A Min-Max Theorem for Transversal Submodular Functions and Its Implications

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The present talk will

1. introduce a new concept of **transversal submodular function**, a generalization of ordinary submodular set function,

2. show a **min-max relation** between the minimum of a transversal submodular function and the maximum of the negative of a norm composed of $\ell_1$ and $\ell_\infty$ norms, and

3. based on the min-max relation, give a **unifying view over the recent results** on generalizations of submodular set functions:


\( V \): a nonempty finite set

\( \mathcal{U} \equiv \{U_1, U_2, \ldots, U_n\} \): a **partition** of \( V \)

\[ \begin{array}{cccc}
U_1 & U_2 & \cdots & U_n \\
\end{array} \]

\( T(\subseteq V) \): a **subtransversal** (or **partial transversal**) of \( \mathcal{U} \)

\(|T \cap U| \leq 1\) for all \( U \in \mathcal{U} \)

\( \mathcal{T} \): the set of all subtransversals of \( \mathcal{U} \)

\( \mathcal{U}(T) = \{U \in \mathcal{U} \mid U \cap T \neq \emptyset\} \) \hspace{1em} (\forall T \in \mathcal{T})

\( U(v) \): the unique \( U \in \mathcal{U} \) that contains \( v \in V \)

We consider a function \( f : \mathcal{T} \to \mathbb{R} \).
Consider two \textbf{binary operations} $\triangledown$ and $\triangle$ on $\mathcal{T}$ satisfying the condition that for all $T_1, T_2 \in \mathcal{T}$

\[
T_1 \triangledown T_2 \in \mathcal{T}, \quad \mathcal{U}(T_1 \triangledown T_2) \subseteq \mathcal{U}(T_1) \cup \mathcal{U}(T_2),
\]
\[
T_1 \triangle T_2 \in \mathcal{T}, \quad \mathcal{U}(T_1 \triangle T_2) \subseteq \mathcal{U}(T_1) \cap \mathcal{U}(T_2).
\]

Define a function $f : \mathcal{T} \to \mathbb{R}$ with $f(\emptyset) = 0$ satisfying

\[
f(T_1) + f(T_2) \geq f(T_1 \triangledown T_2) + f(T_1 \triangle T_2) \quad (\forall T_1, T_2 \in \mathcal{T}).
\]

We call $f$ a \textbf{transversal submodular function} or a \textbf{t-submodular function}, for short.
Example 1: \textit{k-submodular functions} due to Huber and Kolmogorov (ISCO 2012).

For any \( T, T' \in \mathcal{T} \) define \textbf{binary operations} \( \sqcup \) and \( \sqcap \) on \( \mathcal{T} \) by

\[
T \sqcup T' = (T \cup T') \setminus \bigcup\{U \in \mathcal{U} \mid |U \cap (T \cup T')| = 2\},
\]

\[
T \sqcap T' = T \cap T'.
\]

Let \( k = \max\{|U| \mid U \in \mathcal{U}\} \).
A function \( f : \mathcal{T} \to \mathbb{R} \) is called \textbf{k-submodular} if

\[
f(T) + f(T') \geq f(T \sqcup T') + f(T \sqcap T') \quad (\forall T, T' \in \mathcal{T}).
\]

We assume \( f(\emptyset) = 0 \).
**Example 1:** $k$-submodular functions due to Huber and Kolmogorov (ISCO 2012).

For any $T, T' \in \mathcal{T}$ define **binary operations** $\sqcup$ and $\sqcap$ on $\mathcal{T}$ by

\[
T \sqcup T' = (T \cup T') \setminus \bigcup \{U \in \mathcal{U} \mid |U \cap (T \cup T')| = 2\}, \\
T \sqcap T' = T \cap T'.
\]

Let $k = \max\{|U| \mid U \in \mathcal{U}\}$.

A function $f : \mathcal{T} \to \mathbb{R}$ is called **$k$-submodular** if

\[
f(T) + f(T') \geq f(T \sqcup T') + f(T \sqcap T') \quad (\forall T, T' \in \mathcal{T}).
\]

We assume $f(\emptyset) = 0$.

**Remark:** Bouchet (1997) considered $k$-submodular functions (monotone nondecreasing and unit-increasing) to define a set system called a **multimatroid** as a generalization of delta-matroids.
**Example 2:** Submodular functions on product lattices and, in particular, diamonds due to Kuivinen (*Discrete Optimization*, 2011).

0\(_U\): a new element for each \( U \in \mathcal{U} \)

Put \( \hat{U} = U \cup \{0_U\} \) for each \( U \in \mathcal{U} \).

An arbitrary lattice \( \mathcal{L}_U = (\hat{U}, \lor_U, \land_U) \) with lattice operations, join \( \lor_U \) and meet \( \land_U \), for each \( U \in \mathcal{U} \)

0\(_U\): the minimum element of \( \mathcal{L}_U \).

