Discrete Newton Algorithms for Budgeted Network Problems

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Sauder School of Business, UBC; U. Rome Tor Vergata; Aachen

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Outline

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What is Network Interdiction?

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Now suppose that an interdictor can damage the network:

In general we have a flow player who wants to push as much post-damage flow as possible through the network . . .

. . . and an interdiction player who wants to damage the network by removing arcs subject to a budget so as to minimize the post-removal flow.

The Formal Network Interdiction Model

- We start with an ordinary max flow min cut network with source s, sink t, and capacities c. The capacity of cut S is $\mathrm{cap}_c(S)$.

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- Finally, we have a budget $B \geq 0$ to spend on destroying arcs. Our objective is to spend at most B (maybe fractionally) in a way that minimizes the value of the residual flow.
  - In Min Cut we remove arcs until there is zero flow left, but here we remove only as much as we can under the budget.

Removing arcs greedily

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- So let's get some idea of how much flow we can remove by destroying arcs from a fixed cut S.

The interdiction curve for a fixed cut S

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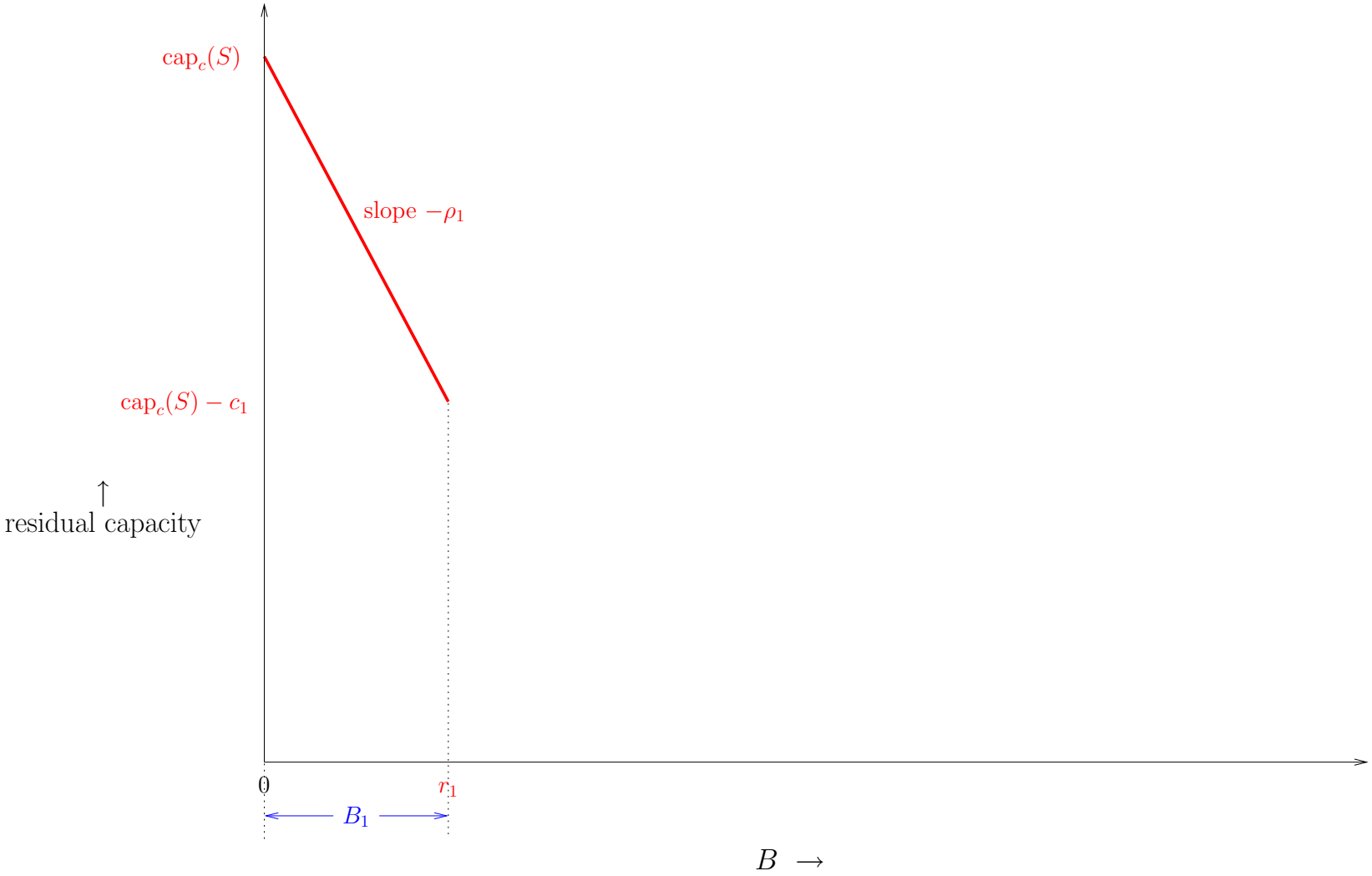

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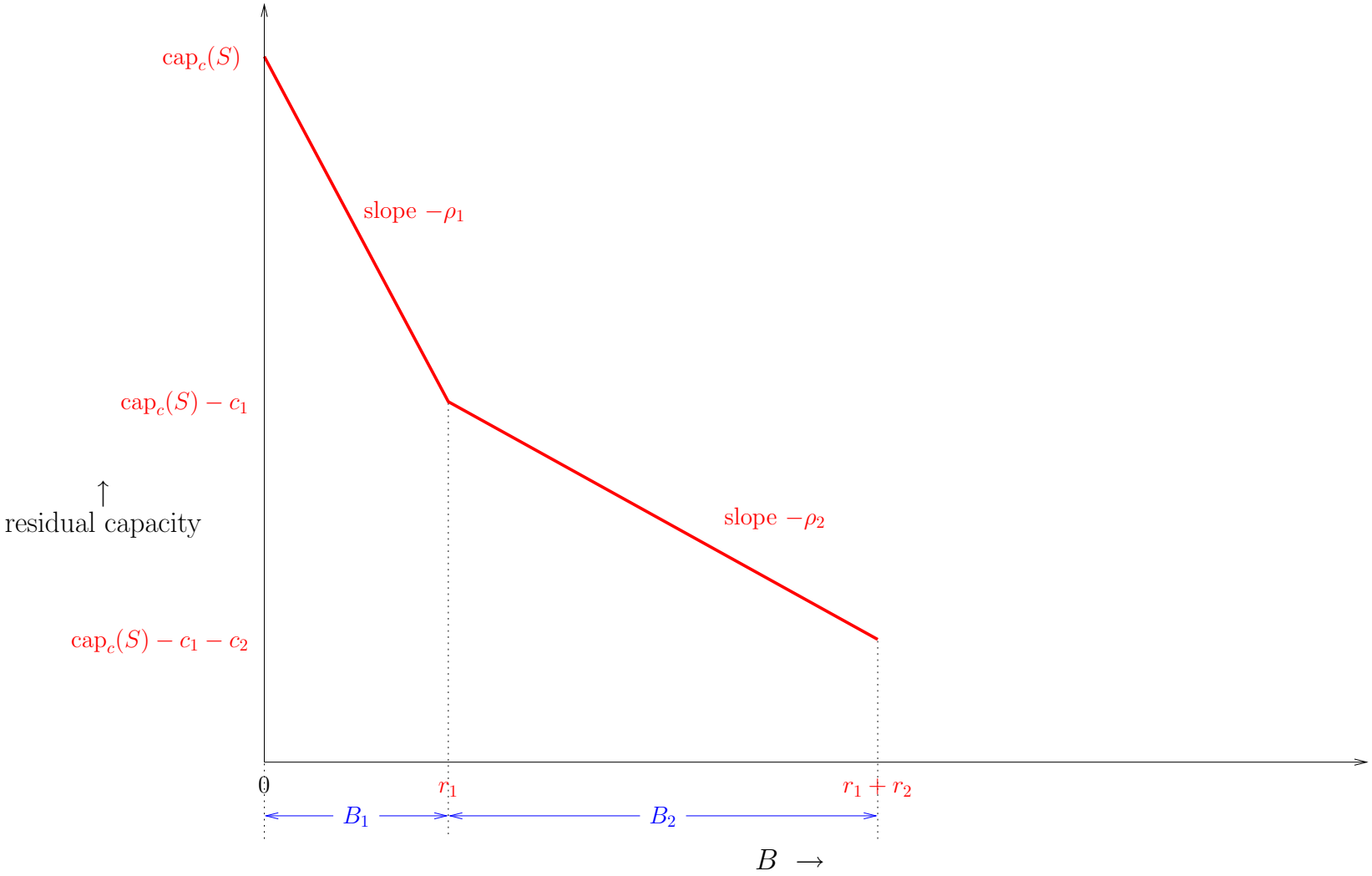

