

Valid Inequalities for the 1-Restricted Simple 2-Matching Polytope

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Outline

- **Definitions and known results:**
 - Simple 2-matching and *k-restricted* simple 2-matching
 - A simple IP formulation for 1-restricted simple 2-matchings
 - Extreme points of the LP relaxation
- **Valid inequalities for the integral polytope:**
 - *1-restricted blossom*
 - *r-1 blossom inequalities* and *r-2 blossom inequalities*
- **Characterization of the facet-inducing**
 - *r-1 blossom inequalities*
 - *r-2 blossom inequalities*
- **Depth of the facet-inducing *r-2* blossom inequalities**
- **Open questions**

Definitions and Known Results

- **Simple 2-matching:**
 - Maximum weight simple 2-matching in a general graph
 - Polyhedral characterization (Edmonds, 1965) and polynomial-time algorithm (Johnson, 1965)
- **K -restricted simple 2-matching:**
 - A k -restricted simple 2-matching only consists of simple paths and cycles of length more than k .
 - Increasingly accurate approximate solutions to the Hamiltonian paths or cycles as k increases
 - Polynomial-time algorithm for finding a 1-restricted simple 2-matching with max. number of edges (Hartvigsen, 2007)
 - Polynomial-time algorithm for finding a 1-restricted simple 2-matching that covers maximum number of nodes (Hartvigsen, Hell and Szabó, 2006)

A Simple IP Formulation

- **IP formulation of 1-restricted simple 2-matchings:**

$$x(\mathcal{D}(v)) \leq 2 \quad \forall v \in V$$

$$x_e - x(\text{adj}(e)) \leq 0 \quad \forall e \in E$$

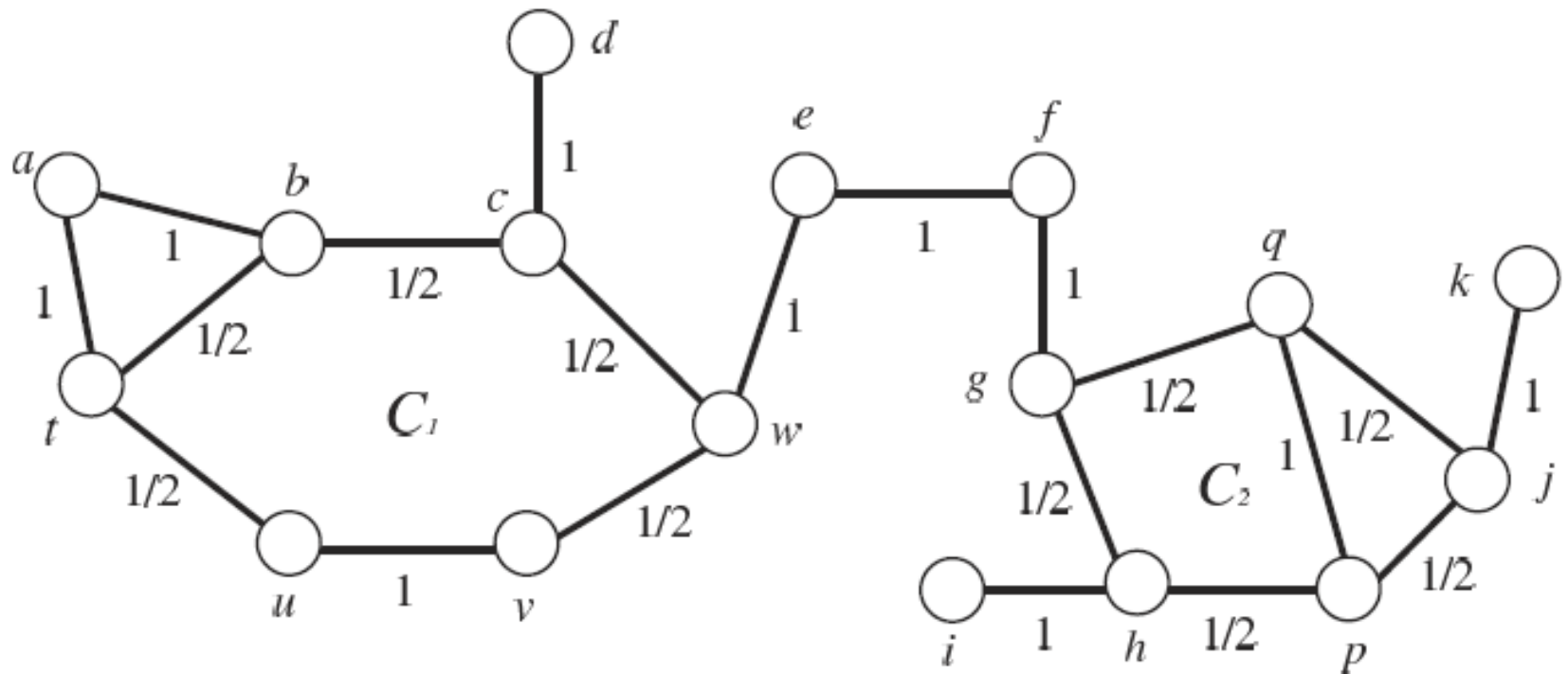
$$x_e \in \{0,1\} \quad \forall e \in E$$

