A Min-Max Theorem for Transversal Submodular Functions and Its Implications

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(Joint work with Shin-ichi Tanigawa, RIMS, Kyoto University)

The present talk will

1. introduce a new concept of transversal submodular function, a generalization of ordinary submodular set function,

2. show a min-max relation between the minimum of a transversal submodular function and the maximum of the negative of a norm composed of ℓ_1 and ℓ_∞ norms, and

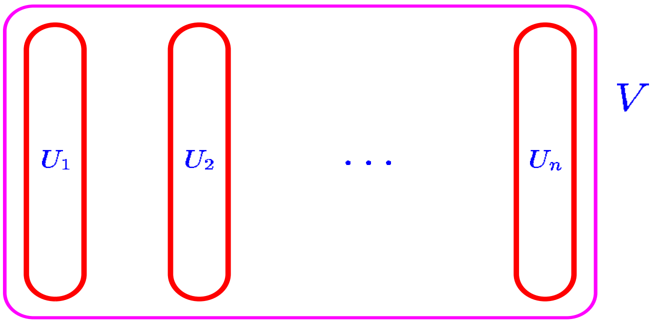
3. based on the min-max relation, give a unifying view over the recent results on generalizations of submodular set functions:

[1] A. Huber and V. Kolmogorov: Towards minimizing k-submodular functions. Proceedings of ISCO 2012, LNCS **7422** (2012) 451–462.

[2] F. Kuivinen: On the complexity of submodular function minimisation on diamonds. *Discrete Optimization* **8** (2011) 459–477.

V: a nonempty finite set

$\mathcal{U} \equiv \{U_1, U_2, \cdots, U_n\}$: a **partition** of V



$T (\subseteq V)$: a **subtransversal** (or **partial transversal**) of $\mathcal{U}$
($|T \cap U| \leq 1$ for all $U \in \mathcal{U}$)

$\mathcal{T}$: the set of all subtransversals of $\mathcal{U}$

$\mathcal{U}(T) = \{U \in \mathcal{U} \mid U \cap T \neq \emptyset\} \quad (\forall T \in \mathcal{T})$

$U(v)$: the unique $U \in \mathcal{U}$ that contains $v \in V$

We consider a function $f : \mathcal{T} \to \mathbb{R}$.

$\rightarrow$

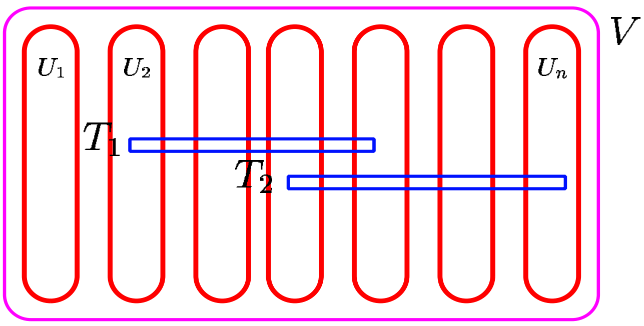
Consider two **binary operations** $\bigtriangledown$ **and** $\bigtriangleup$ **on** $\mathcal{T}$ satisfying the condition that for all $T_1,\ T_2 \in \mathcal{T}$

$$\begin{aligned} &T_1 \bigtriangledown T_2 \in \mathcal{T}, \qquad && \mathcal{U}(T_1 \bigtriangledown T_2) \subseteq \mathcal{U}(T_1) \cup \mathcal{U}(T_2), \\ &T_1 \bigtriangleup T_2 \in \mathcal{T}, \qquad && \mathcal{U}(T_1 \bigtriangleup T_2) \subseteq \mathcal{U}(T_1) \cap \mathcal{U}(T_2). \end{aligned}$$

Define a function $f : \mathcal{T} \to \mathbb{R}$ with $f(\emptyset) = 0$ satisfying

$$f(T_1) + f(T_2) \geq f(T_1 \bigtriangledown T_2) + f(T_1 \bigtriangleup T_2) \qquad (\forall T_1, T_2 \in \mathcal{T}).$$

We call f a **transversal submodular function** or a **t-submodular function**, for short.