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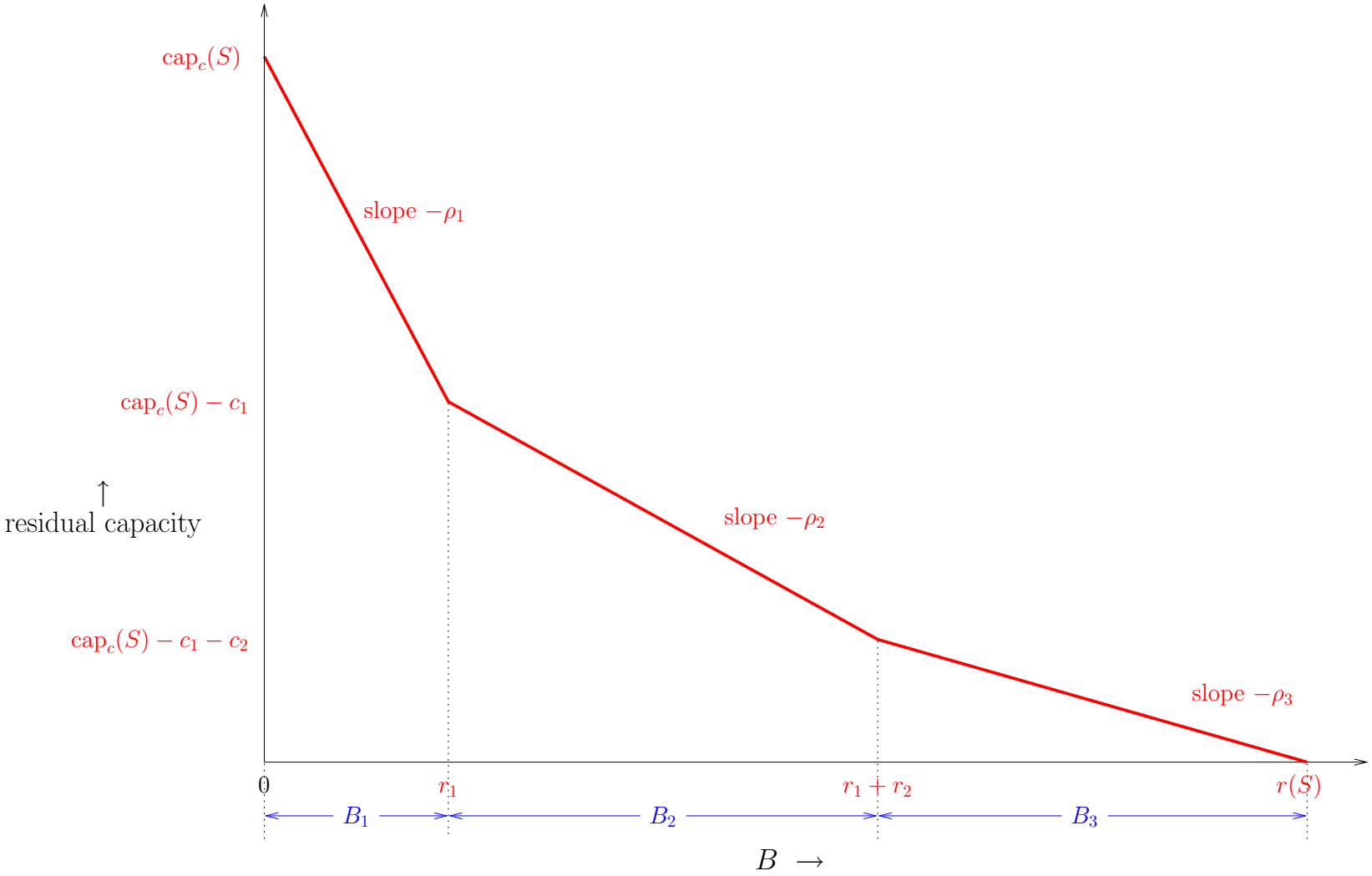
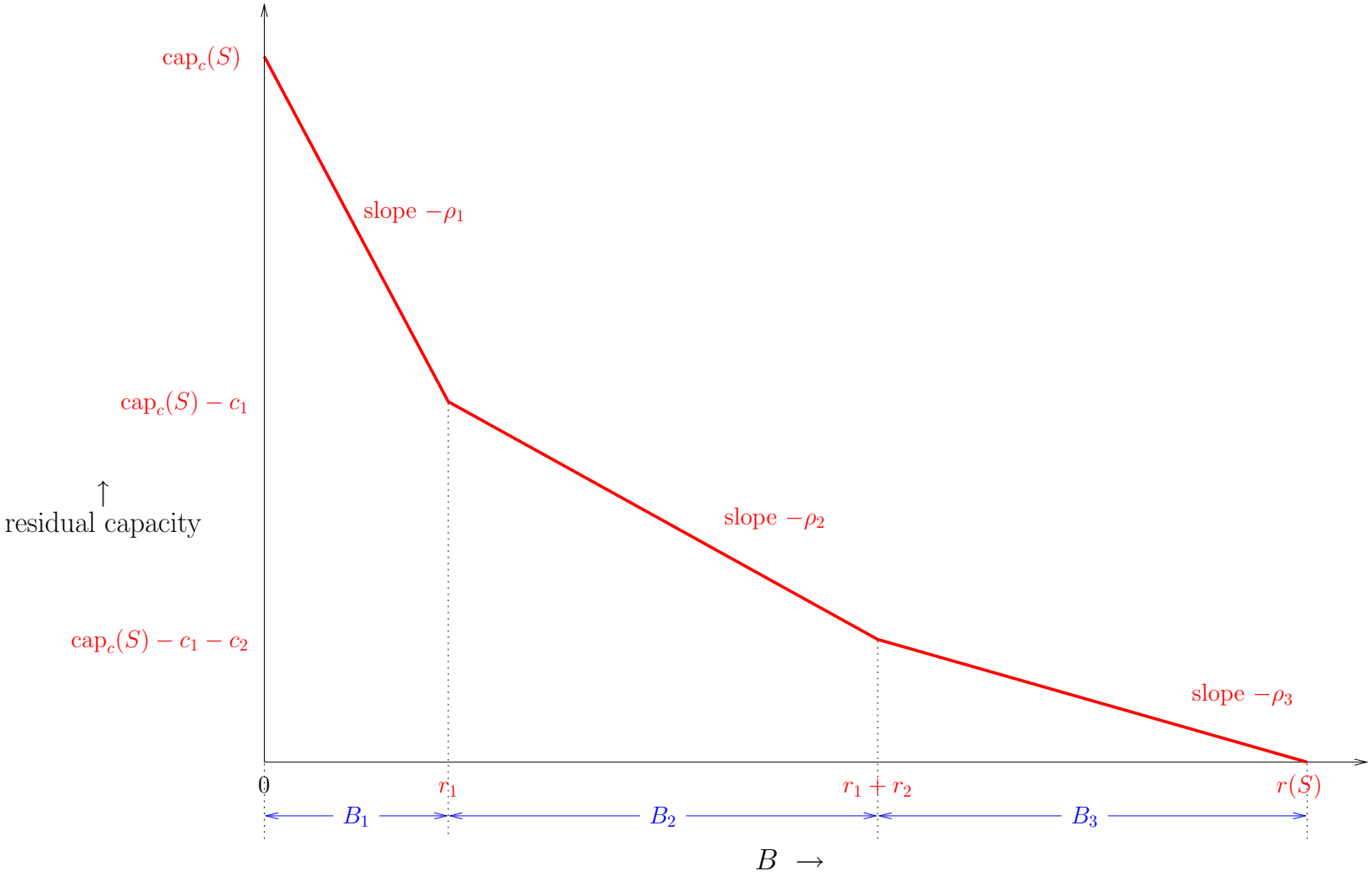


The interdiction curve for a fixed cut S

This curve is piecewise linear convex.



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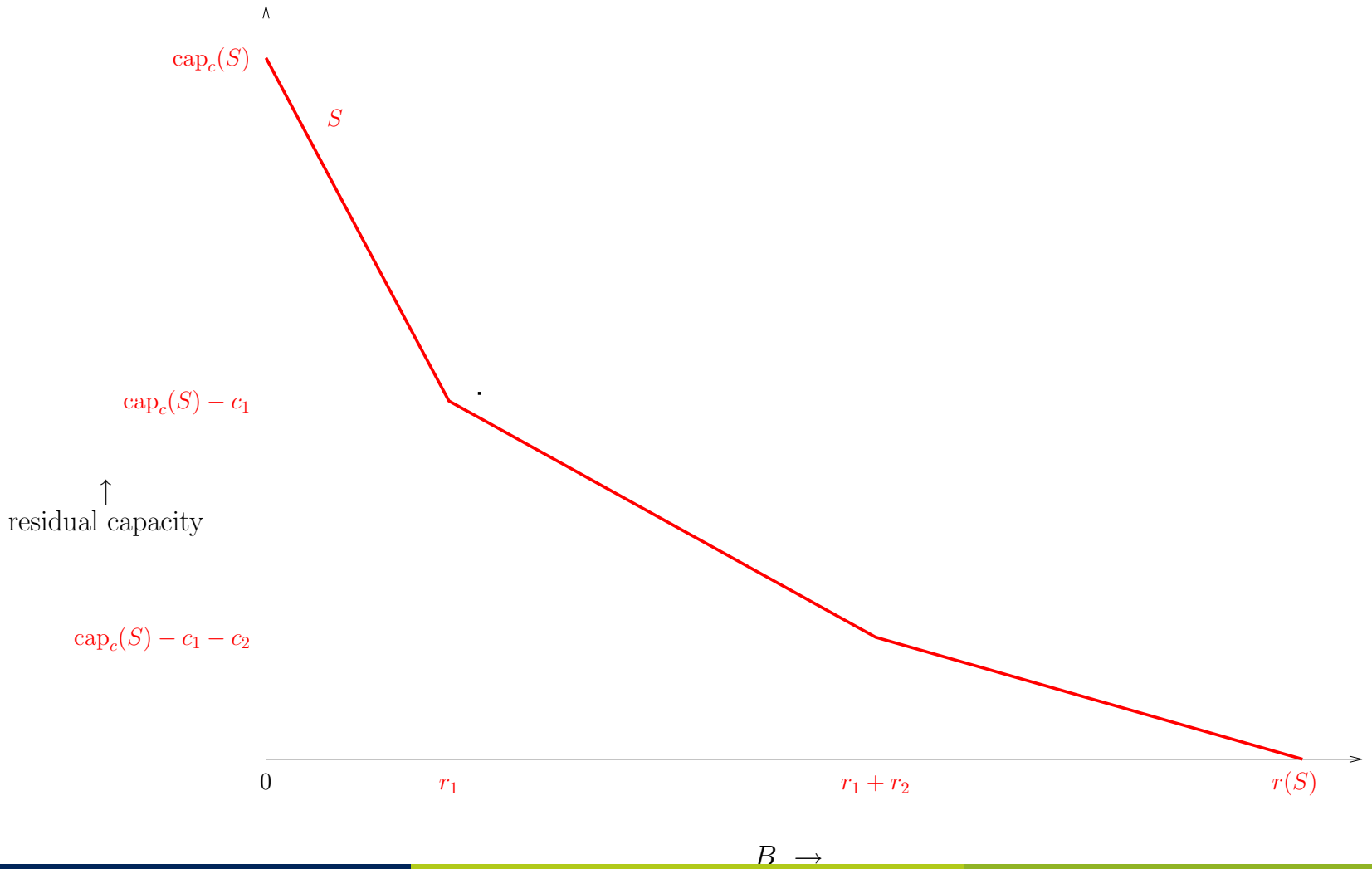

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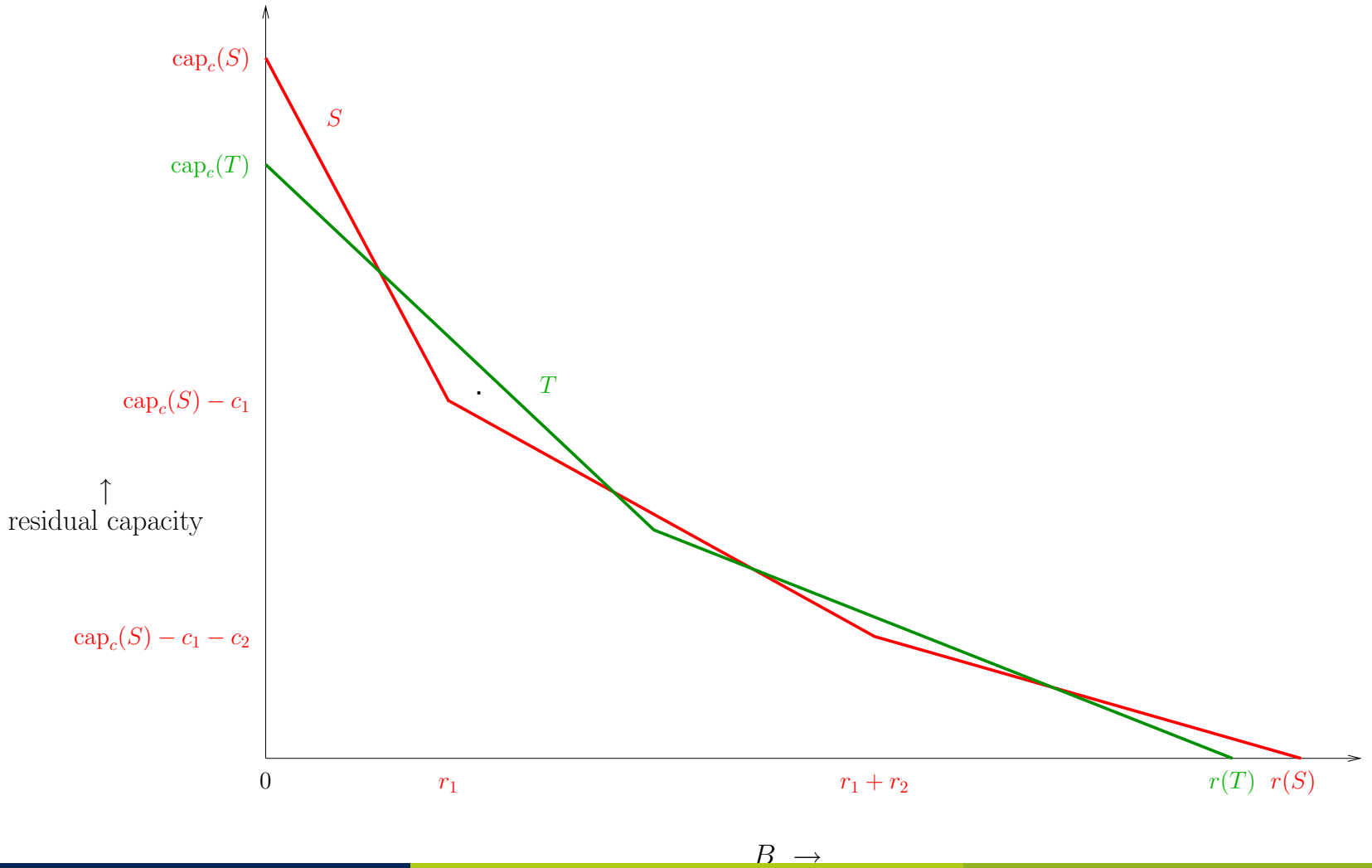

