

A Min-Max Theorem for Transversal Submodular Functions and Its Implications

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(Joint work with [Shin-ichi Tanigawa](#), RIMS, Kyoto University)

The present talk will

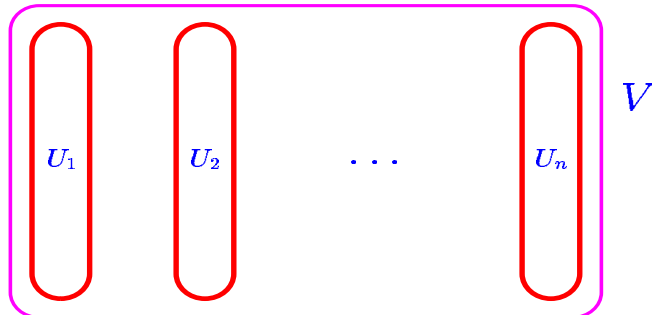
1. introduce a new concept of **transversal submodular function**, a generalization of ordinary submodular set function,
2. show a **min-max relation between the minimum of a transversal submodular function and the maximum of the negative of a norm composed of ℓ_1 and ℓ_∞ norms**, and
3. based on the min-max relation, give a **unifying view over the recent results** on generalizations of submodular set functions:

[1] A. Huber and V. Kolmogorov: Towards minimizing k -submodular functions. Proceedings of ISCO 2012, LNCS **7422** (2012) 451–462.

[2] F. Kuivinen: On the complexity of submodular function minimisation on diamonds. *Discrete Optimization* **8** (2011) 459–477.

V : a nonempty finite set

$\mathcal{U} \equiv \{U_1, U_2, \dots, U_n\}$: a **partition** of V



$T(\subseteq V)$: a **subtransversal** (or **partial transversal**) of \mathcal{U}
($|T \cap U| \leq 1$ for all $U \in \mathcal{U}$)

\mathcal{T} : the set of all subtransversals of \mathcal{U}

$\mathcal{U}(T) = \{U \in \mathcal{U} \mid U \cap T \neq \emptyset\}$ ($\forall T \in \mathcal{T}$)

$U(v)$: the unique $U \in \mathcal{U}$ that contains $v \in V$

We consider a function $f : \mathcal{T} \rightarrow \mathbb{R}$.

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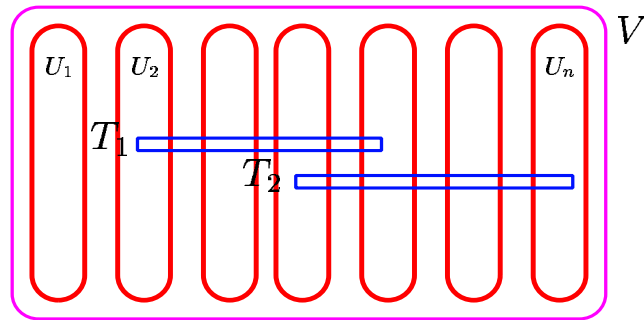
Consider two **binary operations** ∇ and Δ on \mathcal{T} satisfying the condition that for all $T_1, T_2 \in \mathcal{T}$

$$\begin{aligned} T_1 \nabla T_2 &\in \mathcal{T}, & \mathcal{U}(T_1 \nabla T_2) &\subseteq \mathcal{U}(T_1) \cup \mathcal{U}(T_2), \\ T_1 \Delta T_2 &\in \mathcal{T}, & \mathcal{U}(T_1 \Delta T_2) &\subseteq \mathcal{U}(T_1) \cap \mathcal{U}(T_2). \end{aligned}$$

Define a function $f : \mathcal{T} \rightarrow \mathbb{R}$ with $f(\emptyset) = 0$ satisfying

$$f(T_1) + f(T_2) \geq f(T_1 \nabla T_2) + f(T_1 \Delta T_2) \quad (\forall T_1, T_2 \in \mathcal{T}).$$

We call f a **transversal submodular function** or a **t-submodular function**, for short.