- **$H(\mathbf{G})$** : convex hull of the integer solutions.
- **$R(\mathbf{G})$** : LP relaxation of the IP formulation.

Extreme Points of $R(G)$

- **1-tight edge** ($e = uv$): If e has value 1, one edge incident to u has value $\frac{1}{2}$, one edge incident to v has value $\frac{1}{2}$, and the remaining edges adjacent to e have value 0.
- **Characterization of the extreme points** (Hartvigsen and Li, 2008):
 1. The point x only has values 0, $\frac{1}{2}$, and 1.
 2. Let G_c be the graph obtained from G by contracting all 1-tight edges. Then the edges of G_c with value $\frac{1}{2}$ form node-disjoint odd cycles, and every real node in such an odd cycle is incident with an edge with value 1.

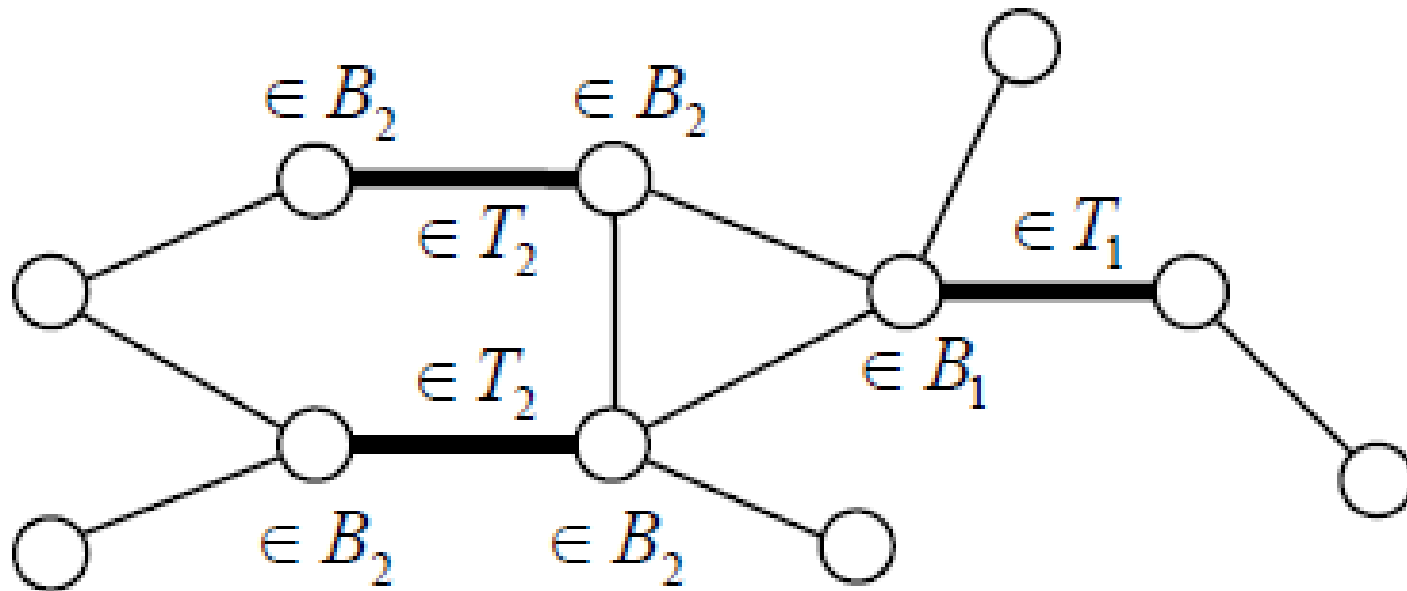
Example: An Extreme Point of $R(G)$



1-Restricted Blossom

- B : a subset of $V(G)$ with at least three nodes.
- T : a 1-matching in G with odd cardinality such that each edge in T is incident with at least one node in B .
- T_i ($i = 1, 2$): the set of edges in T incident with exactly i nodes of B .