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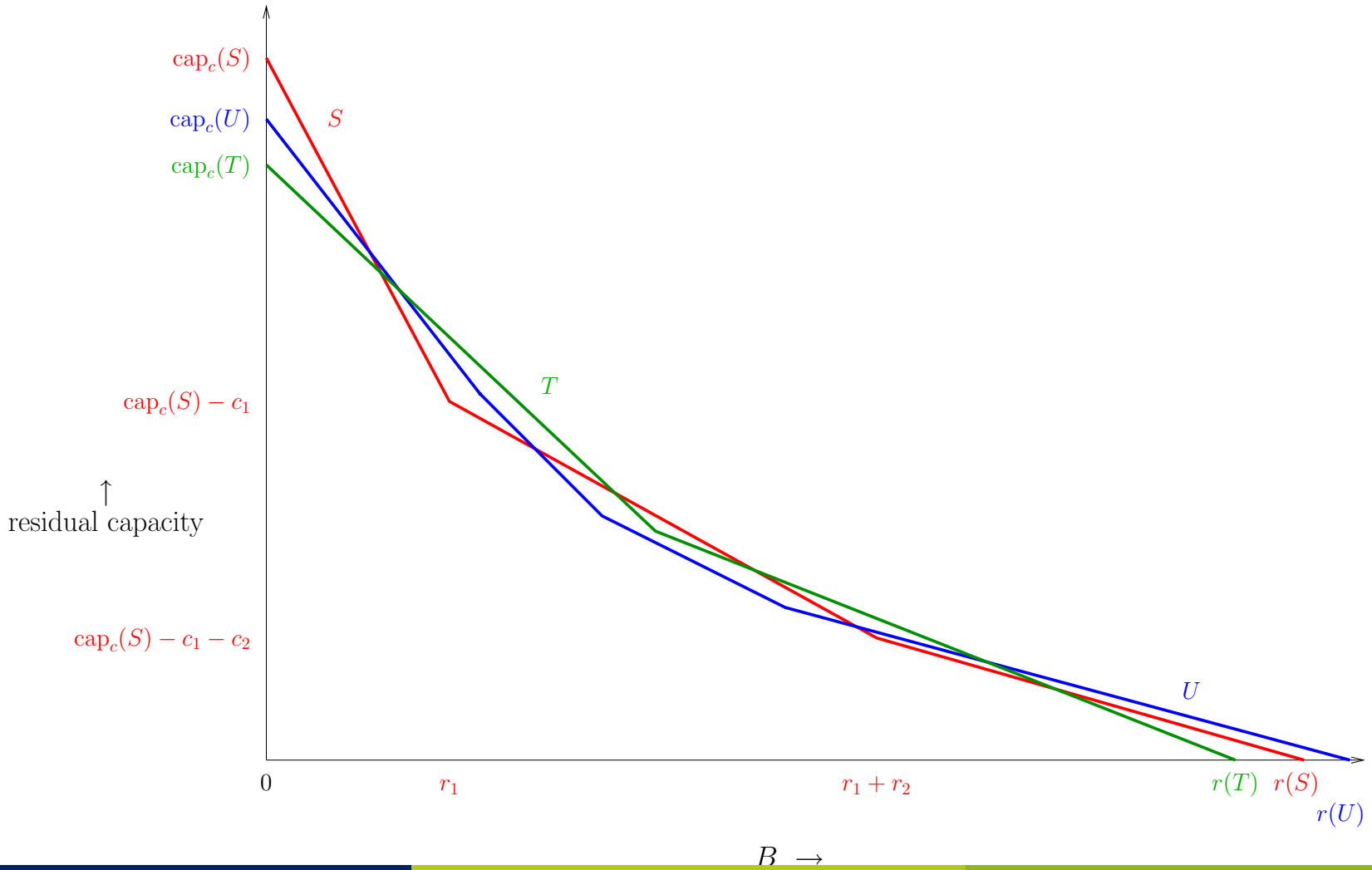

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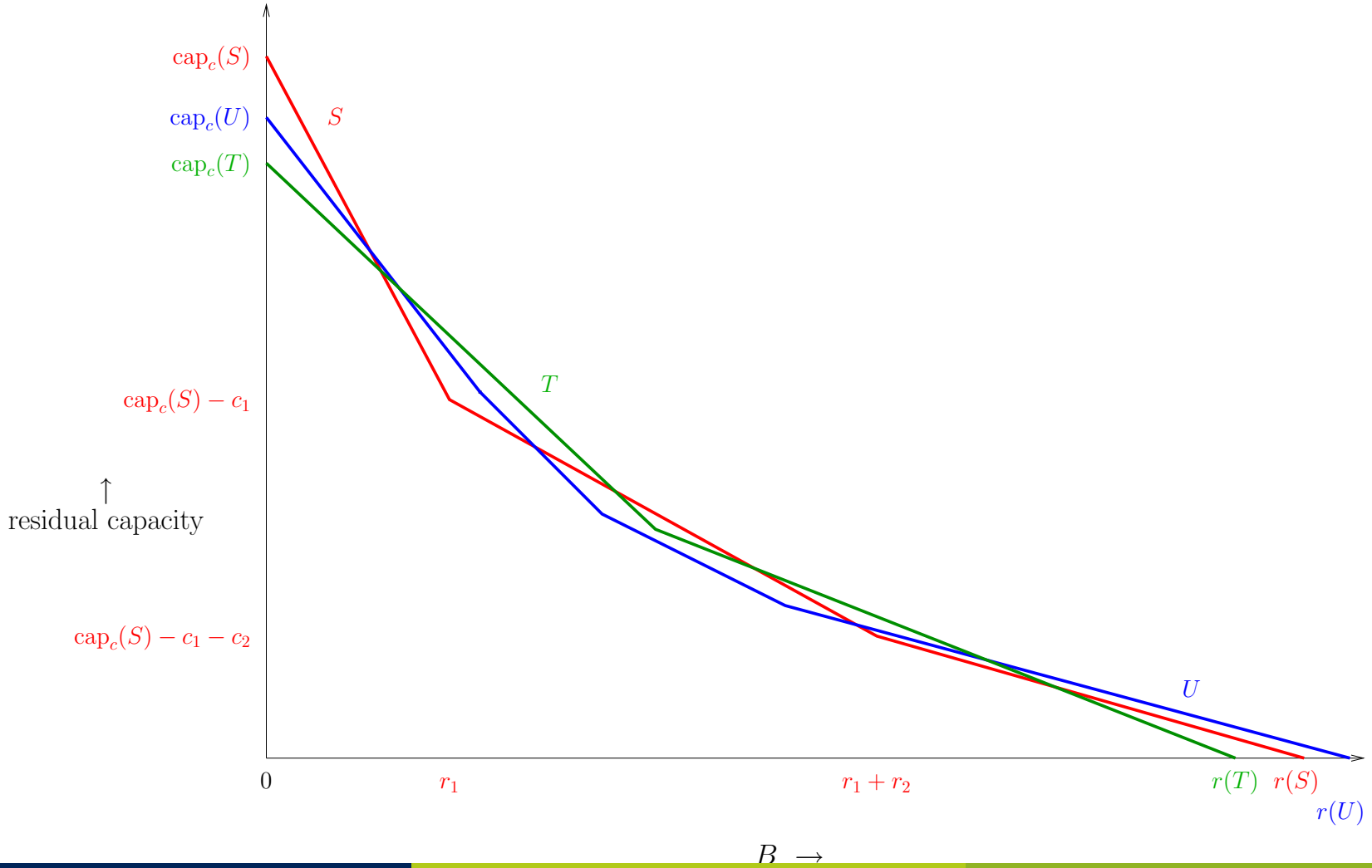
For a given value of B, we just select which S gives the minimum value at B, so the overall curve is the minimum of all the cut-wise curves.



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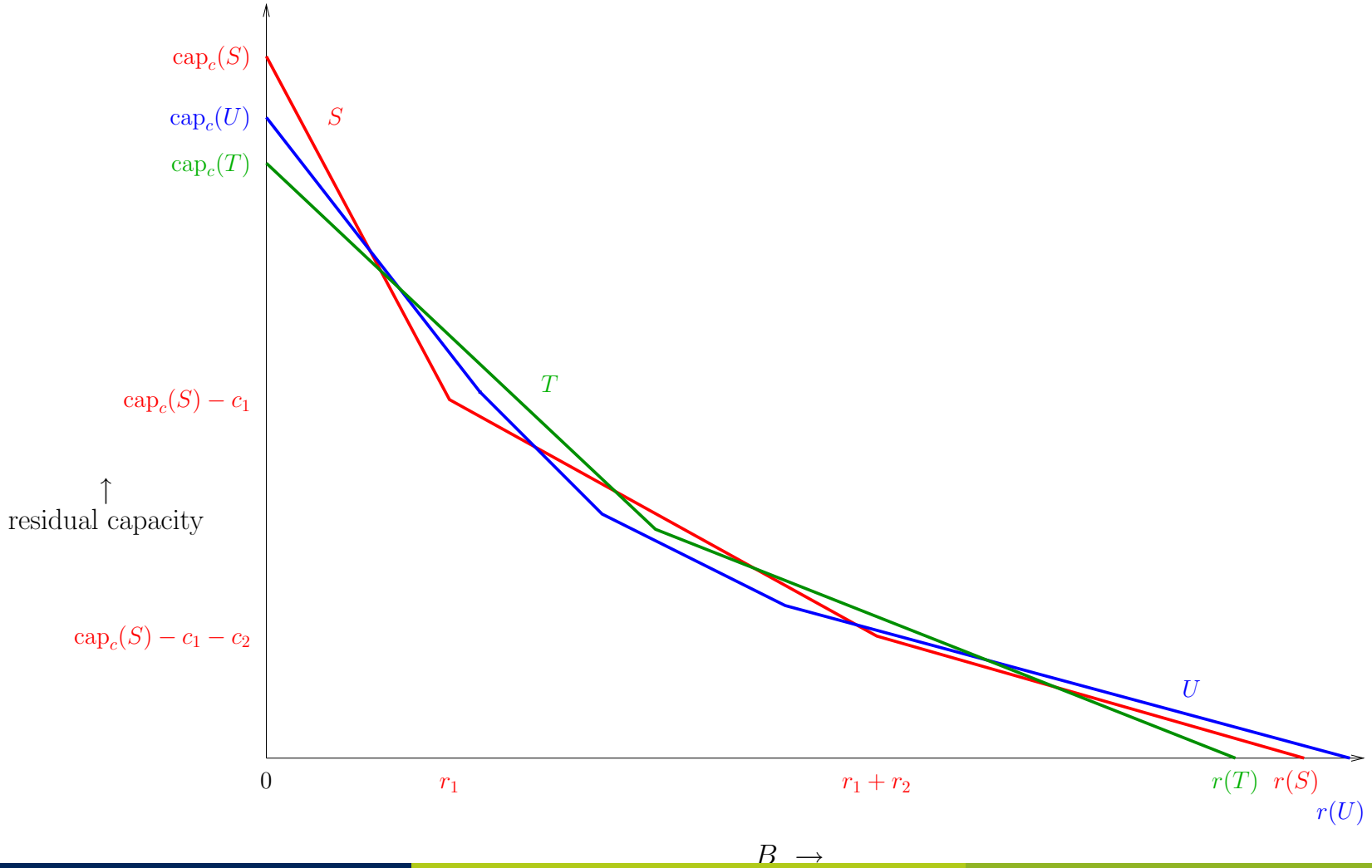
Unfortunately, the minimum of a bunch of convex curves is not in general convex.



