Valid Inequalities for the 1-Restricted Simple 2-Matching Polytope

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Outline

- **Definitions and known results:**
  - Simple 2-matching and *k-restricted* simple 2-matching
  - A simple IP formulation for 1-restricted simple 2-matchings
  - Extreme points of the LP relaxation
- **Valid inequalities for the integral polytope:**
  - *1-restricted blossom*
  - *r-1 blossom inequalities* and *r-2 blossom inequalities*
- **Characterization of the facet-inducing**
  - *r-1* blossom inequalities
  - *r-2* blossom inequalities
- **Depth of the facet-inducing *r-2* blossom inequalities**
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Definitions and Known Results

- **Simple 2-matching:**
  - ❑ Maximum weight simple 2-matching in a general graph
  - ❑ Polyhedral characterization (Edmonds, 1965) and polynomial-time algorithm (Johnson, 1965)

- ***K*-restricted simple 2-matching:**
  - ❑ A *k*-restricted simple 2-matching only consists of simple paths and cycles of length more than *k*.
  - ❑ Increasingly accurate approximate solutions to the Hamiltonian paths or cycles as *k* increases
  - ❑ Polynomial-time algorithm for finding a 1-restricted simple 2-matching with max. number of edges (Hartvigsen, 2007)
  - ❑ Polynomial-time algorithm for finding a 1-restricted simple 2-matching that covers maximum number of nodes (Hartvigsen, Hell and Szabó, 2006)

A Simple IP Formulation

- **IP formulation of 1-restricted simple 2-matchings:**

$$x(\delta(v)) \le 2 \quad \forall\, v \in V$$

$$x_e - x(\mathrm{adj}(e)) \le 0 \quad \forall\, e \in E$$

$$x_e \in \{0,1\} \quad \forall\, e \in E$$

- ***H*(*G*)**: convex hull of the integer solutions.

- ***R*(*G*)**: LP relaxation of the IP formulation.

Extreme Points of *R*(*G*)

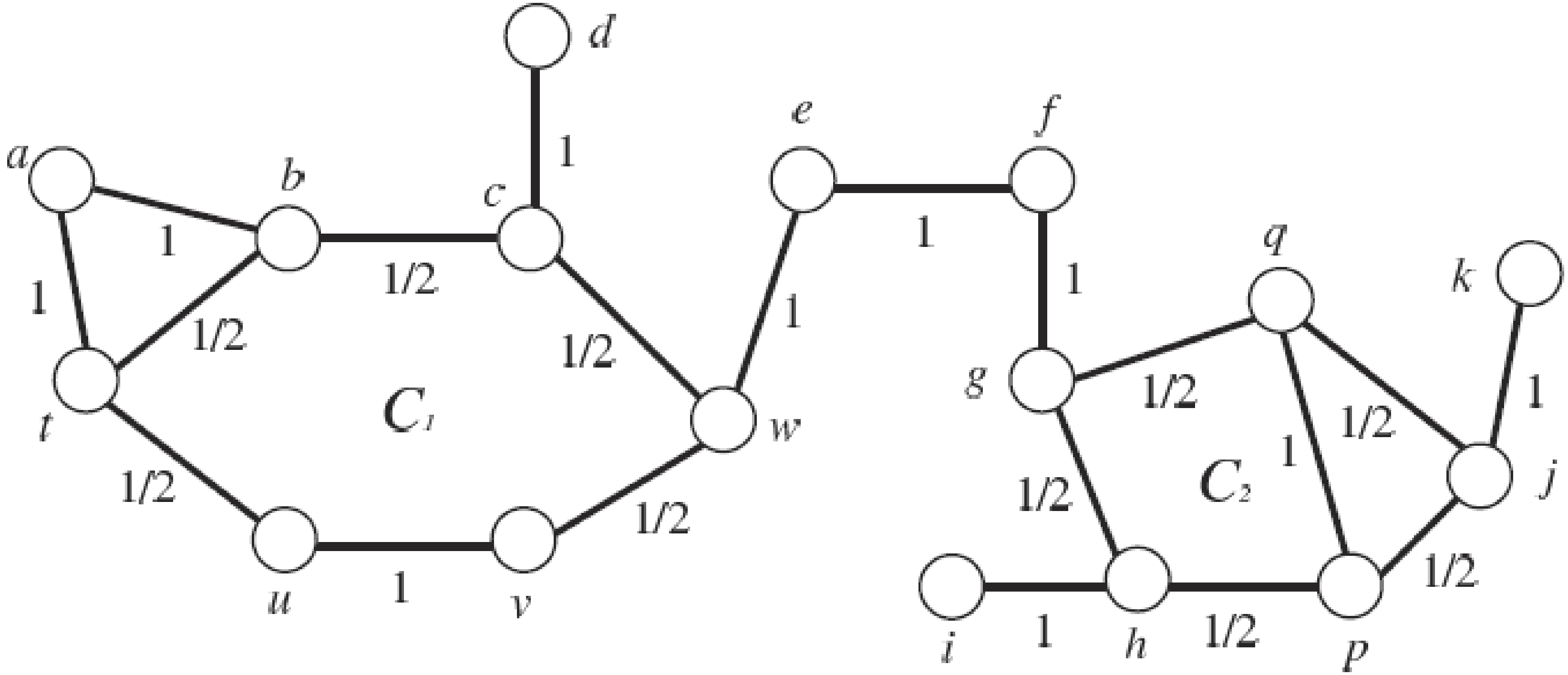
- ***1-tight* edge** (*e* = *uv*)**:** If *e* has value 1, one edge incident to *u* has value ½, one edge incident to *v* has value ½, and the remaining edges adjacent to *e* have value 0.