- B_i ($i = 1, 2$): the set of nodes in B incident with an edge in T_i .
- $B = B_1 \cup B_2$.
- **1-restricted blossom**: denoted by $G(B, T)$, the subgraph induced by the edges in T and their adjacent edges.

Example: 1-Restricted Blossom



Valid Inequalities for $H(G)$ Defined on 1-Restricted Blossom

- ***r-1 blossom inequality*** ($ax \leq b$):

$$a_{uv} = \begin{cases} +1, & \text{if } uv \in T; \text{ or } u, v \in B_1 \\ -1, & \text{if } u \in B_2, v \notin B; \text{ or } u, v \in B_2, uv \notin T_2 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{and } b = |T_1| + \frac{|T| - 1}{2}.$$

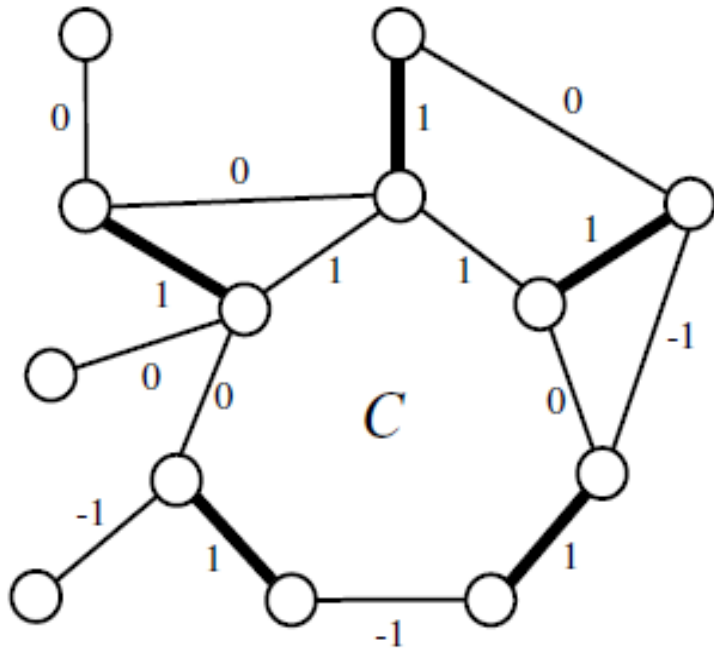
Valid Inequalities for $H(G)$ Defined on 1-Restricted Blossom

- ***r-2 blossom inequality*** ($cx \leq d$):

$$c_{uv} = \begin{cases} +2, & \text{if } uv \in T_2 \\ +1, & \text{if } uv \in T_1 \\ -1, & \text{if } u \in V(T_1), v \notin B, uv \notin T_1; \text{ or } u \in B_1, v \in B_2 \\ -2, & \text{if } u \in B_2, v \notin B; \text{ or } u, v \in B_2, uv \notin T_2 \\ 0, & \text{otherwise} \end{cases}$$