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Example 1: *k*-submodular functions due to Huber and Kolmogorov (ISCO 2012).

For any $T, T' \in \mathcal{T}$ define **binary operations** \sqcup and \sqcap on \mathcal{T} by

$$\begin{aligned} T \sqcup T' &= (T \cup T') \setminus \bigcup \{U \in \mathcal{U} \mid |U \cap (T \cup T')| = 2\}, \\ T \sqcap T' &= T \cap T'. \end{aligned}$$

Let $k = \max\{|U| \mid U \in \mathcal{U}\}$.

A function $f : \mathcal{T} \rightarrow \mathbb{R}$ is called ***k*-submodular** if

$$f(T) + f(T') \geq f(T \sqcup T') + f(T \sqcap T') \quad (\forall T, T' \in \mathcal{T}).$$

We assume $f(\emptyset) = 0$.

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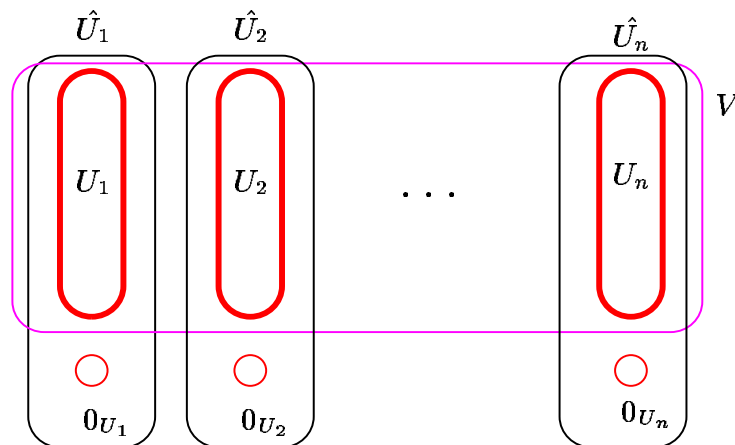
Remark: Bouchet (1997) considered k -submodular functions (monotone nondecreasing and unit-increasing) to define a set system called a **multimatroid** as a generalization of delta-matroids.

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Example 2: Submodular functions on product lattices and, in particular, diamonds due to Kuivinen (*Discrete Optimization*, 2011).

0_U : a **new element** for each $U \in \mathcal{U}$

Put $\hat{U} = U \cup \{0_U\}$ for each $U \in \mathcal{U}$.



An **arbitrary lattice** $\mathcal{L}_U = (\hat{U}, \vee_U, \wedge_U)$ with lattice operations, join \vee_U and meet \wedge_U , for each $U \in \mathcal{U}$

0_U : the **minimum element** of \mathcal{L}_U .

1_U : the **maximum element** of \mathcal{L}_U .

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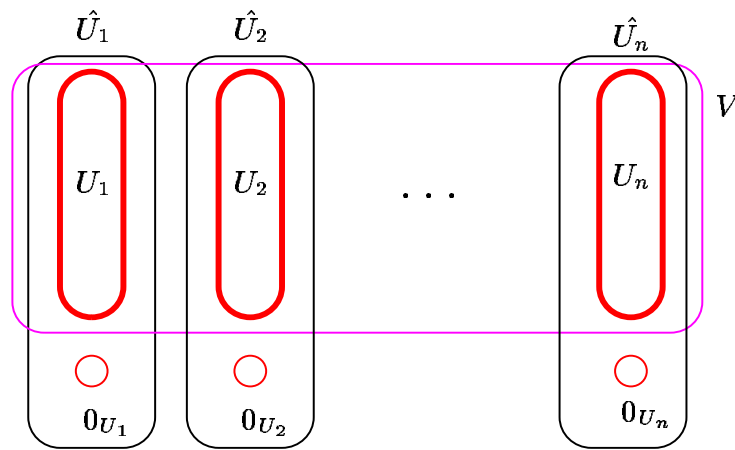
Let $\mathcal{L} = \otimes_{U \in \mathcal{U}} \mathcal{L}_U (= (\otimes_{U \in \mathcal{U}} \hat{U}, \vee, \wedge))$ be the **product of lattices** $\mathcal{L}_U = (\hat{U}, \vee_U, \wedge_U)$ for $U \in \mathcal{U}$.

