fact ψ_f itself) depends on the position of f in the interior. So, *a priori*, the same lattice-free set B might have the unique-lifting property when paired with one f_1, and have the multiple-lifting property when paired with a different f_2.

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Let B be a maximal lattice-free polytope in $\mathbb{R}^q (q \geq 2)$. Then B has the unique-lifting property for all $f \in \mathbf{int}(B)$, or B has the multiple-lifting property for all $f \in \mathbf{int}(B)$.

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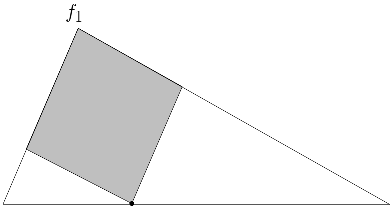

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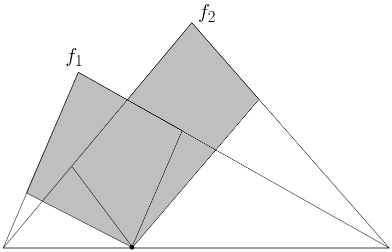

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Proof Ingredient 2 : Take care of intersections due to lattice translations: We find a closed form integral expression for $\mathrm{vol}_{\mathbb{T}^q}(R_\psi/\mathbb{Z}^q)$.

Operations that preserve that Unique-lifting property

Pyramid Construction. Let $B \subseteq \mathbb{R}^q$ be a maximal lattice-free polytope. Consider B as embedded in $\mathbb{R}^{q+1}$, i.e., $B \subseteq \mathbb{R}^q \times \{0\} \subseteq \mathbb{R}^{q+1}$. Let $v \in \mathbb{R}^{q+1} \setminus (\mathbb{R}^q \times \{0\})$. Let $C(B, v)$ be the cone formed with $B - v$ as base. We define

$$\mathrm{Pyr}(B, v) = (C(B, v) + v) \cap \{x \in \mathbb{R}^{q+1} : x_{q+1} \geq -1\}.$$

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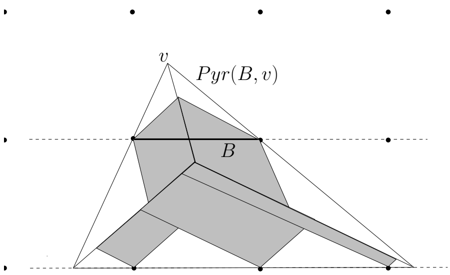

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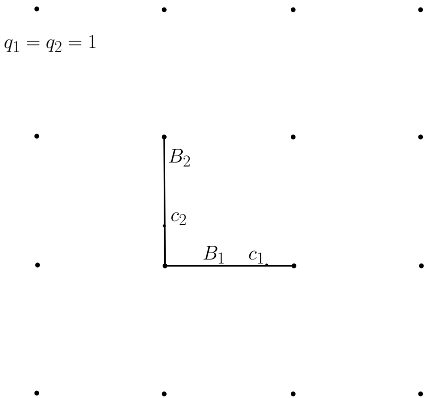

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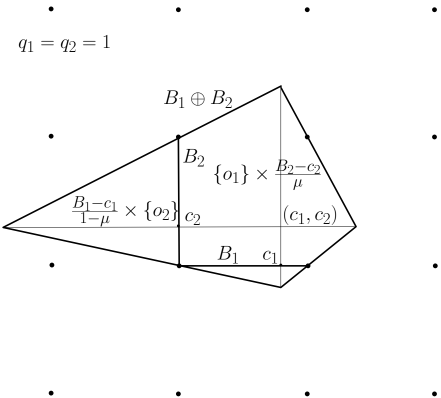

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Let $a = (a_1, \ldots, a_q)$ be an q-tuple of real numbers such that $\frac{1}{a_1} + \ldots + \frac{1}{a_q} = 1$. Then
$S(a) := \mathrm{conv}\{0, a_1 e^1, a_2 e^2, \ldots, a_q e^q\} \subseteq \mathbb{R}^q$ is a maximal lattice-free simplex.

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Unique-Lifting Property of Pyramids

THEOREM Averkov, Basu 2013

Let P be a maximal lattice-free pyramid in $\mathbb{R}^q$ such that every facet of P contains exactly one integer point in its relative interior. P has the unique-lifting property if and only if P is an affine unimodular transformation of $\mathrm{conv}\{0, qe^1, \ldots, qe^q\}$.

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5. Minkowski's Convex Body theorem + Minkowski-Hajós theorem $\Rightarrow$ P is an affine unimodular transformation of $\mathrm{conv}\{0, qe^1, \ldots, qe^q\}$.

Conclusions

- Maximal Lattice-free polytopes with the unique-lifting property give closed form formulas for minimal cut generating functions.
- Detecting the unique-lifting property can be coverted into a geometric question, i.e., covering by lattice translates. Can leverage a lot of research done in Geometry of Numbers and Discrete Geometry.
- Characterize maximal lattice-free polytopes with the unique-lifting property.
- Invariance of Unique-lifting property for S-free sets.

THANK YOU !

Questions/Comments ?