A Two-Variable Analysis of the Two-Trust-Region Subproblem

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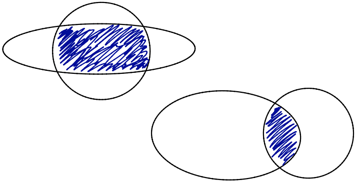
The Combinatorial Optimization Workshop
CNRS Centre Paul Langevin
Aussois, France

January 7, 2014

Outline



1. Quadratically Constrained Quadratic Programming
2. Trust-Region Subproblems (including TTRS)
3. Relaxations of TTRS
4. Nonnegative Quadratics for TTRS ($n = 2$)
5. Conclusions

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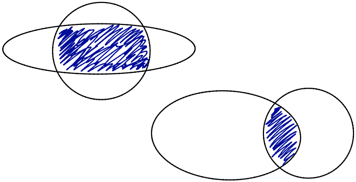

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QCQP

$$\begin{aligned} \min \quad & x^T C x + 2\, c^T x \\ \text{s.t.} \quad & x^T A_i x + 2\, a_i^T x \le b_i \;\; \forall\, i \end{aligned}$$

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So we study

$$\overline{\text{conv}}\{(x, X) : x \in F,\ X = x x^T\}$$

QCQP (cont'd)

- A quadratic lifting . . . powerful but computationally challenging
- Analogous to convex-hull studies in MILP (i.e., difficult)
- I focus on continuous case
  - Closer to my own background
  - Everything here is valid for the mixed-integer case
  - Even for simple looking continuous problems, not much is known

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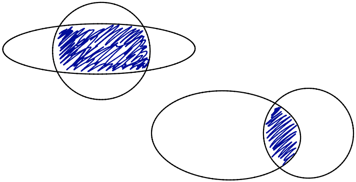

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Classical Trust-Region Subproblem

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\begin{aligned}
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\text{s.t.} \quad & \|x\| \le 1
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\qquad \text{(TRS)}
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Classical Trust-Region Subproblem

$$\begin{array}{ll} \min & x^T C x + 2\, c^T x \\ \text{s. t.} & \|x\| \le 1 \end{array} \qquad \text{(TRS)}$$

- Widely studied, especially by NLP community
  - More and Sorensen (1983)
  - Ye (1992)
  - Rendl and Wolkowicz (1997)
  - Fu, Luo and Ye (1998)
  - Gould, Lucidi, Roma and Toint (1999)
  - Conn, Gould and Toint (2000)
- Polynomial-time solvable via several techniques

Classical Trust-Region Subproblem (cont'd)

Relax xx^T to X...

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- Every extreme point of SDP feasible region satisfies $X = xx^T$
- So have the desired convex hull, and SDP relaxation is tight

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- Linear constraints (Sturm-Zhang 2003)

$$\min \left\{ x^T Q x + 2\, c^T x \; : \; \begin{array}{l} \|x\| \le 1 \\ a_i^T x \le b_i \;\; (i \in \mathcal{I}) \end{array} \right\} \qquad \text{(LTRS)}$$

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- Another ellipsoid (Celis-Dennis-Tapia 1985)

$$\min \left\{ x^T Q x + 2\, c^T x \;:\; \begin{array}{l} \|x\| \le 1 \\ (x-a)^T A (x-a) \le 1 \end{array} \right\} \quad \text{(TTRS)}$$

Extensions of TRS (cont'd)

For practical use in NLP, just solve extension locally (Toint 2013, private communication)

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- LTRS
  - direct algo for a few linear ineq (Bienstock-Michalka 2013)
  - convex hull when linear ineq are "non-intersecting" (Yang-B 2013, B-Anstreicher 2013, Ye-Zhang 2003)

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- TTRS
  - direct algorithm (Bienstock 2013)
  - convex hull?