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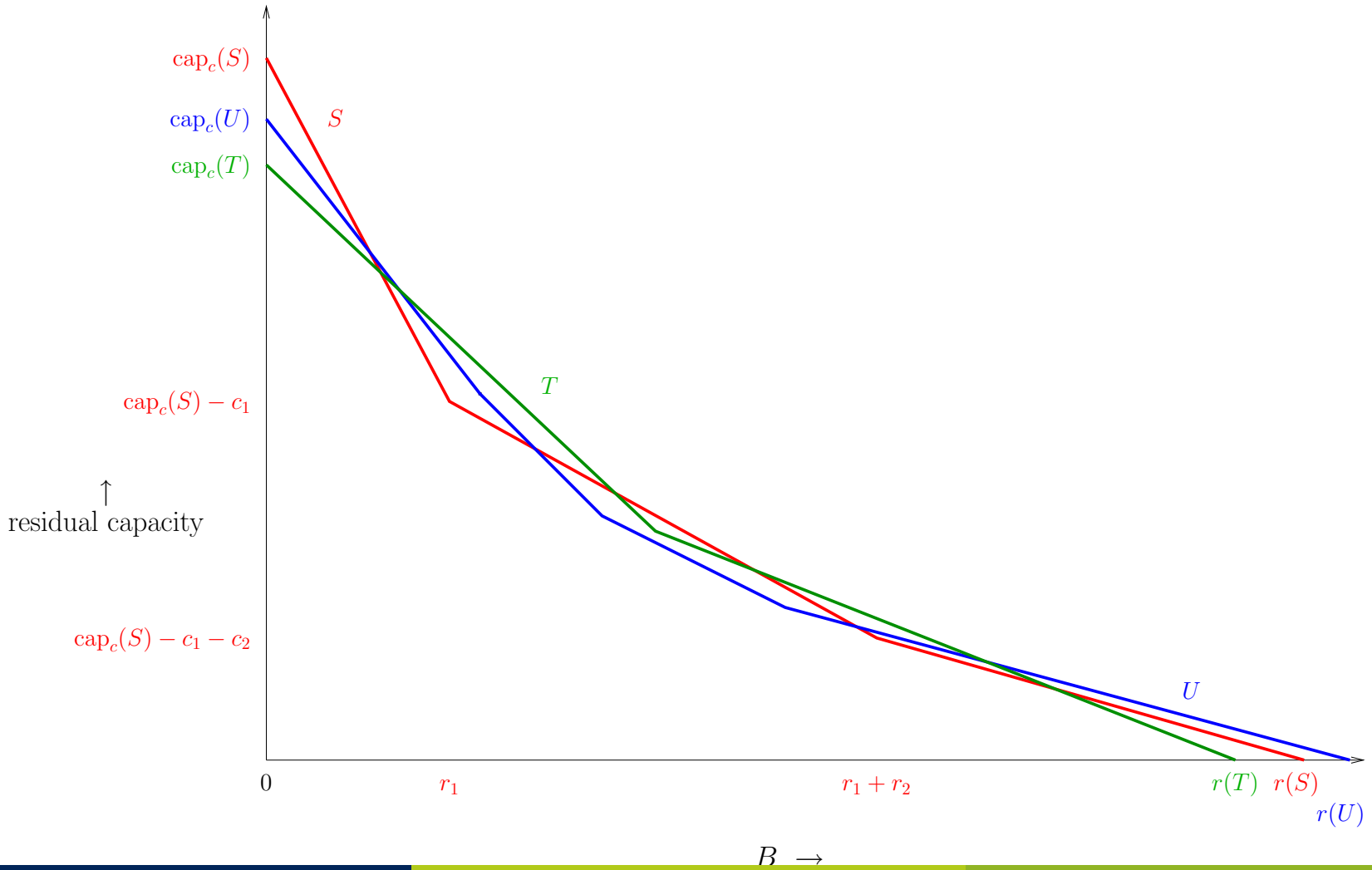
This is why Network Interdiction is NP Hard (Phillips '93; Wood '93).



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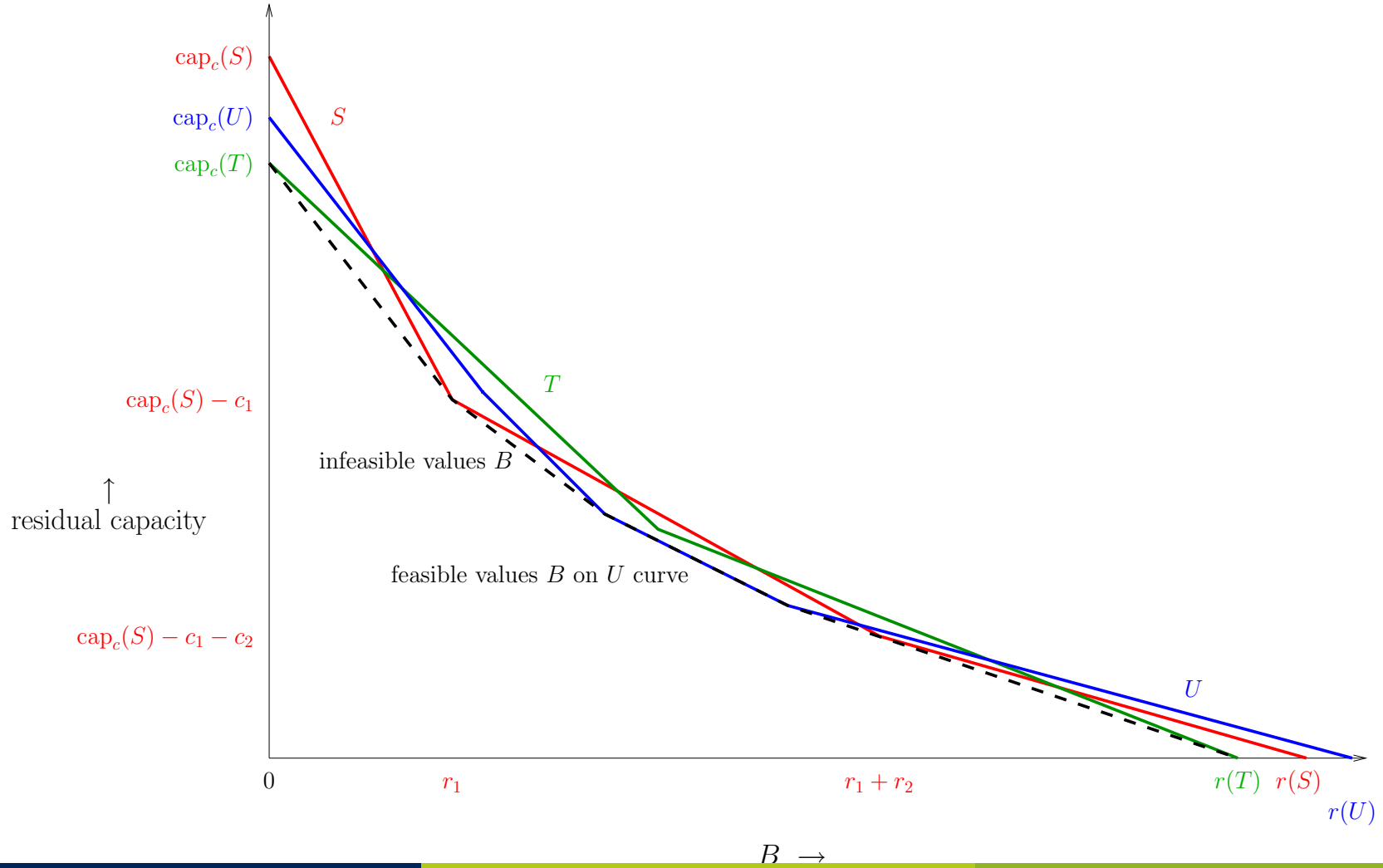
If we take the lower envelope, or convex hull, of the overall interdiction curve, we get something tractable, the B-profile.



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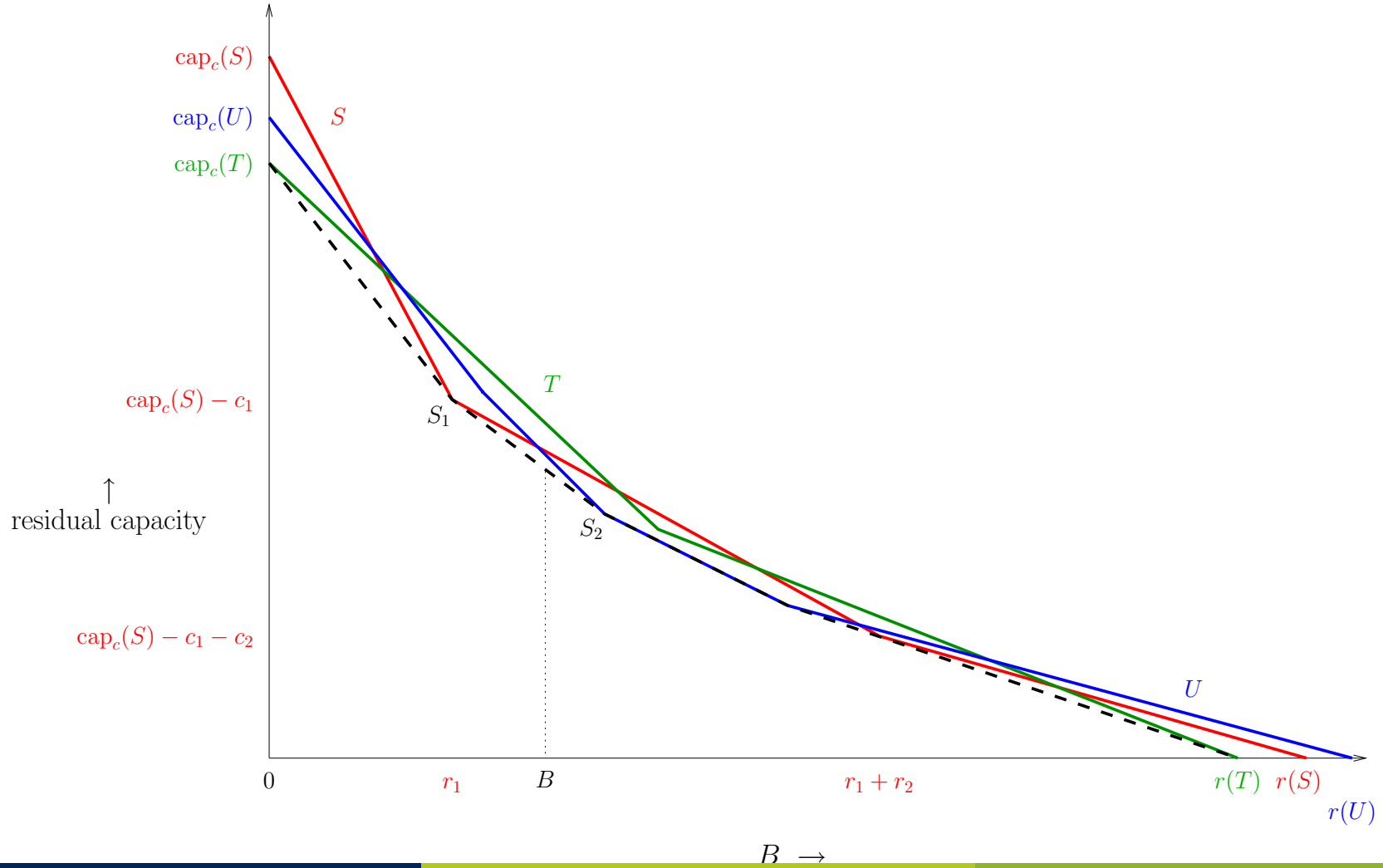
Now budget B corresponds to a convex combination of points coming from the interdiction curves of (one or) two cuts, $S_1 = S$ and $S_2 = U$.



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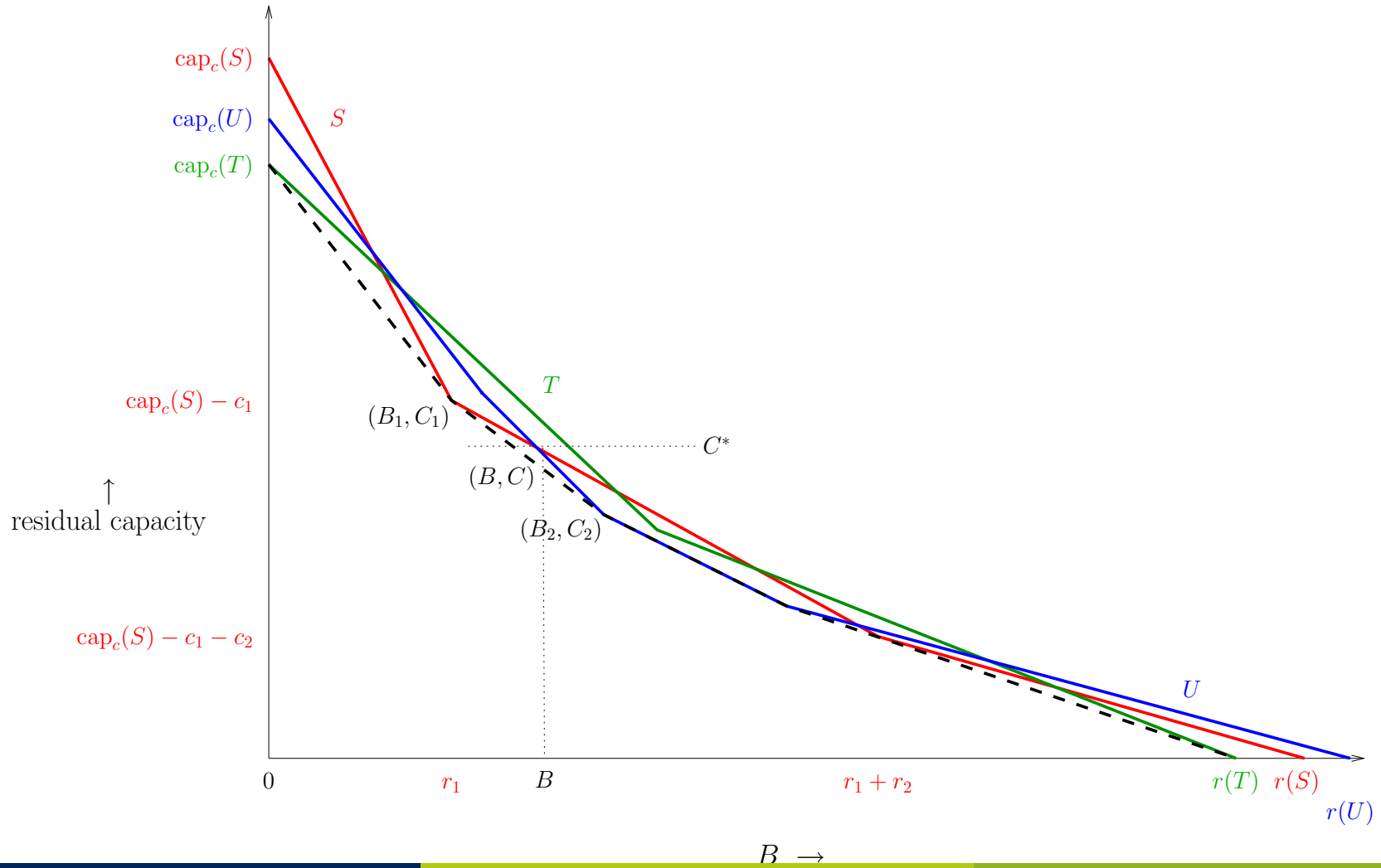
S_1 corresponds to breakpoint (B_1, C_1), S_2 to (B_2, C_2), and we have λ s.t. $B = \lambda_1 B_1 + \lambda_2 B_2$; define $C = \lambda_1 C_1 + \lambda_2 C_2 \leq C^* =$ opt. resid. capacity.