- **Characterization of the extreme points** (Hartvigsen and Li, 2008)**:**

1. The point *x* only has values 0, ½, and 1.

2. Let G_c be the graph obtained from G by contracting all 1-tight edges. Then the edges of G_c with value ½ form node-disjoint odd cycles, and every real node in such an odd cycle is incident with an edge with value 1.

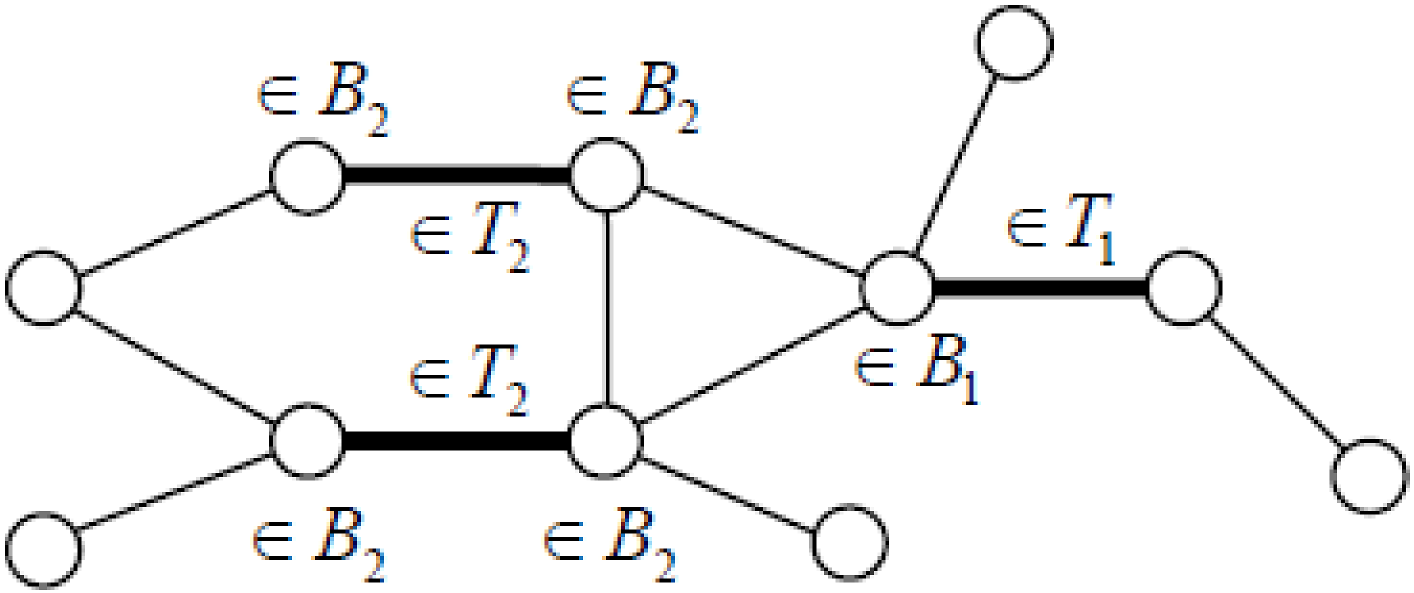
Example: An Extreme Point of *R*(*G*)



