Indicator Constraints in Mixed-Integer Programming

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Indicator (bigM's) constraints

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- It is a well-known modeling trick in Mixed-Integer Linear Programming (MILP) to use a binary variable y multiplied by a sufficiently big (non-negative) constant M in order to deactivate constraint (1)

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- It is also well known the risk of such a modeling trick, namely
  - weak Linear Programming (LP) relaxations, and
  - numerical issues.

Complementarity Reformulation

- An alternative for logical implications and general deactivations is given by the *complementary* reformulation

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- The obvious drawback of the above reformulation is its *nonconvexity*.
- Thus, the complementary reformulation has been used so far if (and only if) the problem at hand was already nonconvex, as it is often the case, for example, in Chemical Engineering applications.

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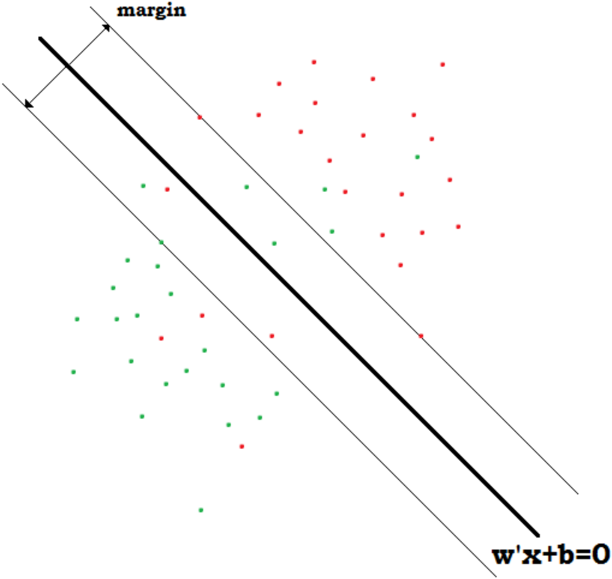
- We do that by exposing a class of Mixed-Integer Convex Quadratic Programming (MIQP) problems arising in *Supervised Classification* where the Global Optimization (GO) solver `Couenne` using reformulation (3) is consistently faster than virtually any state-of-the-art commercial MIQP solver like `IBM-Cplex`, `Gurobi` and `Xpress`.

- This is quite counter-intuitive because, in general, convex MIQPs admit more efficient solution techniques both in theory and in practice, especially by benefiting of virtually all machinery of MILP solvers.

Support Vector Machine (SVM)



The input data

- Ω: the population.
- Population is partitioned into two classes, $\{-1, +1\}$.
- For each object in Ω, we have
  - $x = (x^1, \ldots, x^d) \in X \subset \mathbb{R}^d$: predictor variables.
  - $y \in \{-1, +1\}$: class membership.

- The goal is to find a hyperplane $\omega^\top x + b = 0$ that aims at separating, if possible, the two classes.
- Future objects will be classified as

$$\begin{aligned} y = +1 \quad & \text{if} \quad \omega^\top x + b > 0 \\ y = -1 \quad & \text{if} \quad \omega^\top x + b < 0 \end{aligned} \tag{4}$$

Soft-margin approach

$$\min \frac{\omega^\top \omega}{2} + \sum_{i=1}^{n} g(\xi_i)$$

subject to

$$\begin{aligned} y_i(\omega^\top x_i + b) &\geq 1 - \xi_i \quad i = 1, \ldots, n \\ \xi_i &\geq 0 \quad i = 1, \ldots, n \\ \omega \in \mathbb{R}^d, b \in \mathbb{R} \end{aligned}$$

where n is the size of the sample and $g(\xi_i) = \frac{C}{n}\xi_i$ the most popular choice for the loss function.

Ramp Loss Model (Brooks, *OR*, 2011)

- Ramp Loss Function $g(t) = (\min\{t, 2\})^+$ yielding the Ψ-learning approach, with $(a)^+ = \max\{a, 0\}$.

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$$\min \frac{\omega^\top \omega}{2} + \frac{C}{n}(\sum_{i=1}^n \xi_i + 2\sum_{i=1}^n z_i)$$

s.t. (RLM)

$$\begin{aligned} y_i(\omega^\top x_i + b) &\geq 1 - \xi_i - Mz_i & \forall i = 1, \ldots, n \\ 0 \leq \xi_i &\leq 2 & \forall i = 1, \ldots, n \\ z &\in \{0, 1\}^n & \\ \omega \in \mathbb{R}^d, b &\in \mathbb{R} & \end{aligned}$$

with $M > 0$ big enough constant.

Expectations (and Troubles)

- In principle, RLM is a tractable Mixed-Integer Convex Quadratic Problem that nowadays commercial (and even some noncommercial) solvers should be able to solve:
  - convex objective function,
  - linear constraints, and
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  not much more difficult than a standard Mixed-Integer *Linear* Problem.
- However, the bigM constraints in the above model destroy the chances of the solver to consistently succeed for $n > 50$.

Solving the MIQP by `IBM-Cplex`

- 23 instances from Brooks, `Type B`, $n = 100$, time limit of 3,600 CPU seconds.