A function $f : \otimes_{U \in \mathcal{U}} \hat{U} \rightarrow \mathbb{R}$ is called a **submodular function on product lattice \mathcal{L}** if

$$f(\hat{T}) + f(\hat{T}') \geq f(\hat{T} \vee \hat{T}') + f(\hat{T} \wedge \hat{T}')$$

for all $\hat{T}, \hat{T}' \in \otimes_{U \in \mathcal{U}} \hat{U}$.

This function can be regarded as a special case of t-submodular functions by discarding minimum elements 0_U for all $U \in \mathcal{U}$.



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A Min-Max Theorem for T-submodular Functions

Let $f : \mathcal{T} \rightarrow \mathbb{R}$ be a **t-submodular function**.

Define a function $F : 2^{\mathcal{U}} \rightarrow \mathbb{R}$ as follows.

$$F(\mathcal{X}) = \min\{f(T) \mid T \in \mathcal{T}, \mathcal{U}(T) \subseteq \mathcal{X}\} \quad (\forall \mathcal{X} \subseteq \mathcal{U}).$$

Lemma 1: $F : 2^{\mathcal{U}} \rightarrow \mathbb{R}$ is a submodular function on $2^{\mathcal{U}}$ with $F(\emptyset)=0$.

(Proof) For any $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{U}$ there exist $T_{\mathcal{X}}, T_{\mathcal{Y}} \in \mathcal{T}$ such that $\mathcal{U}(T_{\mathcal{X}}) \subseteq \mathcal{X}$, $\mathcal{U}(T_{\mathcal{Y}}) \subseteq \mathcal{Y}$, $F(\mathcal{X}) = f(T_{\mathcal{X}})$, $F(\mathcal{Y}) = f(T_{\mathcal{Y}})$.

Hence we have

$$\begin{aligned} F(\mathcal{X}) + F(\mathcal{Y}) &= f(T_{\mathcal{X}}) + f(T_{\mathcal{Y}}) \\ &\geq f(T_{\mathcal{X}} \nabla T_{\mathcal{Y}}) + f(T_{\mathcal{X}} \Delta T_{\mathcal{Y}}) \\ &\geq F(\mathcal{X} \cup \mathcal{Y}) + F(\mathcal{X} \cap \mathcal{Y}). \end{aligned}$$

We also have $F(\emptyset) = f(\emptyset) = 0$. □

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We can easily see that

$$\min\{f(T) \mid T \in \mathcal{T}\} = \min\{F(\mathcal{X}) \mid \mathcal{X} \subseteq \mathcal{U}\}.$$

Hence we have the following.

Lemma 2:

$$\min\{f(T) \mid T \in \mathcal{T}\} = \max\{x(\mathcal{U}) \mid x \leq \mathbf{0}, x \in P(F)\},$$

where $P(F) = \{x \in \mathbb{R}^{\mathcal{U}} \mid \forall \mathcal{X} \subseteq \mathcal{U} : x(\mathcal{X}) \leq F(\mathcal{X})\}$, the submodular polyhedron associated with submodular function F and $x(\mathcal{X}) = \sum_{U \in \mathcal{X}} x(U)$.

(Proof) This follows from Edmonds' min-max theorem for submodular function minimization. \square

It should be noted that since F is monotone non-increasing, every $x \in P(F)$ is nonpositive, so that we may suppress the condition $x \leq \mathbf{0}$ appearing in Lemma 2.