1-Restricted Blossom

- ***B***: a subset of $V(G)$ with at least three nodes.
- ***T***: a 1-matching in G with odd cardinality such that each edge in T is incident with at least one node in B.
- $\boldsymbol{T_i}$ (i = 1, 2): the set of edges in T incident with exactly i nodes of B.
- $\boldsymbol{B_i}$ (i = 1, 2): the set of nodes in B incident with an edge in T_i.
- $B = B_1 \cup B_2$.
- ***1-restricted blossom***: denoted by $G(B, T)$, the subgraph induced by the edges in T and their adjacent edges.

Example: 1-Restricted Blossom



Valid Inequalities for *H*(*G*) Defined on 1-Restricted Blossom

- ***r-1 blossom inequality*** $(\boldsymbol{ax} \leq \boldsymbol{b})$:

$$a_{uv} = \begin{cases} +1, & \text{if } uv \in T; \text{ or } u, v \in B_1 \\ -1, & \text{if } u \in B_2, v \notin B; \text{ or } u, v \in B_2, uv \notin T_2 \\ \ \ 0, & \text{otherwise} \end{cases}$$

$$\text{and } b = |T_1| + \frac{|T| - 1}{2}.$$

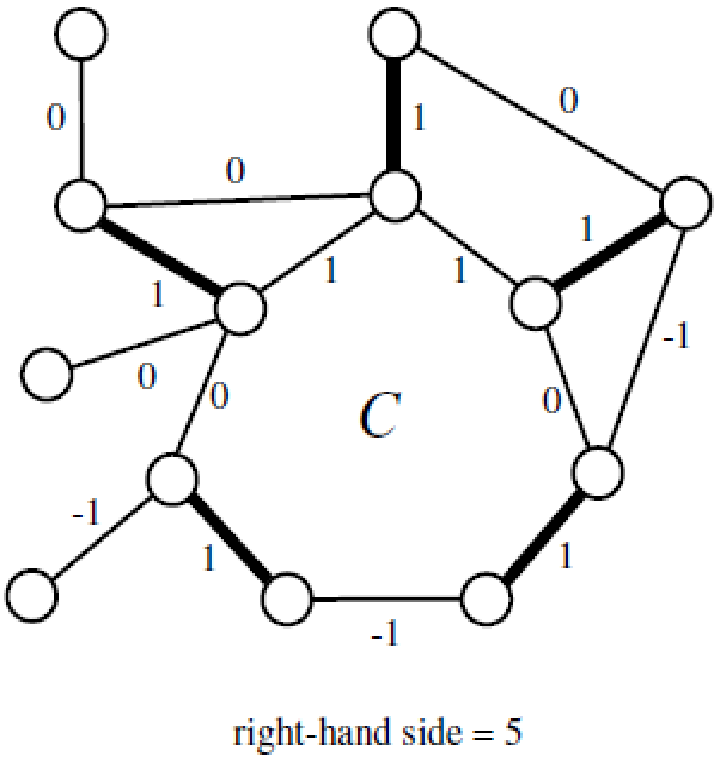
Valid Inequalities for *H*(*G*) Defined on 1-Restricted Blossom