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Example 1: k**-submodular functions** due to Huber and Kolmogorov (ISCO 2012).

For any $T, T' \in \mathcal{T}$ define **binary operations** $\sqcup$ **and** $\sqcap$ **on** $\mathcal{T}$ by

$$\begin{aligned} T \sqcup T' &= (T \cup T') \setminus \bigcup \{U \in \mathcal{U} \mid |U \cap (T \cup T')| = 2\}, \\ T \sqcap T' &= T \cap T'. \end{aligned}$$

Let $k = \max\{|U| \mid U \in \mathcal{U}\}$.

A function $f : \mathcal{T} \to \mathbb{R}$ is called k**-submodular** if

$$f(T) + f(T') \geq f(T \sqcup T') + f(T \sqcap T') \qquad (\forall T,\ T' \in \mathcal{T}).$$

We assume $f(\emptyset) = 0$.

$\rightarrow$

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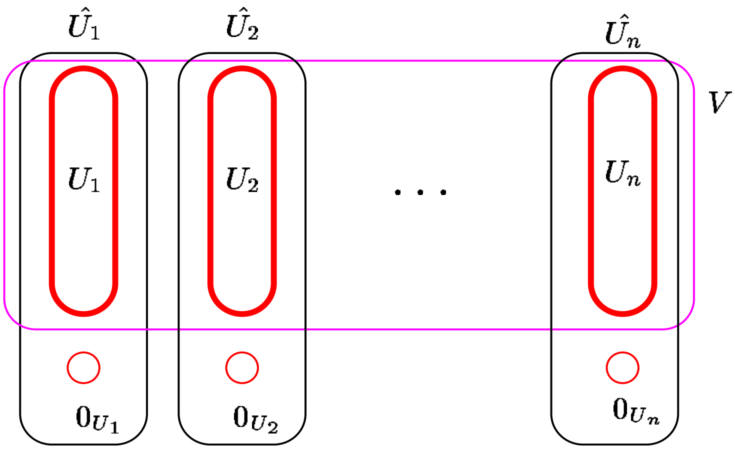
We assume $f(\emptyset) = 0$.

Remark: Bouchet (1997) considered k-submodular functions (monotone nondecreasing and unit-increasing) to define a set system called a **multimatroid** as a generalization of delta-matroids.

$\rightarrow$

Example 2: Submodular functions on product lattices and, in particular, diamonds due to Kuivinen (*Discrete Optimization*, 2011).

0_U: a new element for each $U \in \mathcal{U}$
Put $\hat{U} = U \cup \{0_U\}$ for each $U \in \mathcal{U}$.



An arbitrary **lattice** $\mathcal{L}_U = (\hat{U}, \vee_U, \wedge_U)$ with lattice operations, join $\vee_U$ and meet $\wedge_U$, for each $U \in \mathcal{U}$

0_U: the minimum element of $\mathcal{L}_U$.
1_U: the maximum element of $\mathcal{L}_U$.

$\rightarrow$

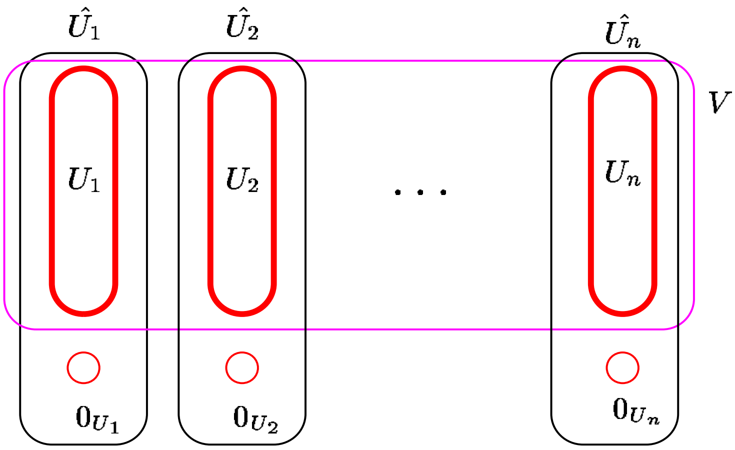
Let $\mathcal{L} = \otimes_{U \in \mathcal{U}} \mathcal{L}_U (= (\otimes_{U \in \mathcal{U}} \hat{U}, \vee, \wedge))$ be the **product of lattices** $\mathcal{L}_U = (\hat{U}, \vee_U, \wedge_U)$ for $U \in \mathcal{U}$.