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For any $x \in \mathbb{R}^{\mathcal{U}}$ define $z_x \in \mathbb{R}^V$ by

$$z_x(v) = x(U(v)) \quad (\forall v \in V).$$

Here it should be noted that $x(U(v))$ is the value of $x \in \mathbb{R}^{\mathcal{U}}$ for the coordinate $U(v) \in \mathcal{U}$.

Lemma 3: Suppose we are given a nonpositive $x \in \mathbb{R}^{\mathcal{U}}$, i.e., $x \leq \mathbf{0}$. Then, we have $x \in P(F)$ if and only if $z_x \in P(f)$, where

$$P(f) = \{z \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : z(T) (\equiv \sum_{v \in T} z(v)) \leq f(T)\}.$$

(Proof) Suppose $x \in P(F)$. Then, for any $T \in \mathcal{T}$

$$z_x(T) = x(\mathcal{U}(T)) \leq F(\mathcal{U}(T)) \leq f(T).$$

Hence $z_x \in P(f)$. Conversely, suppose $z_x \in P(f)$ for $x \in \mathbb{R}^{\mathcal{U}}$ with $x \leq \mathbf{0}$. Then, for any $\mathcal{X} \subseteq \mathcal{U}$ and any $T \in \mathcal{T}$ such that $\mathcal{U}(T) \subseteq \mathcal{X}$ we have

$$x(\mathcal{X}) \leq x(\mathcal{U}(T)) = z_x(T) \leq f(T),$$

where the first inequality holds since $x \leq \mathbf{0}$. This implies

$$x(\mathcal{X}) \leq \min\{f(T) \mid T \in \mathcal{T}, \mathcal{U}(T) \subseteq \mathcal{X}\} = F(\mathcal{X}).$$

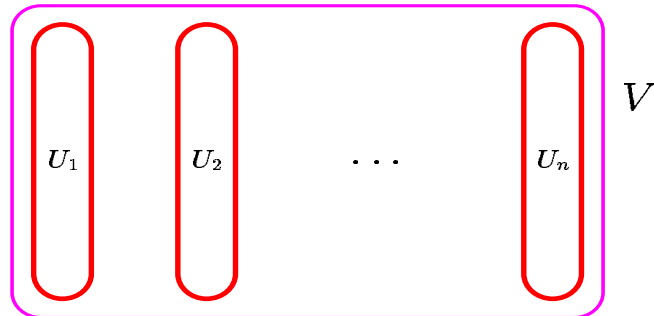
Hence $x \in P(F)$. □

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For any $z \in \mathbb{R}^V$ define

$$\|z\|_{1,\infty} = \sum_{i=1}^n \max_{u \in U_i} |z(u)|.$$

This defines a **norm on \mathbb{R}^V** , which is a composition of ℓ_1 and ℓ_∞ norms.



Remark: $\|\cdot\|_{1,\infty} = \|\cdot\|_1$ if $|U_i| = 1$ for all $i = 1, \dots, n$,
and $\|\cdot\|_{1,\infty} = \|\cdot\|_\infty$ if $n = 1$.

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We are now ready to show the following.

Theorem 4: For any t-submodular function f with $f(\emptyset) = 0$ we have the following min-max relation.

$$\min\{f(T) \mid T \in \mathcal{T}\} = \max\{-\|z\|_{1,\infty} \mid z \in P(f)\}.$$

Moreover, if f is integer-valued, there exists an integral vector z that attains the maximum on the right-hand side.