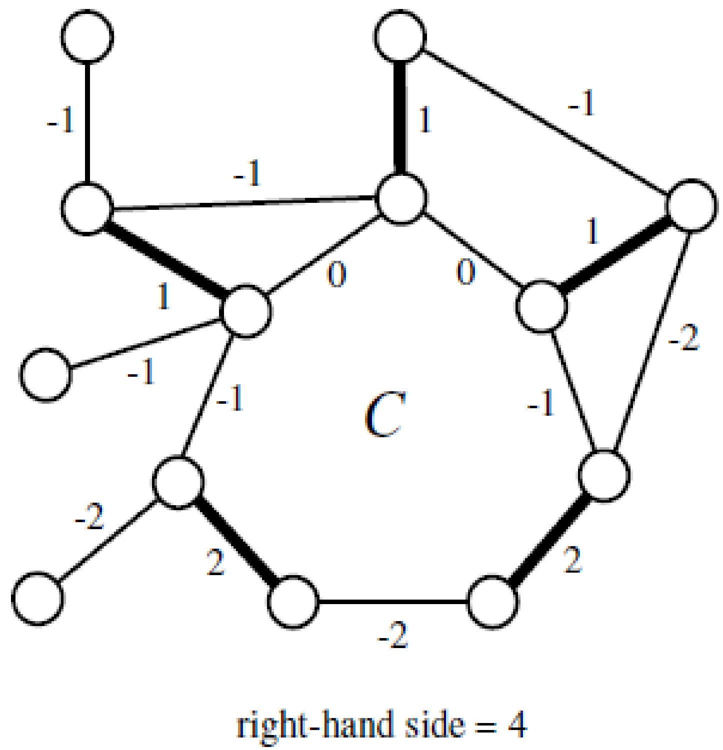
- ***r-2 blossom inequality*** $(\boldsymbol{cx} \leq \boldsymbol{d})$:

$$c_{uv} = \begin{cases} +2, \ \text{if } uv \in T_2 \\ +1, \ \text{if } uv \in T_1 \\ -1, \ \text{if } u \in V(T_1), v \notin B, uv \notin T_1; \ \text{or } u \in B_1, v \in B_2 \\ -2, \ \text{if } u \in B_2, v \notin B; \ \text{or } u, v \in B_2, uv \notin T_2 \\ \ \ 0, \ \text{otherwise} \end{cases}$$

and $d = |T| - 1.$

Example: *r-1* and *r-2* Blossom Inequalities





r-1 blossom inequality

r-2 blossom inequality

Derivation of the Valid Inequalities

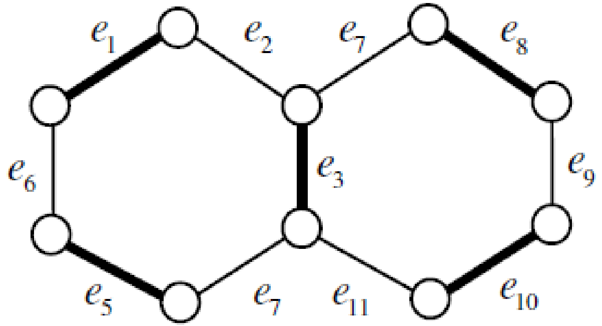
- ***r-1* blossom inequality**:
  - Sum up the node-degree inequalities at the B_1 nodes, the edge-adjacency inequalities at the T_2 edges, and the upper-bound inequalities at the T edges.
  - Divide both sides by 2, and round down the numbers.

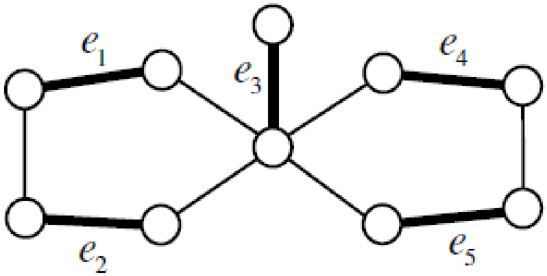
- ***r-2* blossom inequality**:
  - Sum up two times the edge-adjacency inequalities at the T_2 edges, two times the *r-1* blossom inequality, the edge-adjacency inequalities at the T_1 edges, and two times the upper-bound inequalities at the T_2 edges.
  - Divide both sides by 3, and round down the numbers.

More about 1-Restricted Blossom

- $\boldsymbol{G(B)}$: subgraph of *G* induced by the nodes in *B*.
- $\boldsymbol{G_s(B, T_2)}$: the graph obtained from $G(B)$ by sub-dividing each edge of T_2.
- A 1-restricted blossom is ***T-connected***:

  for every edge in *T*, the graph obtained from $G(B)$ by deleting the nodes of the edge is connected.





Two 1-restricted blossoms that are not *T*-connected

Hypomatchable Graph

- A graph *G* is ***hypomatchable***: for every node v of *G*, there is a 1-matching that has degree 1 at every node except for v, where it has degree 0.