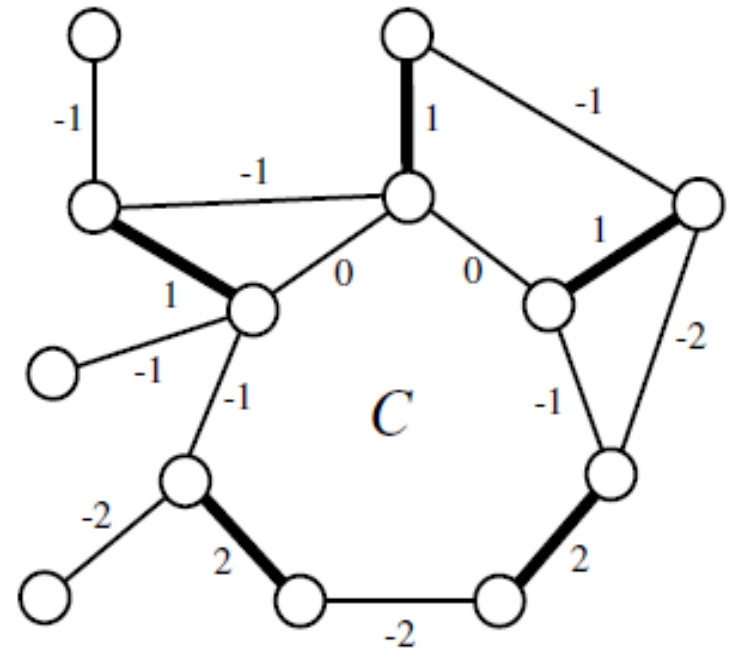
and $d = |T| - 1$.

Example: $r-1$ and $r-2$ Blossom Inequalities



right-hand side = 5

$r-1$ blossom inequality



right-hand side = 4

$r-2$ blossom inequality

Derivation of the Valid Inequalities

- **$r-1$ blossom inequality:**

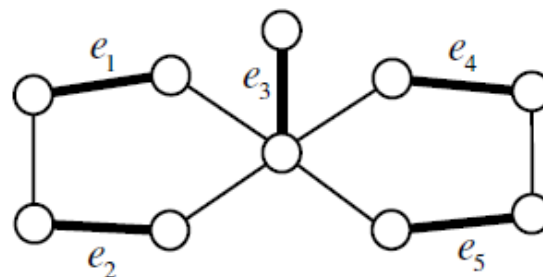
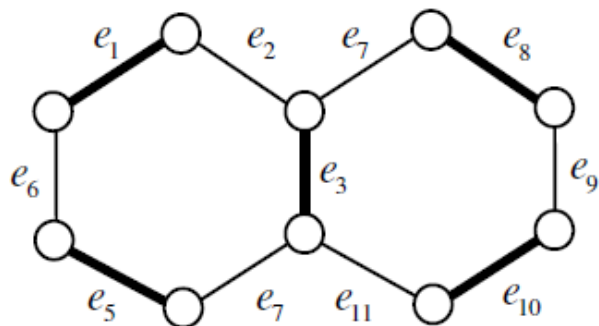
- Sum up the node-degree inequalities at the B_1 nodes, the edge-adjacency inequalities at the T_2 edges, and the upper-bound inequalities at the T edges.
- Divide both sides by 2, and round down the numbers.

- **$r-2$ blossom inequality:**

- Sum up two times the edge-adjacency inequalities at the T_2 edges, two times the $r-1$ blossom inequality, the edge-adjacency inequalities at the T_1 edges, and two times the upper-bound inequalities at the T_2 edges.
- Divide both sides by 3, and round down the numbers.

More about 1-Restricted Blossom

- $G(B)$: subgraph of G induced by the nodes in B .
- $G_s(B, T_2)$: the graph obtained from $G(B)$ by subdividing each edge of T_2 .
- A 1-restricted blossom is ***T-connected***:
for every edge in T , the graph obtained from $G(B)$ by deleting the nodes of the edge is connected.