The linear program and its dual

- The normal min cut dual LP is

$$\begin{array}{lrcl} \min \sum_{u \to v} c_{uv} y_{uv} & & & \\ \text{s.t. } d_u - d_v + y_{uv} & \geq & 0 & \text{for } u \to v \neq t \to s, \\ d_t - d_s + y_{ts} & \geq & 1 & \\ y_{uv} & \geq & 0 & \text{all } u \to v. \end{array}$$

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- Prize-collecting with a budget in place of a penalty.

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- Repeat the new LP with dual variables:

$$\begin{array}{rrrcll} & & \min \sum_{u\to v} c_{uv} y_{uv} & & & \\ x_{uv}: & \text{s.t. } d_u - d_v + y_{uv} + z_{uv} & & \geq & 0 & \text{for } u \to v \neq t \to s, \\ x_{ts}: & & d_t - d_s + y_{ts} & \geq & 1 & \\ \lambda: & & \sum_{u\to v} r_{uv} z_{uv} & \leq & B & \\ & & y_{uv},\ z_{uv} & \geq & 0 & \text{all } u \to v. \end{array}$$

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- The primal interdiction LP is

$$\begin{array}{rrcl} & \max_{x,\lambda} (x_{ts} - B\lambda) & & \\ d: & \text{s.t. conservation} & & \\ y_{uv}: & 0 \leq x_{uv} & \leq & c_{uv} \\ z_{uv}: & x_{uv} - r_{uv}\lambda & \leq & 0. \end{array}$$

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- Repeat the primal interdiction LP and highlight the two capacities:

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$$x_{uv} \leq \min(c_{uv}, \lambda r_{uv}),$$

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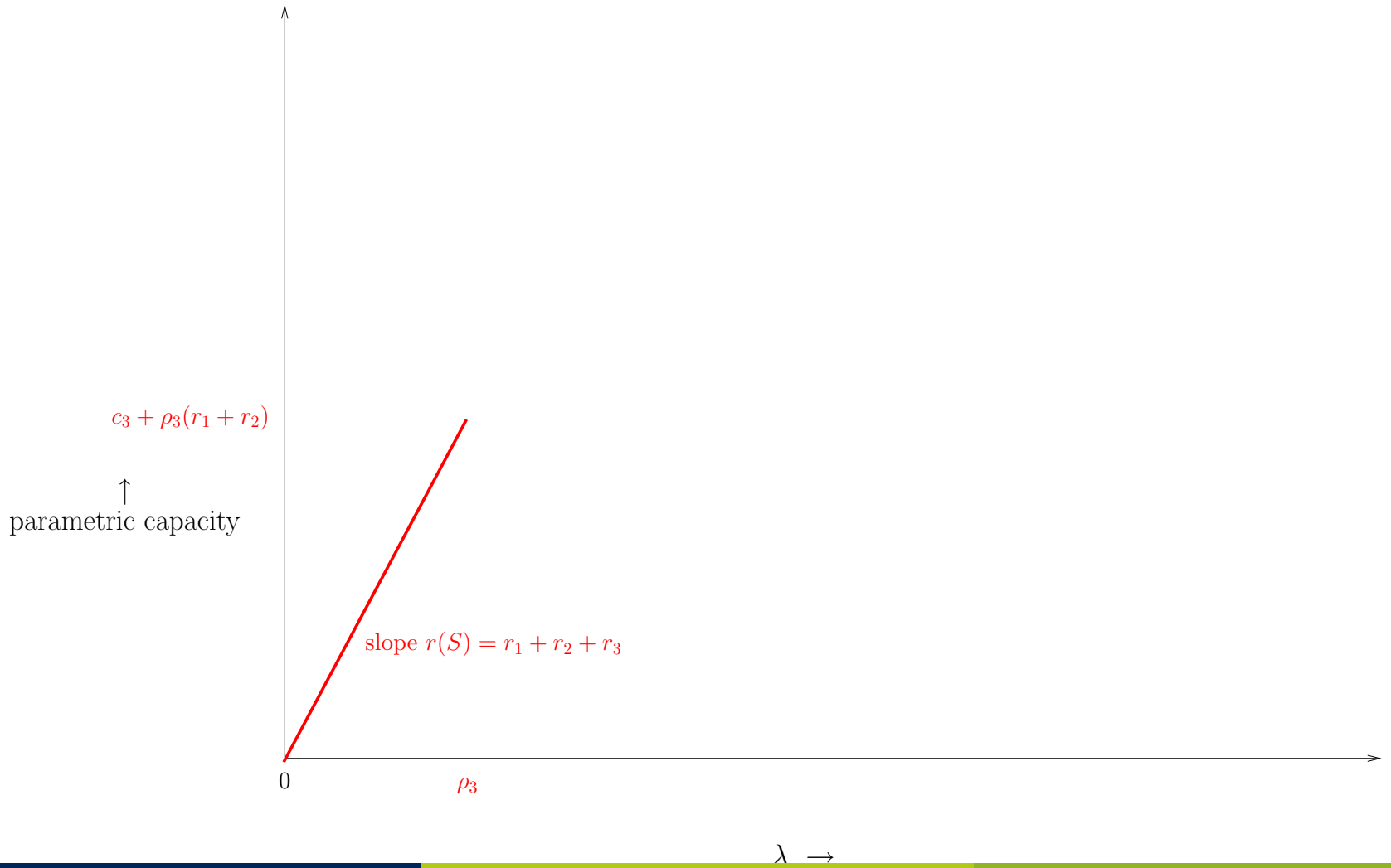
- So let's investigate the behavior of this parametric min cut problem.

Parametric capacity of fixed cut S

When λ is small, $\mathrm{cap}(S, \lambda) = \lambda \mathrm{cap}_r(S)$.



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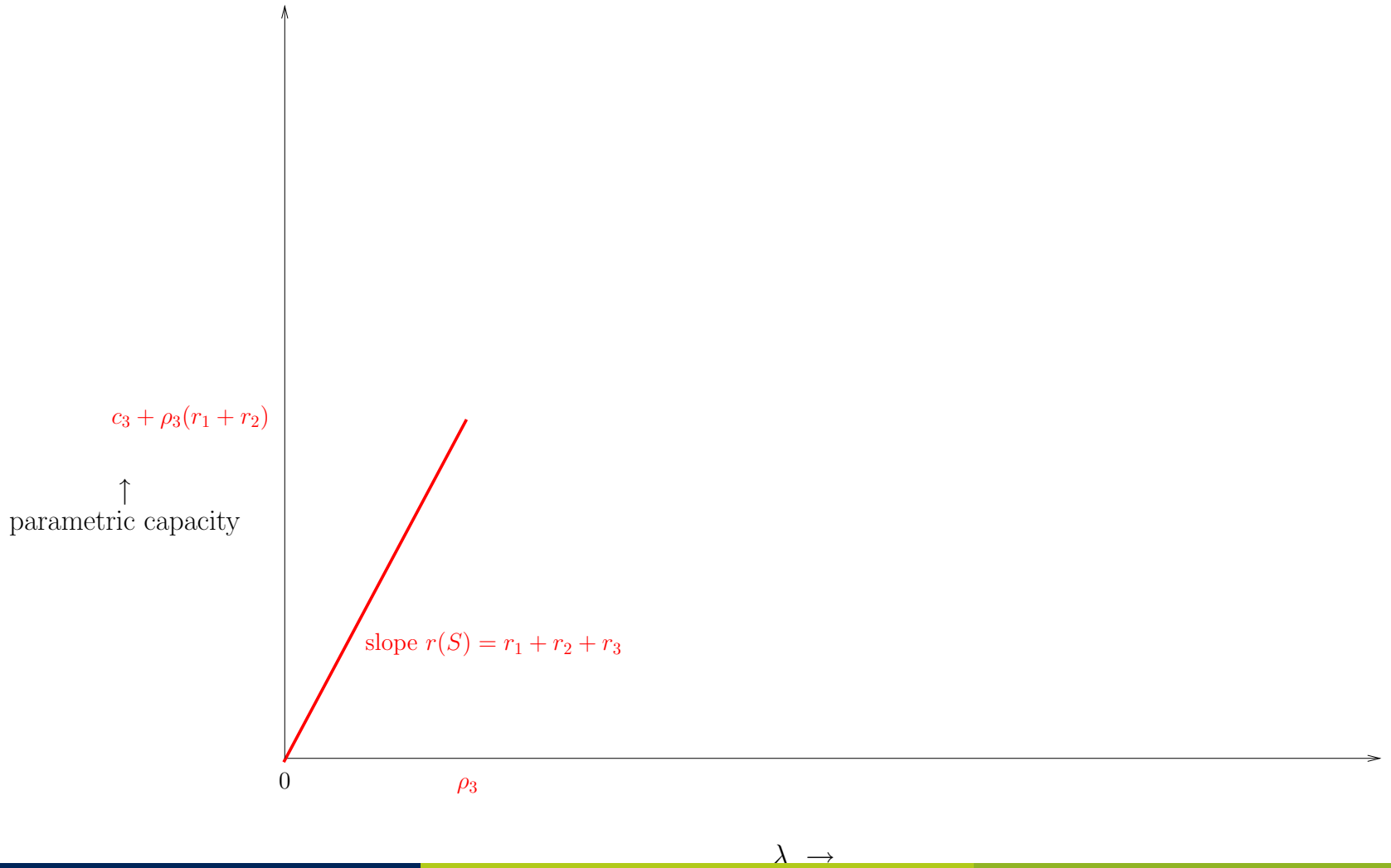
This continues as long as $\lambda r_{uv} \leq c_{uv}$ for all $u \to v \in \delta^+(S)$, or $\lambda \leq \rho_{uv}$.