A function $f : \otimes_{U \in \mathcal{U}} \hat{U} \to \mathbb{R}$ is called a **submodular function on product lattice** $\mathcal{L}$ if

$$f(\hat{T}) + f(\hat{T}') \geq f(\hat{T} \vee \hat{T}') + f(\hat{T} \wedge \hat{T}')$$

for all $\hat{T}, \hat{T}' \in \otimes_{U \in \mathcal{U}} \hat{U}$.

This function can be regarded as a special case of t-submodular functions by discarding minimum elements 0_U for all $U \in \mathcal{U}$.



$\rightarrow$

A Min-Max Theorem for T-submodular Functions

Let $f : \mathcal{T} \to \mathbb{R}$ be a **t-submodular function**.
Define a function $F : 2^{\mathcal{U}} \to \mathbb{R}$ as follows.

$$F(\mathcal{X}) = \min\{f(T) \mid T \in \mathcal{T},\, \mathcal{U}(T) \subseteq \mathcal{X}\} \quad (\forall \mathcal{X} \subseteq \mathcal{U}).$$

Lemma 1: *$F : 2^{\mathcal{U}} \to \mathbb{R}$ is a submodular function on $2^{\mathcal{U}}$ with $F(\emptyset)$=0.*

(Proof) For any $\mathcal{X},\ \mathcal{Y} \subseteq \mathcal{U}$ there exist $T_{\mathcal{X}},\ T_{\mathcal{Y}} \in \mathcal{T}$ such that

$$\mathcal{U}(T_{\mathcal{X}}) \subseteq \mathcal{X}, \quad \mathcal{U}(T_{\mathcal{Y}}) \subseteq \mathcal{Y}, \quad F(\mathcal{X}) = f(T_{\mathcal{X}}), \quad F(\mathcal{Y}) = f(T_{\mathcal{Y}}).$$

Hence we have

$$\begin{aligned} F(\mathcal{X}) + F(\mathcal{Y}) &= f(T_{\mathcal{X}}) + f(T_{\mathcal{Y}}) \\ &\geq f(T_{\mathcal{X}} \bigtriangledown T_{\mathcal{Y}}) + f(T_{\mathcal{X}} \bigtriangleup T_{\mathcal{Y}}) \\ &\geq F(\mathcal{X} \cup \mathcal{Y}) + F(\mathcal{X} \cap \mathcal{Y}). \end{aligned}$$

We also have $F(\emptyset) = f(\emptyset) = 0$. $\square$

$\rightarrow$

We can easily see that

$$\min\{f(T) \mid T \in \mathcal{T}\} = \min\{F(\mathcal{X}) \mid \mathcal{X} \subseteq \mathcal{U}\}.$$

Hence we have the following.

Lemma 2:

$$\min\{f(T) \mid T \in \mathcal{T}\} = \max\{x(\mathcal{U}) \mid x \leq \mathbf{0},\ x \in \mathrm{P}(F)\},$$

where $\mathrm{P}(F) = \{x \in \mathbb{R}^{\mathcal{U}} \mid \forall \mathcal{X} \subseteq \mathcal{U} : x(\mathcal{X}) \leq F(\mathcal{X})\}$, *the submodular polyhedron associated with submodular function* F *and* $x(\mathcal{X}) = \sum_{U \in \mathcal{X}} x(U)$.

(Proof) This follows from Edmonds' min-max theorem for submodular function minimization. □

It should be noted that since F is monotone non-increasing, every $x \in \mathrm{P}(F)$ is nonpositive, so that we may suppress the condition $x \leq \mathbf{0}$ appearing in Lemma 2.