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IBM-Cplex			
		% gap	
time (sec.)	nodes	ub	lb
3,438.49	16,142,440	–	–
tl	12,841,549	–	23.61
tl	20,070,294	–	37.82
tl	20,809,936	–	9.37
tl	17,105,372	–	26.17
tl	13,865,833	–	22.67
tl	14,619,065	–	21.40
tl	13,347,313	–	14.59
tl	12,257,994	–	22.22
tl	13,054,400	–	23.13
tl	14,805,943	–	12.37
tl	12,777,936	–	21.97
tl	14,075,300	–	23.32
tl	13,994,099	–	12.48
tl	10,671,225	–	23.08
tl	12,984,857	–	22.72
tl	12,564,000	–	14.11
tl	11,217,844	–	23.45
tl	12,854,704	–	22.72
tl	14,018,831	–	12.43
tl	11,727,308	–	23.55
tl	15,482,162	–	18.67
tl	12,258,164	–	14.88

Reformulating by Complementarity

$$\min \frac{\omega^\top \omega}{2} + \frac{C}{n}\left(\sum_{i=1}^{n} \xi_i + 2\sum_{i=1}^{n}(1-\bar{z}_i)\right)$$

$$\begin{aligned}
(y_i(\omega^\top x_i + b) - 1 + \xi_i)\cdot\bar{z}_i \geq 0 &\qquad \forall i = 1,\dots,n \\
0 \leq \xi_i \leq 2 &\qquad \forall i = 1,\dots,n \\
\bar{z} \in \{0,1\}^n & \\
\omega \in \mathbb{R}^d & \\
b \in \mathbb{R}, &
\end{aligned}$$

- where $\bar{z}_i = 1 - z_i$, and
- the resulting model is a Mixed-Integer Nonconvex Quadratically Constrained Problem (MIQCP) that `IBM-Cplex`, like all other commercial solvers initially developed for MILP, cannot solve (yet).

Solving the MIQCP by Couenne

- Despite the nonconvexity of the above MIQCP, there are several options to run the new model as it is and one of them is the open-source solver Couenne belonging to the Coin-OR arsenal.

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Couenne			
		% gap	
time (sec.)	nodes	ub	lb
163.61	17,131	–	–
1,475.68	181,200	–	–
tl	610,069	14.96	15.38
160.85	25,946	–	–
717.20	131,878	–	–
1,855.16	221,618	–	–
482.19	56,710	–	–
491.26	55,292	–	–
1,819.42	216,831	–	–
807.95	89,894	–	–
536.40	62,291	–	–
1,618.79	196,711	–	–
630.18	83,676	–	–
533.77	65,219	–	–
2,007.62	211,157	–	–
641.05	72,617	–	–
728.93	73,142	–	–
1,784.93	193,286	–	–
752.50	84,538	–	–
412.16	48,847	–	–
2,012.62	223,702	–	–
768.73	104,773	–	–
706.39	70,941	–	–

What does Couenne do?

- Although,
  - Convex MIQP should be much easier than nonconvex MIQCP, and
  - IBM-Cplex is by far more sophisticated than Couenne

  one can still argue that a comparison in performance between two different solution methods and computer codes is anyway hard to perform.

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- However, the reported results are rather surprising, especially if one digs into the way in which Couenne solves the problem, namely considering three aspects:
  1. McCormick Linearization,
  2. Branching, and
  3. alternative L_1 norm.

McCormick Linearization

- The most crucial observation is that the complementarity constraints are internally reformulated by `Couenne` through the classical McCormick linearization
  1. $\vartheta_i = y_i(\omega^\top x_i + b) - 1 + \xi_i$, with $\vartheta_i^L \leq \vartheta_i \leq \vartheta_i^U$, and
  2. $u_i = \vartheta_i \bar{z}_i$

  for $i = 1, \ldots, n$.

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  2. $u_i = \vartheta_i \bar{z}_i$

  for $i = 1, \ldots, n$. Then, the product corresponding to each new variable u_i is linearized as

$$u_i \geq 0 \quad (5)$$

$$u_i \geq \vartheta_i^L \, \bar{z}_i \quad (6)$$

$$u_i \geq \vartheta_i + \vartheta_i^U \, \bar{z}_i - \vartheta_i^U \quad (7)$$

$$u_i \leq \vartheta_i + \vartheta_i^L \, \bar{z}_i - \vartheta_i^L \quad (8)$$

$$u_i \leq \vartheta_i^U \, \bar{z}_i \quad (9)$$

  again for $i = 1, \ldots, n$, where (5) are precisely the complementarity constraints and ϑ_i^L plays the role of the bigM.

Branching

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- However, the default version of `Couenne` does not take advantage of this possibility and branches (first) on binary variables z's.

- Thus, again it is surprising that such a branching strategy leads to an improvement over the sophisticated branching framework of `IBM-Cplex`.

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- In order to answer this question we performed an experiment in which the quadratic part of the objective function has been replaced by its L_1 norm making the entire bigM model linear. Ultimately, the absolute value of ω is minimized.
- Computationally, this has no effect and Couenne continues to consistently outperform MILP solvers on this very special (modified) class of problems.

Tightening ω's based on the objective function

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- Among those reductions, we observed that Couenne does a very simple bound tightening (at the root node) based on the computation of an upper bound, i.e., a feasible solution, of value, say U,

$$\omega_i \in \left[-\sqrt{2U}, \sqrt{2U}\right] \qquad \forall i = 1, \ldots, d.$$

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- In other words, when the bigM values are large it is very hard to solve the problem without changing them during search.

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- Switching off in Couenne any of these components leads to a dramatic degradation in the results.

Conclusions

- In a broad sense, we are using the SVM with the ramp loss to investigate the possibility of exploiting tools from (noncomvex) MINLP in MILP or (convex) MIQP, essentially, the reverse of the common path.

- More precisely, we have argued that sophisticated (nonconvex) MINLP tools might be very effective to face one of the most structural issues of MILP, which is dealing with the weak continuous relaxations associated with bigM constraints.

- A lot to be done . . .