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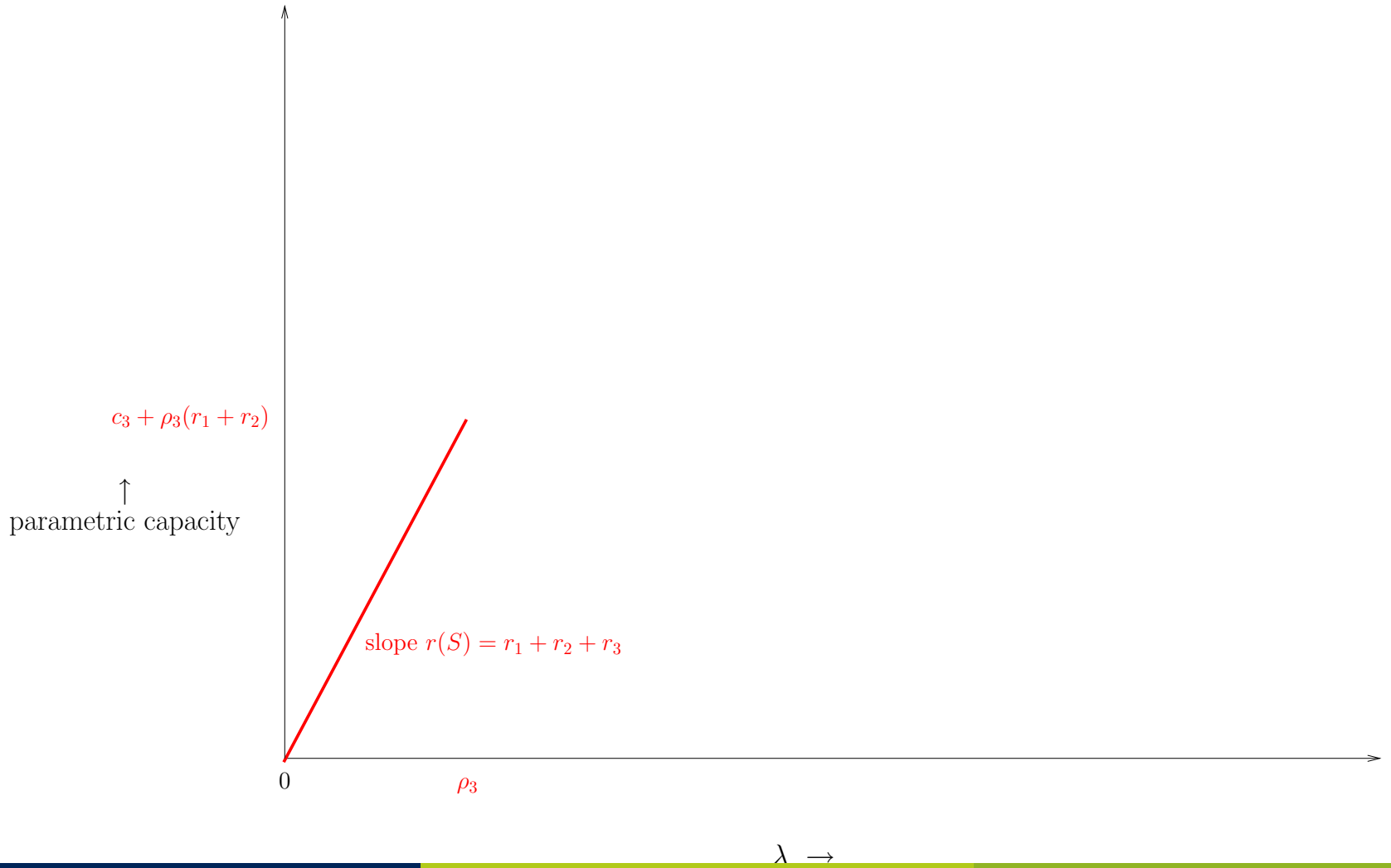
Thus the first breakpoint is when λ hits $\min_{\delta^+(S)} \rho_{uv}$.



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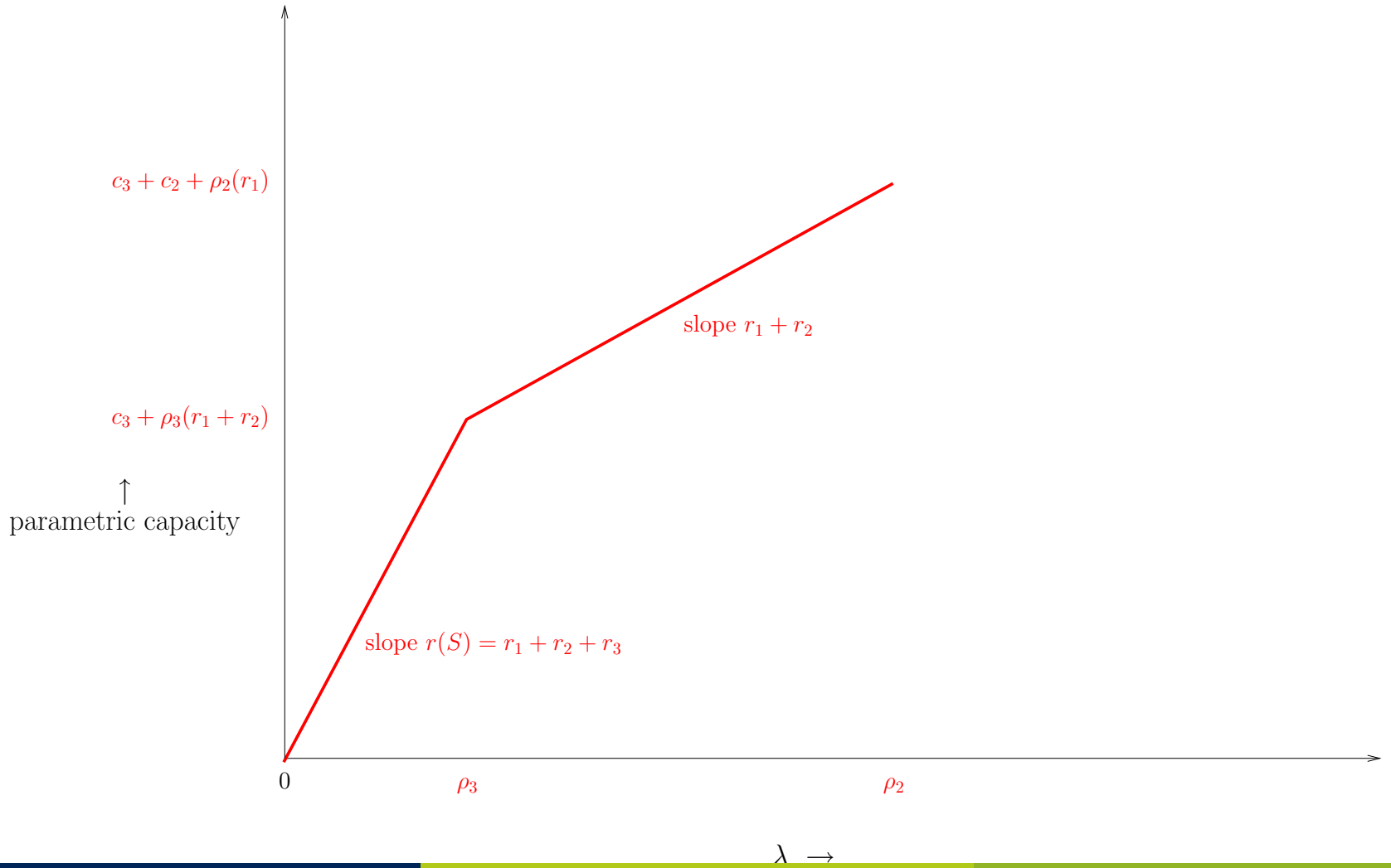

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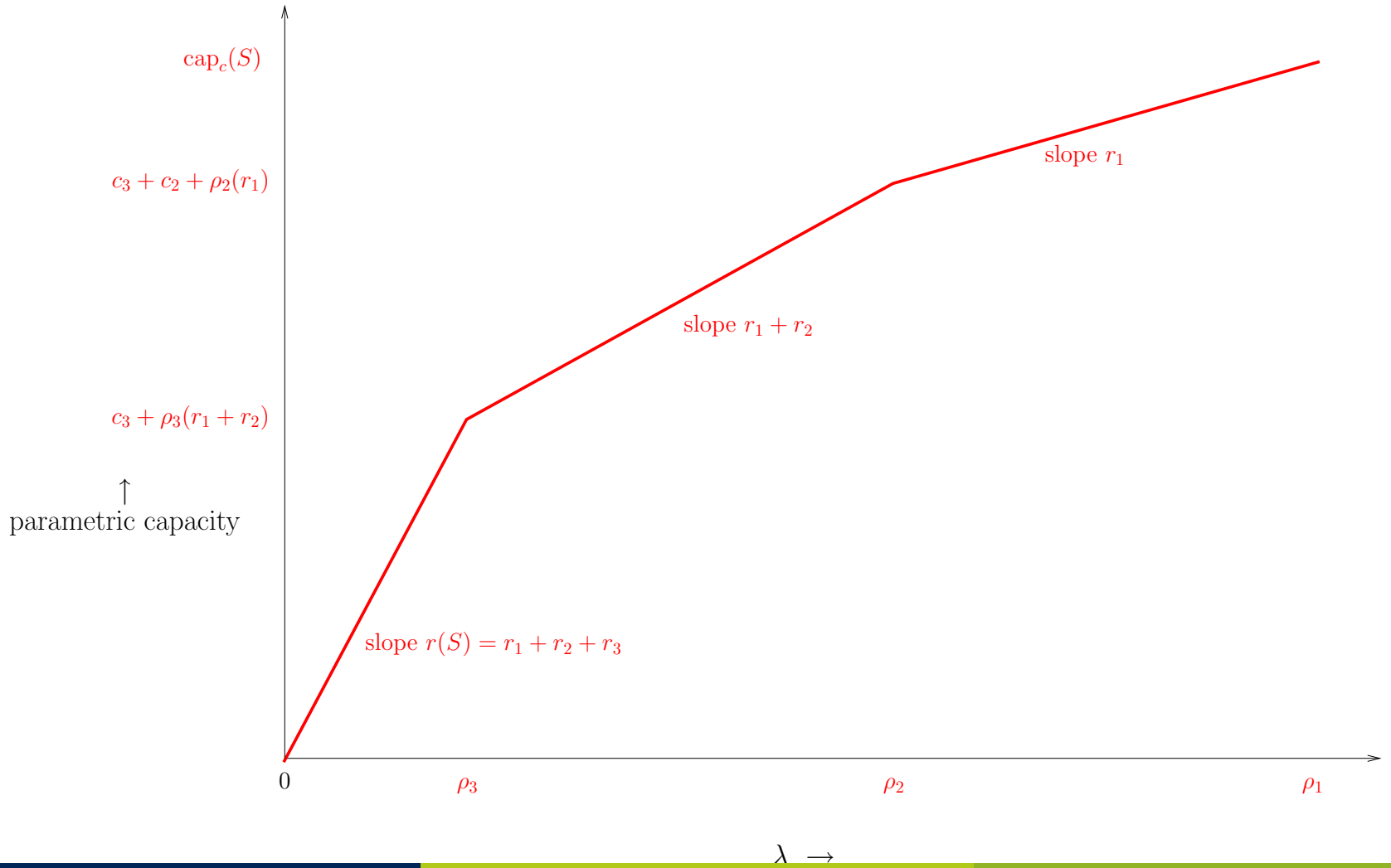

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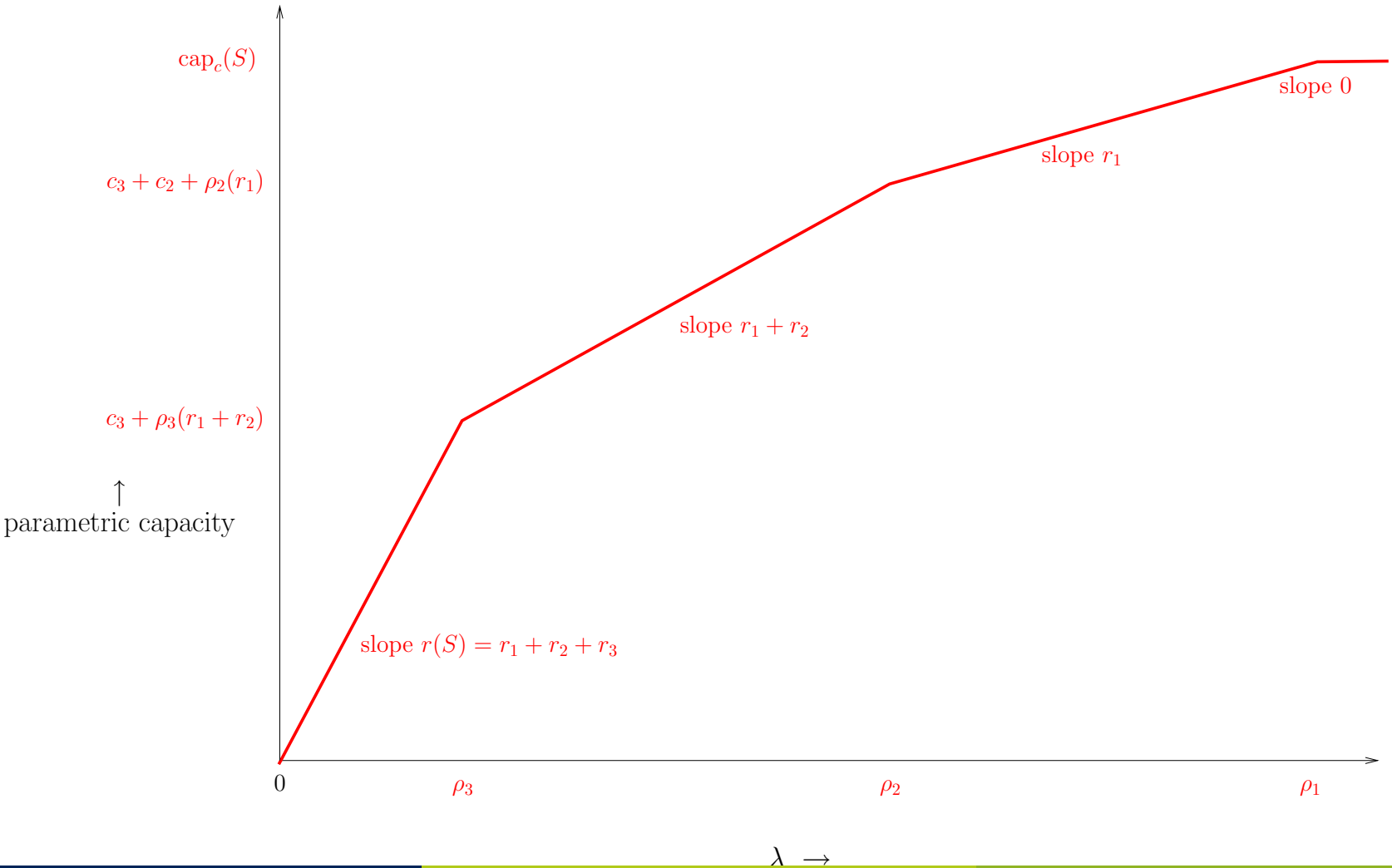