$\rightarrow$

For any $x \in \mathbb{R}^{\mathcal{U}}$ define $z_x \in \mathbb{R}^V$ by

$$z_x(v) = x(U(v)) \qquad (\forall v \in V).$$

Here it should be noted that $x(U(v))$ is the value of $x \in \mathbb{R}^{\mathcal{U}}$ for the coordinate $U(v) \in \mathcal{U}$.

Lemma 3: Suppose we are given a nonpositive $x \in \mathbb{R}^{\mathcal{U}}$, i.e., $x \leq \mathbf{0}$. Then, we have $x \in \mathrm{P}(F)$ if and only if $z_x \in \mathrm{P}(f)$, where

$$\mathrm{P}(f) = \{z \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : z(T)(\equiv \sum_{v \in T} z(v)) \leq f(T)\}.$$

(Proof) Suppose $x \in \mathrm{P}(F)$. Then, for any $T \in \mathcal{T}$

$$z_x(T) = x(\mathcal{U}(T)) \leq F(\mathcal{U}(T)) \leq f(T).$$

Hence $z_x \in \mathrm{P}(f)$. Conversely, suppose $z_x \in \mathrm{P}(f)$ for $x \in \mathbb{R}^{\mathcal{U}}$ with $x \leq \mathbf{0}$. Then, for any $\mathcal{X} \subseteq \mathcal{U}$ and any $T \in \mathcal{T}$ such that $\mathcal{U}(T) \subseteq \mathcal{X}$ we have

$$x(\mathcal{X}) \leq x(\mathcal{U}(T)) = z_x(T) \leq f(T),$$

where the first inequality holds since $x \leq \mathbf{0}$. This implies

$$x(\mathcal{X}) \leq \min\{f(T) \mid T \in \mathcal{T}, \mathcal{U}(T) \subseteq \mathcal{X}\} = F(\mathcal{X}).$$

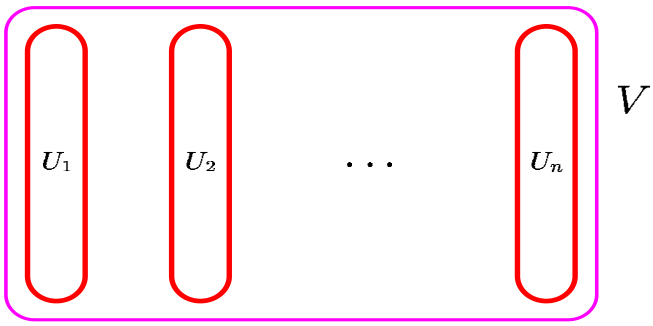
Hence $x \in \mathrm{P}(F)$. $\square$

$\rightarrow$

For any $z \in \mathbb{R}^V$ define

$$||z||_{1,\infty} = \sum_{i=1}^{n} \max_{u \in U_i} |z(u)|.$$

This defines a **norm** on $\mathbb{R}^V$, which is a composition of ℓ_1 and ℓ_∞ norms.



Remark: $||\cdot||_{1,\infty} = ||\cdot||_1$ if $|U_i| = 1$ for all $i = 1, \cdots, n$, and $||\cdot||_{1,\infty} = ||\cdot||_\infty$ if $n = 1$.

$\rightarrow$

We are now ready to show the following.

Theorem 4: For any t-submodular function f with $f(\emptyset) = 0$ we have the following min-max relation.

$$\min\{f(T) \mid T \in \mathcal{T}\} = \max\{-||z||_{1,\infty} \mid z \in \mathrm{P}(f)\}.$$

Moreover, if f is integer-valued, there exists an integral vector z that attains the maximum on the right-hand side.

(Proof) Denote the right-hand side by RHS. It follows from Lemmas 2 and 3 that

$$\begin{aligned}
\mathrm{RHS} &= \max\{-||z||_{1,\infty} \mid z \leq \mathbf{0},\ z \in \mathrm{P}(f)\} \\
&= \max\{-||z_x||_{1,\infty} \mid x \in \mathbb{R}^{\mathcal{U}}, x \leq \mathbf{0},\ z_x \in \mathrm{P}(f)\} \\
&= \max\{x(\mathcal{U}) \mid x \leq \mathbf{0},\ x \in \mathrm{P}(F)\} \\
&= \min\{F(\mathcal{X}) \mid \mathcal{X} \subseteq \mathcal{U}\} \\
&= \min\{f(T) \mid T \in \mathcal{T}\},
\end{aligned}$$

where the first and second equalities are due to the hereditary property of polyhedron $\mathrm{P}(f)$ and the definition of $|| \cdot ||_{1,\infty}$.