1\(_U\): the maximum element of \( \mathcal{L}_U \).
Let $\mathcal{L} = \bigotimes_{U \in \mathcal{U}} \mathcal{L}_U (= (\bigotimes_{U \in \mathcal{U}} \hat{U}, \vee, \wedge))$ be the product of lattices $\mathcal{L}_U = (\hat{U}, \vee_U, \wedge_U)$ for $U \in \mathcal{U}$.

A function $f : \bigotimes_{U \in \mathcal{U}} \hat{U} \to \mathbb{R}$ is called a submodular function on product lattice $\mathcal{L}$ if

$$f(\hat{T}) + f(\hat{T}') \geq f(\hat{T} \vee \hat{T}') + f(\hat{T} \wedge \hat{T}')$$

for all $\hat{T}, \hat{T}' \in \bigotimes_{U \in \mathcal{U}} \hat{U}$.

This function can be regarded as a special case of t-submodular functions by discarding minimum elements $0_U$ for all $U \in \mathcal{U}$.
A Min-Max Theorem for T-submodular Functions

Let $f : \mathcal{T} \to \mathbb{R}$ be a \textit{t-submodular function}.
Define a function $F : 2^\mathcal{U} \to \mathbb{R}$ as follows.

$$
F(\mathcal{X}) = \min \{ f(T) \mid T \in \mathcal{T}, \mathcal{U}(T) \subseteq \mathcal{X} \} \quad (\forall \mathcal{X} \subseteq \mathcal{U}).
$$

\textbf{Lemma 1:} $F : 2^\mathcal{U} \to \mathbb{R}$ is a submodular function on $2^\mathcal{U}$ with $F(\emptyset) = 0$.

(Proof) For any $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{U}$ there exist $T_{\mathcal{X}}, T_{\mathcal{Y}} \in \mathcal{T}$ such that

$\mathcal{U}(T_{\mathcal{X}}) \subseteq \mathcal{X}$, $\mathcal{U}(T_{\mathcal{Y}}) \subseteq \mathcal{Y}$, $F(\mathcal{X}) = f(T_{\mathcal{X}})$, $F(\mathcal{Y}) = f(T_{\mathcal{Y}})$.

Hence we have

$$
F(\mathcal{X}) + F(\mathcal{Y}) = f(T_{\mathcal{X}}) + f(T_{\mathcal{Y}}) \\
\geq f(T_{\mathcal{X}} \triangle T_{\mathcal{Y}}) + f(T_{\mathcal{X}} \triangle T_{\mathcal{Y}}) \\
\geq F(\mathcal{X} \cup \mathcal{Y}) + F(\mathcal{X} \cap \mathcal{Y}).
$$

We also have $F(\emptyset) = f(\emptyset) = 0$.  \hfill  \Box
We can easily see that

\[ \min\{f(T) \mid T \in \mathcal{T}\} = \min\{F(\mathcal{X}) \mid \mathcal{X} \subseteq \mathcal{U}\}. \]

Hence we have the following.

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**Lemma 2:**

\[ \min\{f(T) \mid T \in \mathcal{T}\} = \max\{x(\mathcal{U}) \mid x \leq 0, x \in P(F)\}, \]

where \( P(F) = \{x \in \mathbb{R}^d \mid \forall \mathcal{X} \subseteq \mathcal{U} : x(\mathcal{X}) \leq F(\mathcal{X})\} \), the submodular polyhedron associated with submodular function \( F \) and \( x(\mathcal{X}) = \sum_{U \in \mathcal{X}} x(U) \).

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(Proof) This follows from Edmonds’ min-max theorem for submodular function minimization. \( \square \)

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It should be noted that since \( F \) is monotone non-increasing, every \( x \in P(F) \) is nonpositive, so that we may suppress the condition \( x \leq 0 \) appearing in Lemma 2.
For any \( x \in \mathbb{R}^{U} \) define \( z_x \in \mathbb{R}^{V} \) by
\[
z_x(v) = x(U(v)) \quad (\forall v \in V).
\]
Here it should be noted that \( x(U(v)) \) is the value of \( x \in \mathbb{R}^{U} \) for the coordinate \( U(v) \in U \).