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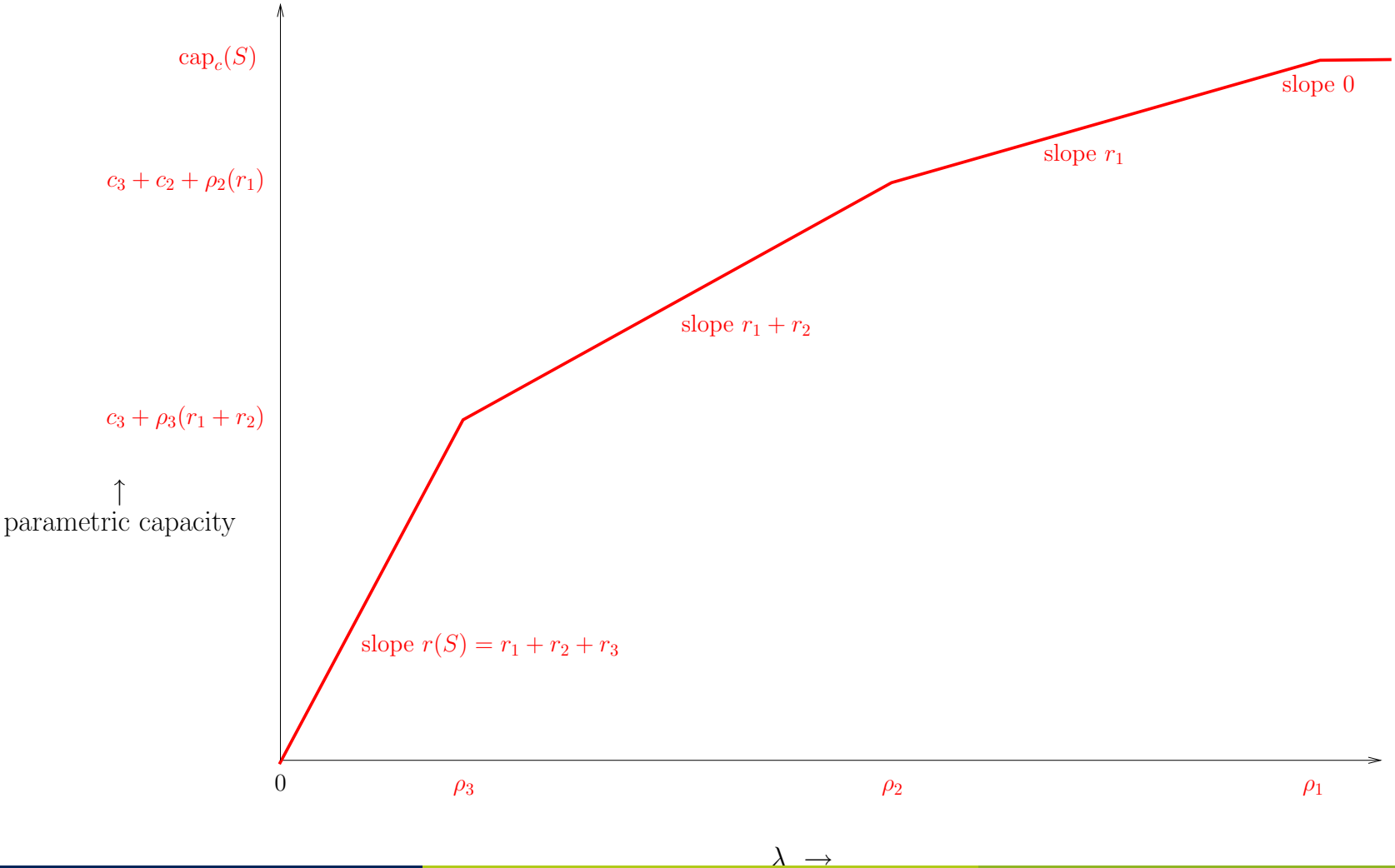
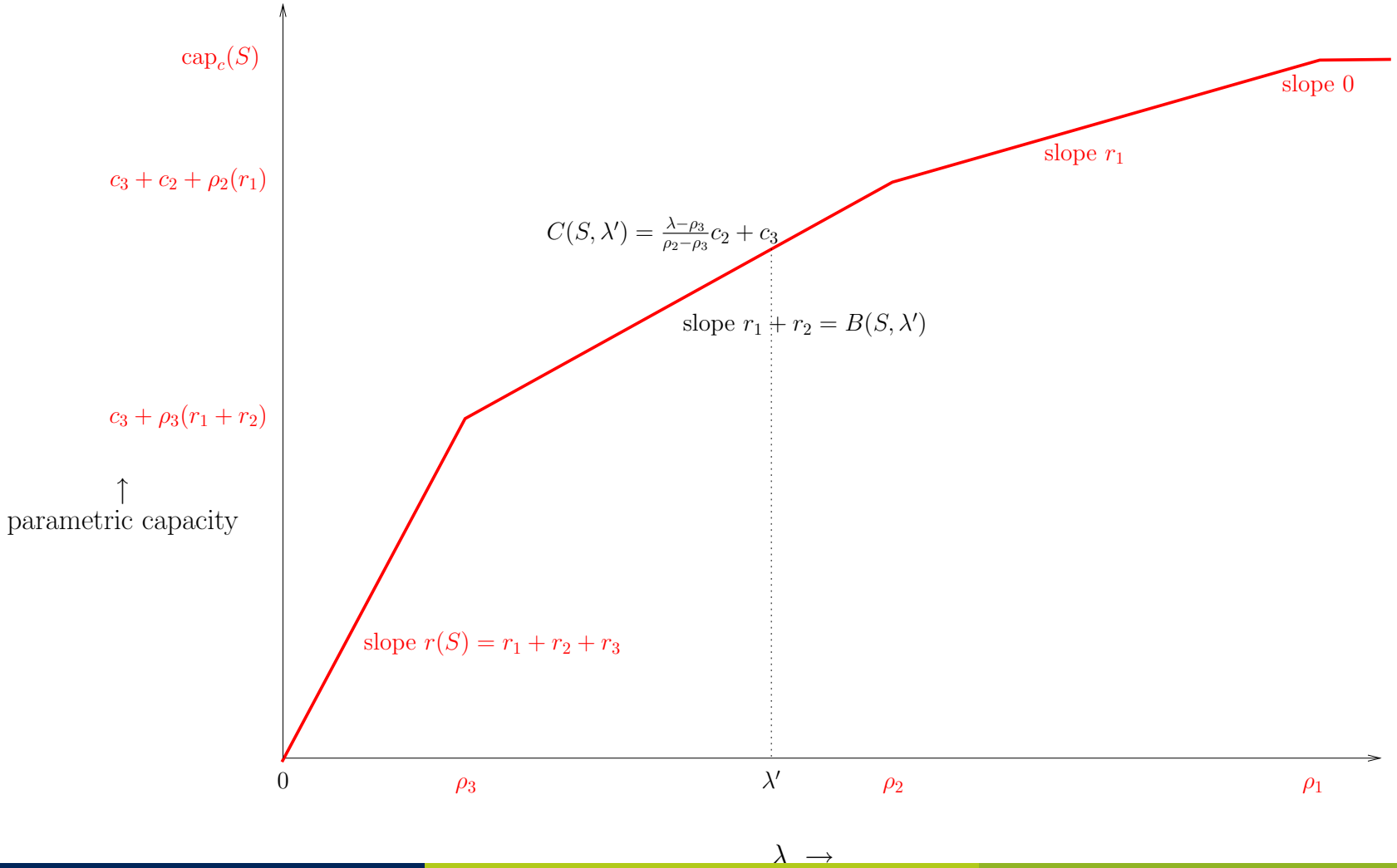
The parametric capacity curve for S is piecewise linear concave.



Parametric capacity of fixed cut S

For a value λ' of λ we also get the local budget $B(S, \lambda')$ and local residual capacity $C(S, \lambda')$.



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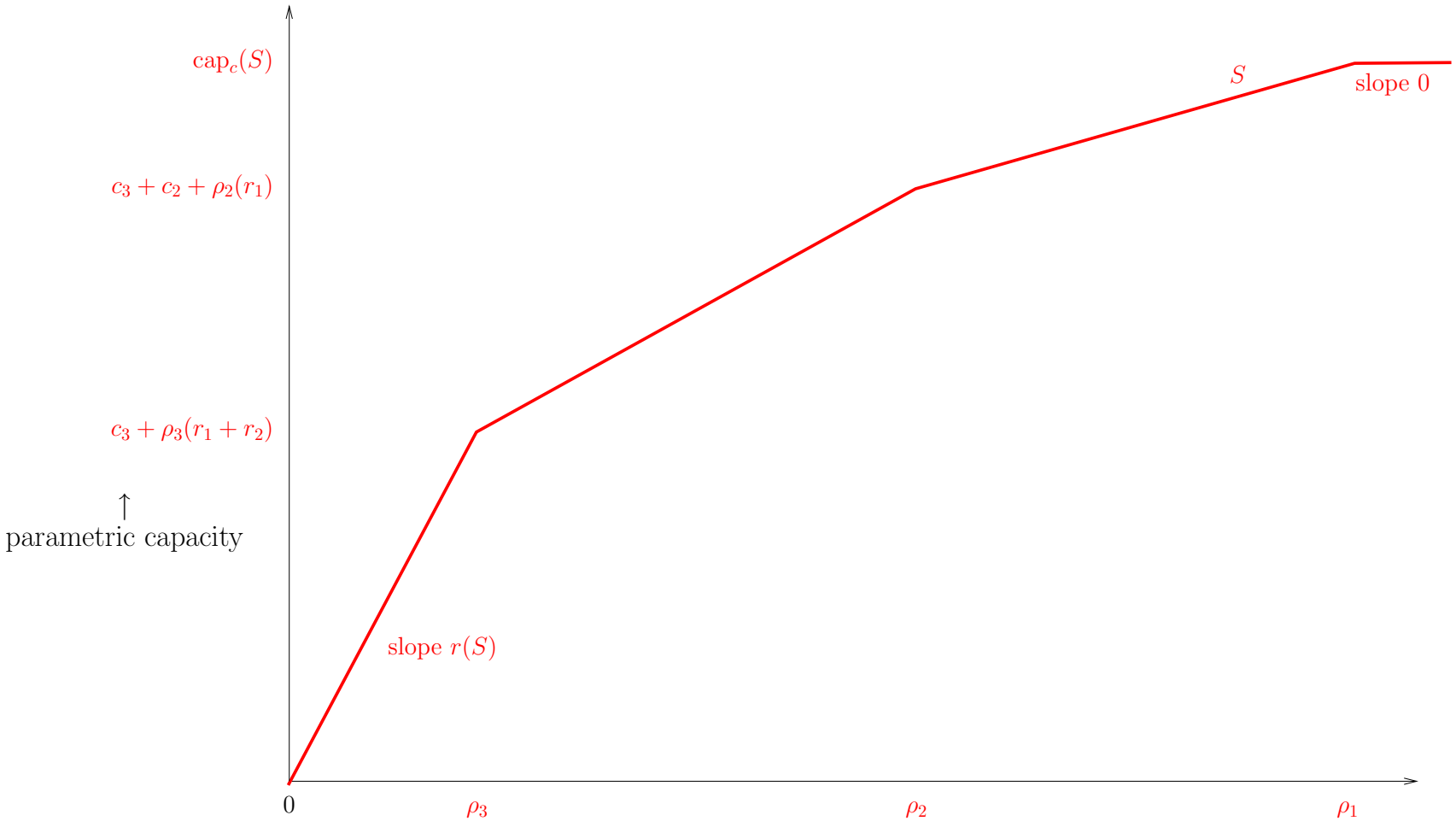