Two 1-restricted blossoms that are not *T*-connected

Hypomatchable Graph

- A graph G is ***hypomatchable***: for every node v of G , there is a 1-matching that has degree 1 at every node except for v , where it has degree 0.
- **Theorem 1** (Pulleyblank and Edmonds, 1974): For a graph G with $|E(G)| \geq 3$ and a node-induced subgraph $G' = (V', E')$ of G , the inequality $\sum_{e \in E'(G')} x_e \leq \frac{|V'(G')| - 1}{2}$ is facet-inducing for the 1-matching polytope of G iff G' is 2-connected and hypomatchable.

Facet-Inducing Inequalities for $H(G)$

- **Theorem 2:** For a T -connected 1-restricted blossom $G(B, T)$ in a connected graph G , the associated $r-1$ blossom inequality is facet-inducing iff $G_s(B, T_2)$ is hypomatchable and every $u \in V(T_1) \setminus B_1$ is connected to some node $v \notin B$ through an edge.
- **Theorem 3:** For a T -connected 1-restricted blossom $G(B, T)$ in a connected graph G , the associated $r-2$ blossom inequality is facet-inducing iff $G_s(B, T_2)$ is hypomatchable.
- We guess that the T -connectedness condition can be dropped out of the above results.

Proofs of Theorem 2 and Theorem 3

- Key components of the proofs:
 - ❑ $G_s(B, T_2)$ is hypomatchable iff $G_c(B, T_2)$ is hypomatchable and $G_c^e(B, T_2)$ has a perfect 1-matching $\forall e \in T_2$.
 - ❑ To prove necessity, we define five types of 1-restricted simple 2-matchings in G . If a $r-1$ ($r-2$, *resp.*) blossom inequality is satisfied at equality by a 1-restricted simple 2-matching M , then M has Type 1, 2, or 3 (Type 3, 4, or 5, *resp.*)
 - ❑ To prove sufficiency, we construct $|E(G)|$ linearly independent 1-restricted simple 2-matchings of G that satisfy the $r-1$ ($r-2$, *resp.*) blossom inequality at equality.
 - ❑ Pulleyblank and Edmonds (1974): Every 2-connected hypomatchable G has $|E(G)|$ linearly independent hypomatchings.

Depth of Valid Inequality

- P_I : convex hull of integer points in polyhedron P .
- **Chvátal-Gomory cut**: $\alpha x \leq \lfloor \beta \rfloor$, if $\alpha x \leq \beta$ is valid for P and α has only integer components.
- **Chvátal closure P' of P** : the set of points in P that satisfy all Chvátal-Gomory cuts.
- Denote $P_0 = P$ and $P_{t+1} = (P_t)'$ for $t \geq 0$.
- Chvátal (1973): For every rational polytope P , there exists a finite $t \geq 0$ such that $P_t = P_I$.
- **Chvátal rank of P** : the smallest t satisfying $P_t = P_I$.
- **Depth of a valid inequality for P_I** : the smallest t such that the inequality is valid for P_t .

Depth of $r-2$ Blossom Inequalities

- **Theorem 4:** The depth of a facet-inducing $r-2$ blossom inequality is 1 if $T_1 = \emptyset$; 2 otherwise.
- **Remark:** T -connectedness is not needed.
- Main idea of the proof:
 - $cx \leq d$ is a facet-inducing $r-2$ blossom inequality $\rightarrow G_S(B, T_2)$ is hypomatchable \rightarrow construct an $x^* \in R(G)$ satisfying $cx^* = d + 1$ (using odd-ear decomposition)
 - $T_1 = \emptyset$: $c = 2a$ and $d = 2b$, where $ax \leq b$ is a $r-1$ blossom inequality with depth 1.
 - $T_1 \neq \emptyset$: The *g.c.f.* of the components of c is 1, and d cannot be obtained from rounding down d' , where $cx \leq d'$ is a valid inequality for $R(G)$.

Open Questions

- Is there a complete description of $H(G)$ for a general graph G ?
- Can the current facet-inducing blossom inequalities be generalized by relaxing the condition of T -connectedness ?
- Do the facet-inducing $r-1$ and $r-2$ blossom inequalities induce facets of the k -restricted simple 2-matching polytope for $k \geq 2$ and facets of the Hamiltonian path polytope ?