Moreover, if f is integer-valued, then so is the corresponding submodular function $F : 2^{\mathcal{U}} \to \mathbb{R}$. Therefore, there exists an integral $x \in \mathbb{R}^{\mathcal{U}}$ that attains the maximum on the right-hand side. Then $z_x \in \mathbb{R}^V$ defined for Lemma 3 is an integral maximizer of the right-hand side, due to Lemmas 2 and 3. $\square$

$\rightarrow$

Consider k**-submodular functions** due to Huber and Kolmogorov (2012).

As a corollary of Theorem 4 we get

Corollary 5: For any k-submodular function $f : \mathcal{T} \to \mathbb{R}$ with $f(\emptyset) = 0$

$$\min\{f(T) \mid T \in \mathcal{U}\} = \max\{-||z||_{1,\infty} \mid z \in \mathrm{P}(f)\}.$$

Moreover, if f is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

Huber and Kolmogorov (2012) considered

$$\mathrm{P}_2(f) = \{z \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : \ z(T) \leq f(T), \\ \forall U \in \mathcal{U}, \ \forall X \in \binom{U}{2} : \ z(X) \leq 0\}.$$

Note that we have

$$\mathrm{P}(f) \cap \mathbb{R}^V_{\leq 0} = \mathrm{P}_2(f) \cap \mathbb{R}^V_{\leq 0} \subseteq \mathrm{P}_2(f) \subseteq \mathrm{P}(f), \qquad (1)$$

where $\mathbb{R}^V_{\leq 0}$ is the set of all nonpositive vectors in $\mathbb{R}^V$.

$\rightarrow$

For polyhedron $\mathrm{P}_2(f)$ considered by Huber and Kolmogorov (2012) we have the following.

Theorem 6: For any k-submodular function $f : \mathcal{T} \to \mathbb{R}$ with $f(\emptyset) = 0$

$$\min\{f(T) \mid T \in \mathcal{U}\} = \max\{-||z||_{1,\infty} \mid z \in \mathrm{P}_2(f)\}.$$

Moreover, if f is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

$\rightarrow$

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$$\min\{f(T) \mid T \in \mathcal{U}\} = \max\{-||z||_{1,\infty} \mid z \in \mathrm{P}_2(f)\}.$$

Moreover, if f is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

Remark: A good characterization of minimizing k-submodular functions is open. $\square$

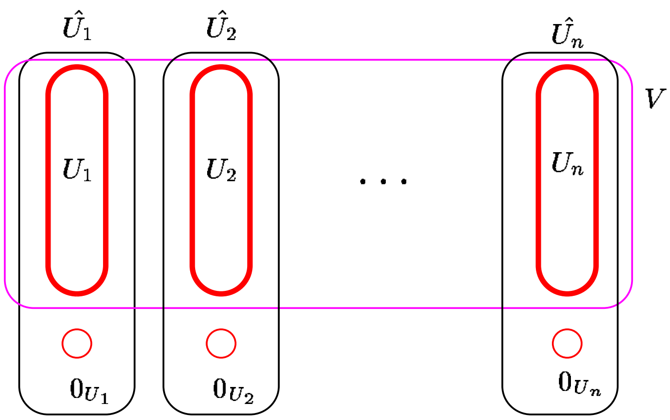
$\rightarrow$

Now, consider a **submodular function f on product lattice** $\mathcal{L} = \otimes_{U \in \mathcal{U}} \mathcal{L}_U$, which is identified with a function $\bar{f}$ on $\mathcal{T}$ defined by

$$\bar{f}(T) = f(\hat{T}) \qquad (\forall T \in \mathcal{T}) \quad (\hat{T} \in \mathcal{L}).$$

We then have function $\bar{f}$ satisfying

$$\bar{f}(T) + \bar{f}(T') \geq \bar{f}(T \vee_0 T') + \bar{f}(T \wedge_0 T') \qquad (\forall T,\, T' \in \mathcal{T}).$$



$\rightarrow$

Define

$$\mathrm{P}(\bar{f}) = \{z \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : z(T) \leq \bar{f}(T)\}.$$

Here we assume $\bar{f}(\emptyset) = 0$.