- **Theorem 1** (Pulleyblank and Edmonds, 1974)**:** For a graph G with $|E(G)| \geq 3$ and a node-induced subgraph $G' = (V', E')$ of G, the inequality $\sum_{e \in E'(G')} x_e \leq \frac{|V'(G')|-1}{2}$ is facet-inducing for the 1-matching polytope of G iff G' is 2-connected and hypomatchable.

Facet-Inducing Inequalities for *H*(*G*)

- **Theorem 2:** For a *T*-connected 1-restricted blossom $G(B,T)$ in a connected graph G, the associated *r-1* blossom inequality is facet-inducing iff $G_s(B,T_2)$ is hypomatchable and every $u \in V(T_1)\backslash B_1$ is connected to some node $v \notin B$ through an edge.

- **Theorem 3:** For a *T*-connected 1-restricted blossom $G(B,T)$ in a connected graph G, the associated *r-2* blossom inequality is facet-inducing iff $G_s(B,T_2)$ is hypomatchable.

- We guess that the *T*-connectedness condition can be dropped out of the above results.

Proofs of Theorem 2 and Theorem 3

- Key components of the proofs:
  - $G_s(B, T_2)$ is hypomatchable iff $G_c(B, T_2)$ is hypomatchable and $G_c^e(B, T_2)$ has a perfect 1-matching $\forall e \in T_2$.
  - To prove necessity, we define five types of 1-restricted simple 2-matchings in G. If a *r-1* (*r-2, resp.*) blossom inequality is satisfied at equality by a 1-restricted simple 2-matching M, then M has Type 1, 2, or 3 (Type 3, 4, or 5, *resp.*)
  - To prove sufficiency, we construct $|E(G)|$ linearly independent 1-restricted simple 2-matchings of G that satisfy the *r-1* (*r-2, resp.*) blossom inequality at equality.
  - Pulleyblank and Edmonds (1974): Every 2-connected hypomatchable G has $|E(G)|$ linearly independent hypomatchings.

Depth of Valid Inequality

- $\boldsymbol{P_I}$: convex hull of integer points in polyhedron P.
- ***Chvátal-Gomory cut***: $\alpha x \leq \lfloor \beta \rfloor$, if $\alpha x \leq \beta$ is valid for P and α has only integer components.
- ***Chvátal closure $\boldsymbol{P'}$ of $\boldsymbol{P}$***: the set of points in P that satisfy all Chvátal-Gomory cuts.
- Denote $P_0 = P$ and $P_{t+1} = (P_t)'$ for $t \geq 0$.
- Chvátal (1973): For every rational polytope P, there exists a finite $t \geq 0$ such that $P_t = P_I$.
- ***Chvátal rank of $\boldsymbol{P}$***: the smallest *t* satisfying $P_t = P_I$.
- ***Depth of a valid inequality for $\boldsymbol{P_I}$***: the smallest ***t*** such that the inequality is valid for P_t.

Depth of *r-2* Blossom Inequalities

- **Theorem 4:** The depth of a facet-inducing *r-2* blossom inequality is 1 if $T_1 = \emptyset$; 2 otherwise.
- **Remark:** *T*-connectedness is not needed.
- Main idea of the proof:
  - $cx \leq d$ is a facet-inducing *r-2* blossom inequality $\longrightarrow$ $G_s(B, T_2)$ is hypomatchable $\longrightarrow$ construct an $x^* \in R(G)$ satisfying $cx^* = d + 1$ (using odd-ear decomposition)
  - $T_1 = \emptyset$: $c = 2a$ and $d = 2b$, where $ax \leq b$ is a *r-1* blossom inequality with depth 1.
  - $T_1 \neq \emptyset$: The *g.c.f.* of the components of *c* is 1, and *d* cannot be obtained from rounding down d', where $cx \leq d'$ is a valid inequality for $R(G)$.

Open Questions

- Is there a complete description of $H(G)$ for a general graph G ?

- Can the current facet-inducing blossom inequalities be generalized by relaxing the condition of *T*-connectedness ?

- Do the facet-inducing *r-1* and *r-2* blossom inequalities induce facets of the *k*-restricted simple 2-matching polytope for $k \geq 2$ and facets of the Hamiltonian path polytope ?