**Lemma 3:** Suppose we are given a nonpositive \( x \in \mathbb{R}^{U} \), i.e., \( x \leq 0 \). Then, we have \( x \in \mathcal{P}(\mathcal{F}) \) if and only if \( z_x \in \mathcal{P}(f) \), where
\[
\mathcal{P}(f) = \{ z \in \mathbb{R}^{V} \mid \forall T \in \mathcal{T} : z(T)(\equiv \sum_{v \in T} z(v)) \leq f(T) \}.
\]

(Proof) Suppose \( x \in \mathcal{P}(\mathcal{F}) \). Then, for any \( T \in \mathcal{T} \)
\[
z_x(T) = x(U(T)) \leq F(U(T)) \leq f(T).
\]
Hence \( z_x \in \mathcal{P}(f) \). Conversely, suppose \( z_x \in \mathcal{P}(f) \) for \( x \in \mathbb{R}^{U} \) with \( x \leq 0 \). Then, for any \( \mathcal{X} \subseteq \mathcal{U} \) and any \( T \in \mathcal{T} \) such that \( U(T) \subseteq \mathcal{X} \) we have
\[
x(\mathcal{X}) \leq x(U(T)) = z_x(T) \leq f(T),
\]
where the first inequality holds since \( x \leq 0 \). This implies
\[
x(\mathcal{X}) \leq \min\{ f(T) \mid T \in \mathcal{T}, \mathcal{U}(T) \subseteq \mathcal{X} \} = F(\mathcal{X}).
\]
Hence \( x \in \mathcal{P}(\mathcal{F}) \). \( \square \)
For any $z \in \mathbb{R}^V$ define

$$||z||_{1,\infty} = \sum_{i=1}^{n} \max_{u \in U_i} |z(u)|.$$  

This defines a norm on $\mathbb{R}^V$, which is a composition of $\ell_1$ and $\ell_\infty$ norms.

**Remark:** $|| \cdot ||_{1,\infty} = || \cdot ||_1$ if $|U_i| = 1$ for all $i = 1, \ldots, n,$ and $|| \cdot ||_{1,\infty} = || \cdot ||_\infty$ if $n = 1.$
We are now ready to show the following.

**Theorem 4**: For any t-submodular function \( f \) with \( f(\emptyset) = 0 \) we have the following min-max relation.

\[
\min\{f(T) \mid T \in \mathcal{T}\} = \max\{-\|z\|_{1,\infty} \mid z \in P(f)\}.
\]

Moreover, if \( f \) is integer-valued, there exists an integral vector \( z \) that attains the maximum on the right-hand side.

(Proof) Denote the right-hand side by RHS. It follows from Lemmas 2 and 3 that

\[
\text{RHS} = \max\{-\|z\|_{1,\infty} \mid z \leq 0, z \in P(f)\} = \max\{-\|z_x\|_{1,\infty} \mid x \in \mathbb{R}^\mathcal{U}, x \leq 0, z_x \in P(f)\} = \max\{x(\mathcal{U}) \mid x \leq 0, x \in P(F)\} = \min\{F(\mathcal{X}) \mid \mathcal{X} \subseteq \mathcal{U}\} = \min\{f(T) \mid T \in \mathcal{T}\},
\]

where the first and second equalities are due to the hereditary property of polyhedron \( P(f) \) and the definition of \( \|\cdot\|_{1,\infty} \).

Moreover, if \( f \) is integer-valued, then so is the corresponding submodular function \( F : 2^\mathcal{U} \to \mathbb{R} \). Therefore, there exists an integral \( x \in \mathbb{R}^\mathcal{U} \) that attains the maximum on the right-hand side. Then \( z_x \in \mathbb{R}^V \) defined for Lemma 3 is an integral maximizer of the right-hand side, due to Lemmas 2 and 3.
Consider \textit{k-submodular functions} due to Huber and Kolmogorov (2012).

As a corollary of Theorem 4 we get

\textbf{Corollary 5:} For any \textit{k}-submodular function \( f : \mathcal{T} \rightarrow \mathbb{R} \) with \( f(\emptyset) = 0 \)

\[ \min \{ f(T) \mid T \in \mathcal{U} \} = \max \{ -\|z\|_{1,\infty} \mid z \in P(f) \}. \]

Moreover, if \( f \) is integer-valued, then there exists an integral \( z \) that attains the maximum on the right-hand side.

Huber and Kolmogorov (2012) considered

\[ P_2(f) = \{ z \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : z(T) \leq f(T), \]

\[ \forall U \in \mathcal{U}, \forall X \in \binom{U}{2} : z(X) \leq 0 \}. \]

Note that we have

\[ P(f) \cap \mathbb{R}^V_{\leq 0} = P_2(f) \cap \mathbb{R}^V_{\leq 0} \subseteq P_2(f) \subseteq P(f), \quad (1) \]

where \( \mathbb{R}^V_{\leq 0} \) is the set of all nonpositive vectors in \( \mathbb{R}^V \).
For polyhedron $P_2(f)$ considered by Huber and Kolmogorov (2012) we have the following.