As a corollary of Theorem 4 we obtain the following.

Corollary 7: For any submodular function f on the product of lattices with $\bar{f}(\emptyset) = 0$

$$\min\{\bar{f}(T) \mid T \in \mathcal{U}\} = \max\{-||z||_{1,\infty} \mid z \in \mathrm{P}(\bar{f})\}.$$

Moreover, if $\bar{f}$ is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

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Consider the following additional constraint:

(K1′) For each $U \in \mathcal{U}$, $z(u) + z(v) \leq z(u \vee_U v) + z(u \wedge_U v)$ for all $\{u, v\} \in \binom{\hat{U}}{2}$, where $z(0_U) = 0$ for all $U \in \mathcal{U}$.

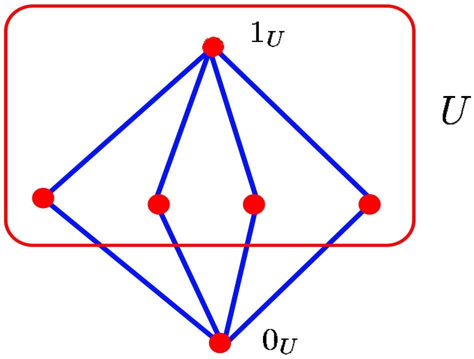
Corollary 8: For any submodular function f on the product of lattices with $\bar{f}(\emptyset) = 0$

$$\min\{\bar{f}(T) \mid T \in \mathcal{U}\} = \max\{-||z||_{1,\infty} \mid z \in \mathrm{P}(\bar{f}), \text{(K1')}\}.$$

Moreover, if $\bar{f}$ is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

$\rightarrow$

We assume that $|U| \geq 3$ and all the elements in $U \setminus \{1_U\}$ are incomparable in $\mathcal{L}_U$ for each $U \in \mathcal{U}$. Then lattice $\mathcal{L}_U$ on $\hat{U} = U \cup \{0_U\}$ is called a **diamond**.



We assume that for each $U \in \mathcal{U}$ $\mathcal{L}_U$ is a diamond.

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Corollary 8 gives a min-max formula for a submodular function on the product lattice of **diamonds**. Note that in this special case (K1′) is simplified to

(K1′) For each $U \in \mathcal{U}$, $z(u)+z(v) \leq z(1_U)$ for all $\{u, v\} \in \binom{\bar{U}}{2}$, where $\bar{U} = U \setminus \{1_U\}$.

Kuivinen (2011) further considered stronger constraints:

(K1) For each $U \in \mathcal{U}$
$z(1_U) = \max\{z(u) + z(v) \mid \{u, v\} \in \binom{\bar{U}}{2}\}$.

(K2) For each $U \in \mathcal{U}$ there exists $p \in \bar{U}$ such that $z(p) \geq z(v)$ for all $v \in \bar{U}$ and $z(u) = z(v)$ for all $u, v \in \bar{U} \setminus \{p\}$. (Such a z is called *unified* by Kuivinen (2011).)

Note that (K1) implies (K1′).

$\rightarrow$

Kuivinen (2011) showed the following.

Theorem 9:

$$\min\{\bar{f}(T) \mid T \in \mathcal{U}\}$$
$$= \max\{\sum_{U \in \mathcal{U}} z(1_U) \mid z \in \mathrm{P}(\bar{f}),\ z \leq \mathbf{0},\ \textbf{(K1)},\ \textbf{(K2)}\}.$$

Moreover, if $\bar{f}$ is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

Remark: Kuivinen (2011) showed that this gives a good characterization for submodular functions on diamonds.

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