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Even if NO, still interested in valid inequalities because TTRS appears as substructure more generally. For example:

- The convex hull of the split disjunction of an ellipsoid yields a structure very similar to TTRS (Dadush et al 2011, Belotti et al 2012, Andersen and Jensen 2013, Kılınç-Karzan and Yıldız 2013)

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Spoiler alert!
Our results are only partial

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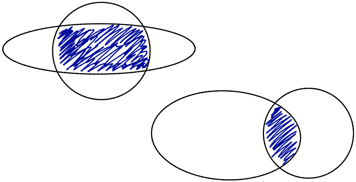

Basic SDP Relaxation

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This is an *SOCRLT ineq* (B-Anstreicher '13, Sturm-Zhang '03)

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Observation. Switching the roles of the ball and the 2^{nd} ellipsoid yields the same SOCRLT inequalities

SOCRLT Constraints (cont'd)

Theorem (B-Anstreicher 2013). The SOCRLT constraints are poly-time separable by solving a classical TRS

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Too bad. This relaxation is not the convex hull (even for $n = 2$). What inequalities are missing?

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Observation. The SOCRLT ineqs are precisely the RLT ineqs (but not obvious how to separate RLT ineqs individually)

Nonnegative-Quadratic Constraints

$$x^T R x + 2\, r^T x + \rho \geq 0 \quad \forall \ \text{feas } x$$

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& R \bullet X + 2\, r^T x + \rho \geq 0 \quad \forall \; (R, r, \rho) \in \mathcal{K}
\end{aligned}$$

Nonnegative-Quadratic Constraints (cont'd)

- Idea goes back to Sturm-Zhang 2003
- Easy to show new inequalities capture the convex hull

Nonnegative-Quadratic Constraints (cont'd)

- Idea goes back to Sturm-Zhang 2003
- Easy to show new inequalities capture the convex hull
- But how to identify those $x^T R x + 2\, r^T x + \rho$ that are nonnegative for all feasible x?
- We are able to classify the generators. . .

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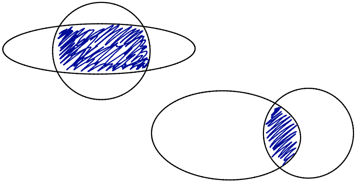

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Classification

For $n = 2$, there are four types of generators for nonneg quadratics:

1. Ellipsoid
2. PSD
3. Lifted RLT
4. Exotic

Ellipsoid Quadratics

$$1 - x^T x \geq 0$$

$$1 - (x - a)^T A(x - a) \geq 0$$

Observation. Already have the corresponding SDP constraints

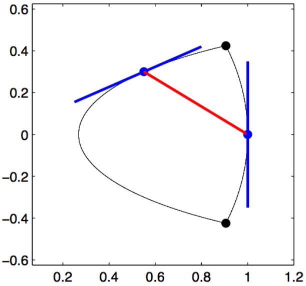
PSD Quadratics

$$x^T R x + 2\, r^T x + \rho \geq 0 \quad \text{where} \quad R \succeq 0$$

Proposition. Already have these constraints due to the ellipsoid constraints and

$$\begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0$$

Lifted RLT Quadratics



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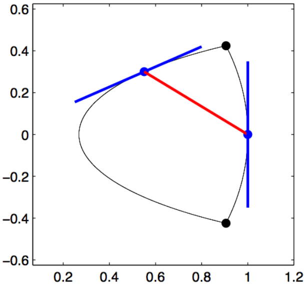
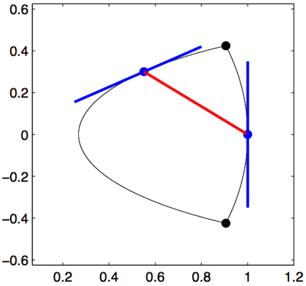


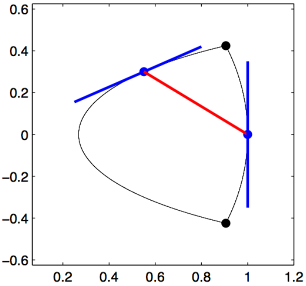
(1st support)(2nd support)

Lifted RLT Quadratics



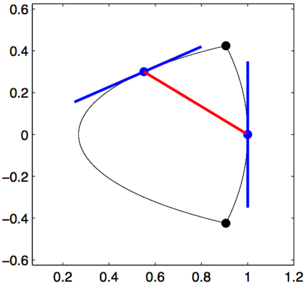
$\big($1st support$\big)\big($2nd support$\big)$ $\qquad$ $\big($support connector$\big)^2$