**Theorem 6**: For any $k$-submodular function $f : \mathcal{T} \to \mathbb{R}$ with $f(\emptyset) = 0$

$$\min\{f(T) \mid T \in \mathcal{U}\} = \max\{-\|z\|_{1,\infty} \mid z \in P_2(f)\}.$$ 

Moreover, if $f$ is integer-valued, then there exists an integral $z$ that attains the maximum on the right-hand side.
For polyhedron $P_2(f)$ considered by Huber and Kolmogorov (2012) we have the following.

Theorem 6: For any $k$-submodular function $f : \mathcal{T} \rightarrow \mathbb{R}$ with $f(\emptyset) = 0$

$$\min\{f(T) \mid T \in \mathcal{U}\} = \max\{-||z||_{1,\infty} \mid z \in P_2(f)\}.$$ 

Moreover, if $f$ is integer-valued, then there exists an integral $z$ that attains the maximum on the right-hand side.

Remark: A good characterization of minimizing $k$-submodular functions is open.
Now, consider a submodular function $f$ on product lattice $L = \bigotimes_{U \in \mathcal{U}} L_U$, which is identified with a function $\bar{f}$ on $T$ defined by

$$\bar{f}(T) = f(\hat{T}) \quad (\forall T \in T) \quad (\hat{T} \in L).$$

We then have function $\bar{f}$ satisfying

$$\bar{f}(T) + \bar{f}(T') \geq \bar{f}(T \lor_0 T') + \bar{f}(T \land_0 T') \quad (\forall T, T' \in T).$$
Define
\[ P(\bar{f}) = \{ z \in \mathbb{R}^V \mid \forall T \in \mathcal{T} : z(T) \leq \bar{f}(T) \}. \]

Here we assume \( \bar{f}(\emptyset) = 0 \).

As a corollary of Theorem 4 we obtain the following.

**Corollary 7**: For any submodular function \( f \) on the product of lattices with \( \bar{f}(\emptyset) = 0 \)

\[ \min \{ \bar{f}(T) \mid T \in \mathcal{U} \} = \max \{ -\|z\|_{1,\infty} \mid z \in P(\bar{f}) \}. \]

Moreover, if \( \bar{f} \) is integer-valued, then there exists an integral \( z \) that attains the maximum on the right-hand side.
Consider the following additional constraint:

\((K1')\) For each \(U \in \mathcal{U}\), \(z(u) + z(v) \leq z(u \lor_U v) + z(u \land_U v)\)
for all \(\{u, v\} \in \binom{\mathcal{U}}{2}\), where \(z(0_U) = 0\) for all \(U \in \mathcal{U}\).

Corollary 8: For any submodular function \(f\) on the product of lattices with \(\bar{f}(\emptyset) = 0\)
\[
\min\{\bar{f}(T) \mid T \in \mathcal{U}\} = \max\{-||z||_{1,\infty} \mid z \in \mathcal{P}(\bar{f}), (K1')\}.
\]
Moreover, if \(\bar{f}\) is integer-valued, then there exists an integral \(z\) that attains the maximum on the right-hand side.
We assume that $|U| \geq 3$ and all the elements in $U \setminus \{1_U\}$ are incomparable in $\mathcal{L}_U$ for each $U \in \mathcal{U}$. Then lattice $\mathcal{L}_U$ on $\hat{U} = U \cup \{0_U\}$ is called a **diamond**.

We assume that for each $U \in \mathcal{U}$ $\mathcal{L}_U$ is a diamond.
Corollary 8 gives a min-max formula for a submodular function on the product lattice of diamonds. Note that in this special case (K1’) is simplified to

(K1’) For each $U \in \mathcal{U}$, $z(u) + z(v) \leq z(1_U)$ for all $\{u, v\} \in \binom{\bar{U}}{2}$, where $\bar{U} = U \setminus \{1_U\}$.

Kuivinen (2011) further considered stronger constraints:

(K1) For each $U \in \mathcal{U}$
$$z(1_U) = \max \{z(u) + z(v) \mid \{u, v\} \in \binom{\bar{U}}{2} \}.$$  

(K2) For each $U \in \mathcal{U}$ there exists $p \in \bar{U}$ such that $z(p) \geq z(v)$ for all $v \in \bar{U}$ and $z(u) = z(v)$ for all $u, v \in \bar{U} \setminus \{p\}$. (Such a $z$ is called unified by Kuivinen (2011).)

Note that (K1) implies (K1’).
Kuivinen (2011) showed the following.

Theorem 9:

$$\min\{\bar{f}(T) \mid T \in \mathcal{U}\} = \max\left\{\sum_{U \in \mathcal{U}} z(1_U) \mid z \in \text{P}(\bar{f}), \ z \leq 0, \ (K1), \ (K2)\right\}.$$ 

Moreover, if $\bar{f}$ is integer-valued, then there exists an integral $z$ that attains the maximum on the right-hand side.

Remark: Kuivinen (2011) showed that this gives a good characterization for submodular functions on diamonds.