(Proof) Denote the right-hand side by RHS. It follows from Lemmas 2 and 3 that

$$\begin{aligned} \text{RHS} &= \max\{-\|z\|_{1,\infty} \mid z \leq \mathbf{0}, z \in P(f)\} \\ &= \max\{-\|z_x\|_{1,\infty} \mid x \in \mathbb{R}^{\mathcal{U}}, x \leq \mathbf{0}, z_x \in P(f)\} \\ &= \max\{x(\mathcal{U}) \mid x \leq \mathbf{0}, x \in P(F)\} \\ &= \min\{F(\mathcal{X}) \mid \mathcal{X} \subseteq \mathcal{U}\} \\ &= \min\{f(T) \mid T \in \mathcal{T}\}, \end{aligned}$$

where the first and second equalities are due to the hereditary property of polyhedron $P(f)$ and the definition of $\|\cdot\|_{1,\infty}$.

Moreover, if f is integer-valued, then so is the corresponding submodular function $F : 2^{\mathcal{U}} \rightarrow \mathbb{R}$. Therefore, there exists an integral $x \in \mathbb{R}^{\mathcal{U}}$ that attains the maximum on the right-hand side. Then $z_x \in \mathbb{R}^{\mathcal{V}}$ defined for Lemma 3 is an integral maximizer of the right-hand side, due to Lemmas 2 and 3. \square

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Consider **k -submodular functions** due to Huber and Kolmogorov (2012).

As a corollary of Theorem 4 we get

Corollary 5: For any k -submodular function $f : \mathcal{T} \rightarrow \mathbb{R}$ with $f(\emptyset) = 0$

$$\min\{f(T) \mid T \in \mathcal{U}\} = \max\{-\|z\|_{1,\infty} \mid z \in P(f)\}.$$

Moreover, if f is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

Huber and Kolmogorov (2012) considered

$$P_2(f) = \{z \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : z(T) \leq f(T), \\ \forall U \in \mathcal{U}, \forall X \in \binom{U}{2} : z(X) \leq 0\}.$$

Note that we have

$$P(f) \cap \mathbb{R}_{\leq 0}^V = P_2(f) \cap \mathbb{R}_{\leq 0}^V \subseteq P_2(f) \subseteq P(f), \quad (1)$$

where $\mathbb{R}_{\leq 0}^V$ is the set of all nonpositive vectors in \mathbb{R}^V .

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For polyhedron $P_2(f)$ considered by Huber and Kolmogorov (2012) we have the following.

Theorem 6: For any k -submodular function $f : \mathcal{T} \rightarrow \mathbb{R}$ with $f(\emptyset) = 0$

$$\min\{f(T) \mid T \in \mathcal{U}\} = \max\{-\|z\|_{1,\infty} \mid z \in P_2(f)\}.$$

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Moreover, if f is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

Remark: A good characterization of minimizing k -submodular functions is open. □

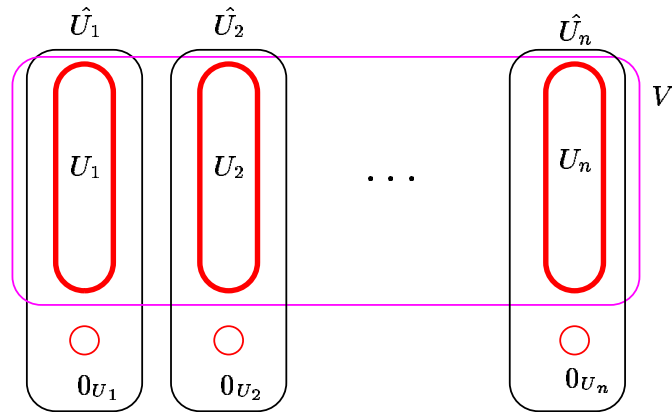
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Now, consider a **submodular function f on product lattice $\mathcal{L} = \otimes_{U \in \mathcal{U}} \mathcal{L}_U$** , which is identified with a function \bar{f} on \mathcal{T} defined by

$$\bar{f}(T) = f(\hat{T}) \quad (\forall T \in \mathcal{T}) \quad (\hat{T} \in \mathcal{L}).$$

We then have function \bar{f} satisfying

$$\bar{f}(T) + \bar{f}(T') \geq \bar{f}(T \vee_0 T') + \bar{f}(T \wedge_0 T') \quad (\forall T, T' \in \mathcal{T}).$$



→

Define

$$P(\bar{f}) = \{z \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : z(T) \leq \bar{f}(T)\}.$$

Here we assume $\bar{f}(\emptyset) = 0$.