Lifted RLT Quadratics



$$\left(1^{\text{st}}\text{ support}\right)\left(2^{\text{nd}}\text{ support}\right) - \varepsilon_{\max}\left(\text{support connector}\right)^2$$

Lifted RLT Quadratics



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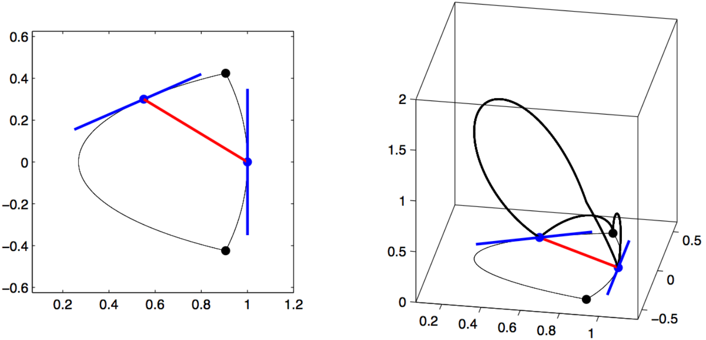

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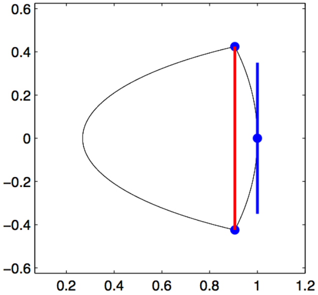
Exotic Quadratics



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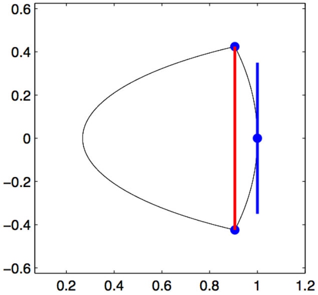

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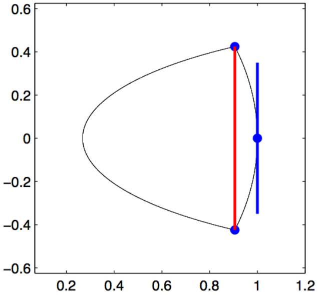

(support) (vertex connector) $\qquad (1 - x^T x)$

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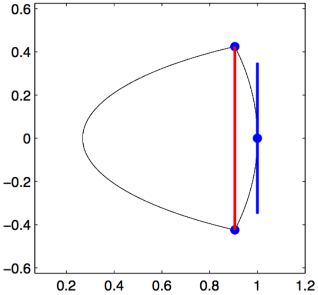

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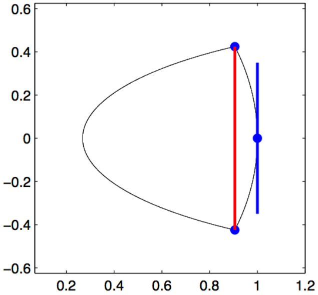

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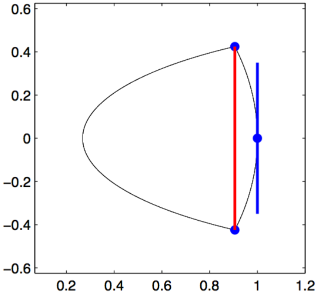

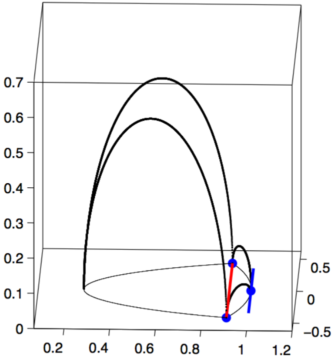

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Some Observations

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  - are specified by two supports and a lifting parameter
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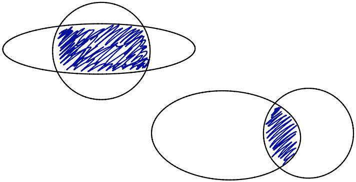

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- Compared to IP, the nonconvex quadratic world is strange
- But if you like extended formulations and convex hulls, there is plenty of work to do
- And nonconvex quadratic IP should be even more fun

Thank you!