As a corollary of Theorem 4 we obtain the following.

Corollary 7: For any submodular function f on the product of lattices with $\bar{f}(\emptyset) = 0$

$$\min\{\bar{f}(T) \mid T \in \mathcal{U}\} = \max\{-\|z\|_{1,\infty} \mid z \in P(\bar{f})\}.$$

Moreover, if \bar{f} is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

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Consider the following additional constraint:

(K1') For each $U \in \mathcal{U}$, $z(u) + z(v) \leq z(u \vee_U v) + z(u \wedge_U v)$
for all $\{u, v\} \in \binom{\hat{U}}{2}$, where $z(0_U) = 0$ for all $U \in \mathcal{U}$.

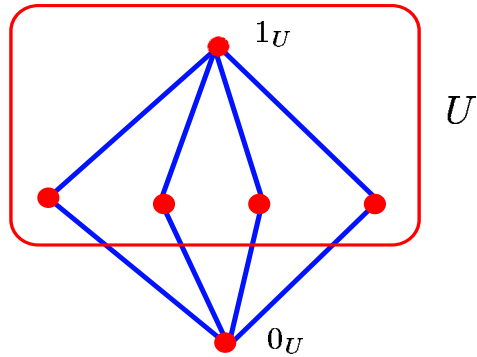
Corollary 8: For any submodular function f on the product of lattices with $\bar{f}(\emptyset) = 0$

$$\min\{\bar{f}(T) \mid T \in \mathcal{U}\} = \max\{-\|z\|_{1,\infty} \mid z \in P(\bar{f}), (\mathbf{K1}')\}.$$

Moreover, if \bar{f} is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

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We assume that $|U| \geq 3$ and all the elements in $U \setminus \{1_U\}$ are incomparable in \mathcal{L}_U for each $U \in \mathcal{U}$. Then lattice \mathcal{L}_U on $\hat{U} = U \cup \{0_U\}$ is called a **diamond**.



We assume that for each $U \in \mathcal{U}$ \mathcal{L}_U is a diamond.

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Corollary 8 gives a min-max formula for a submodular function on the **product lattice of diamonds**. Note that in this special case (K1') is simplified to

(K1') For each $U \in \mathcal{U}$, $z(u) + z(v) \leq z(1_U)$ for all $\{u, v\} \in \binom{\bar{U}}{2}$, where $\bar{U} = U \setminus \{1_U\}$.

Kuivinen (2011) further considered stronger constraints:

(K1) For each $U \in \mathcal{U}$

$$z(1_U) = \max\{z(u) + z(v) \mid \{u, v\} \in \binom{\bar{U}}{2}\}.$$

(K2) For each $U \in \mathcal{U}$ there exists $p \in \bar{U}$ such that $z(p) \geq z(v)$ for all $v \in \bar{U}$ and $z(u) = z(v)$ for all $u, v \in \bar{U} \setminus \{p\}$. (Such a z is called *unified* by Kuivinen (2011).)

Note that (K1) implies (K1').

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Kuivinen (2011) showed the following.

Theorem 9:

$$\begin{aligned} & \min\{\bar{f}(T) \mid T \in \mathcal{U}\} \\ & = \max\left\{\sum_{U \in \mathcal{U}} z(1_U) \mid z \in P(\bar{f}), z \leq \mathbf{0}, (\mathbf{K1}), (\mathbf{K2})\right\}. \end{aligned}$$

Moreover, if \bar{f} is integer-valued, then there exists an integral z that attains the maximum on the right-hand side.

Remark: Kuivinen (2011) showed that this gives a good characterization for submodular functions on diamonds.

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