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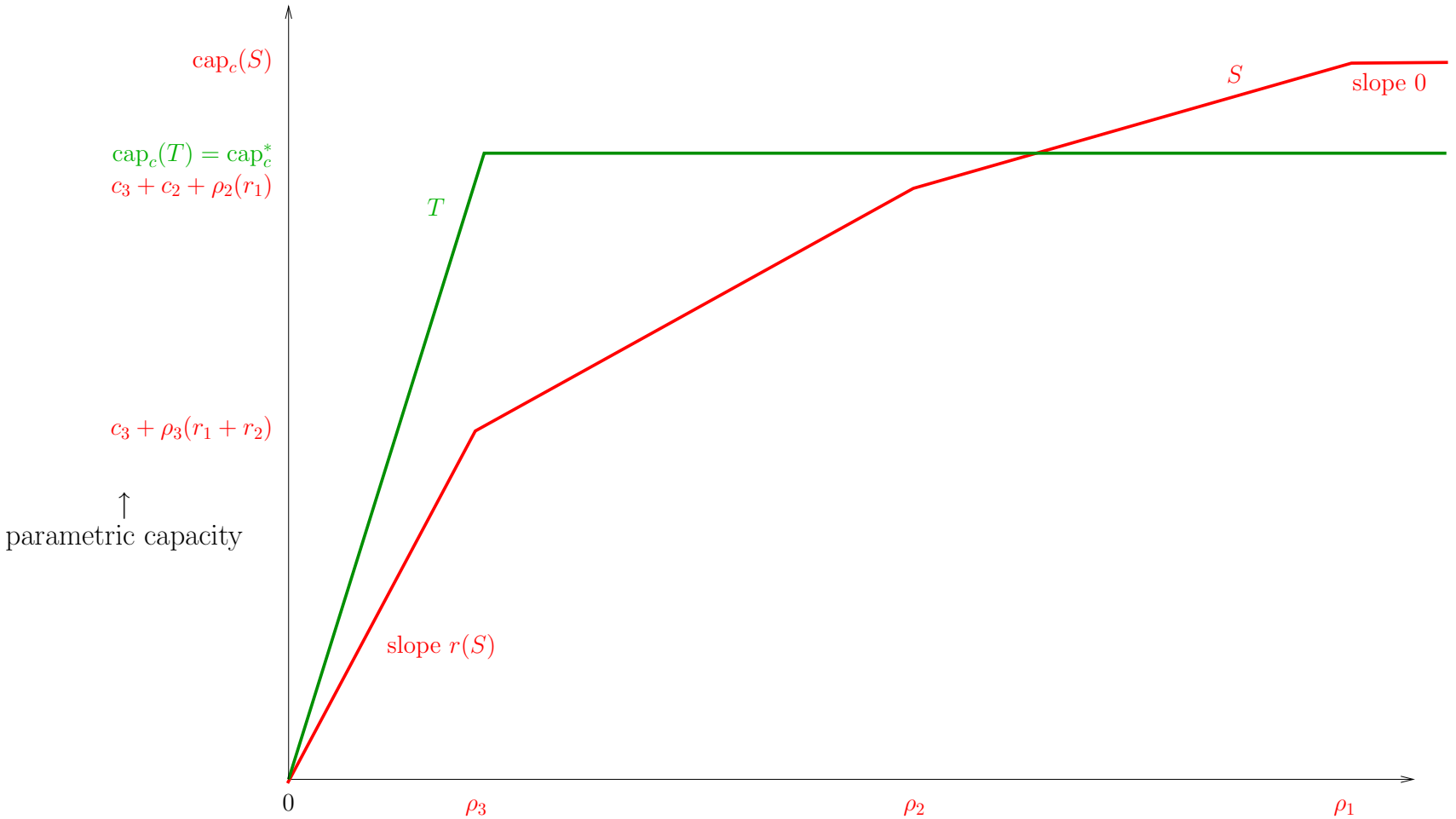

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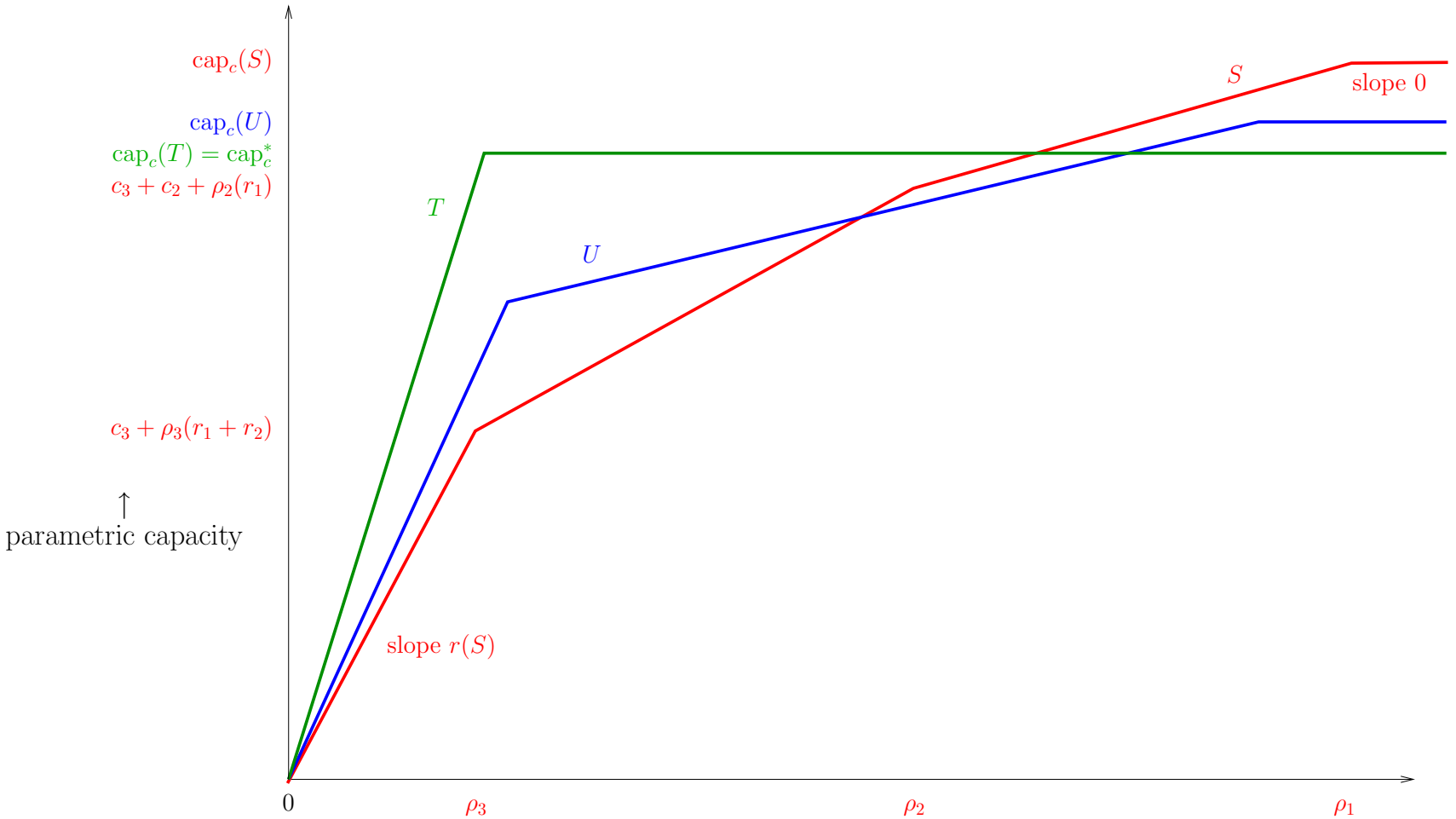

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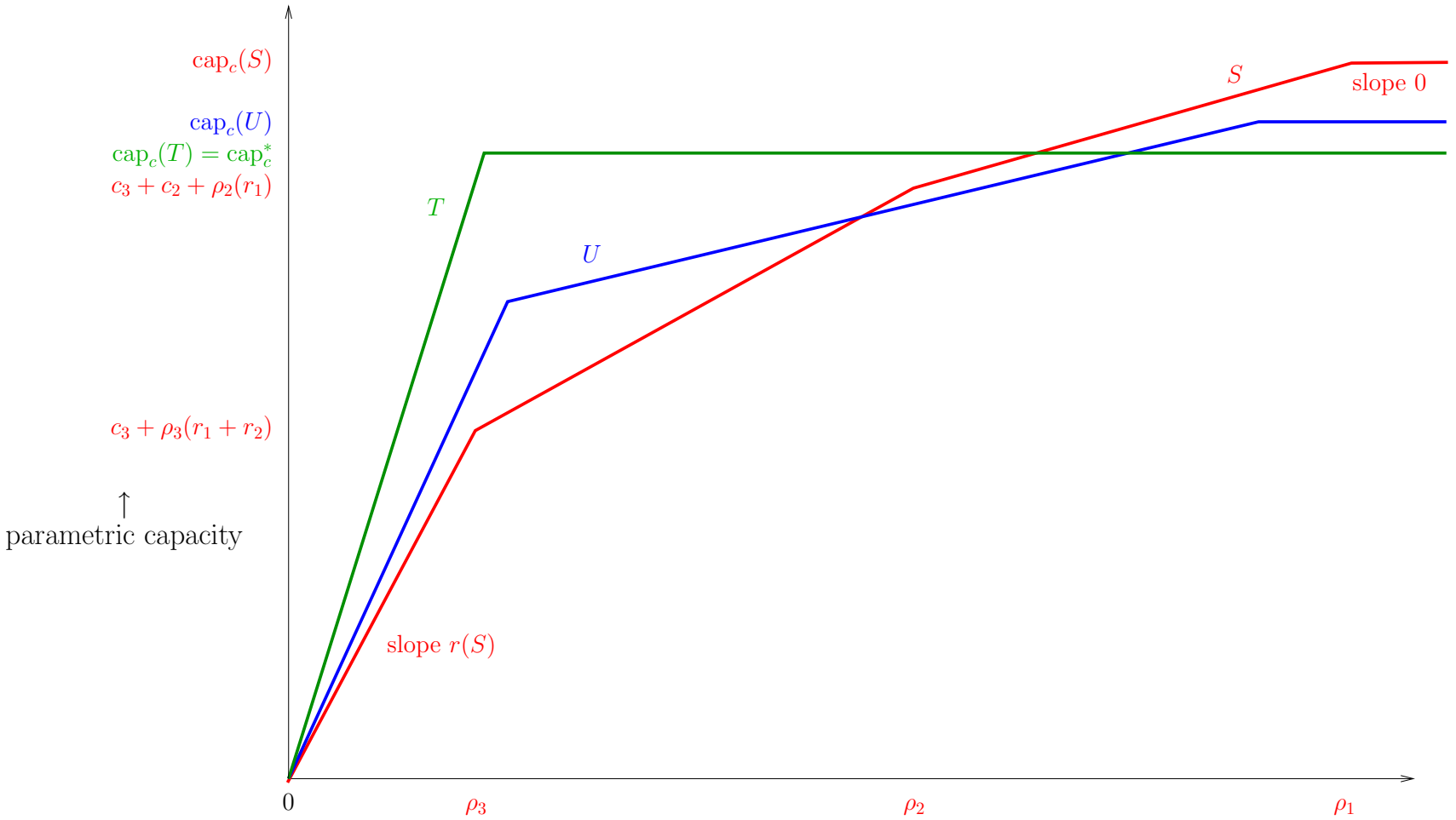

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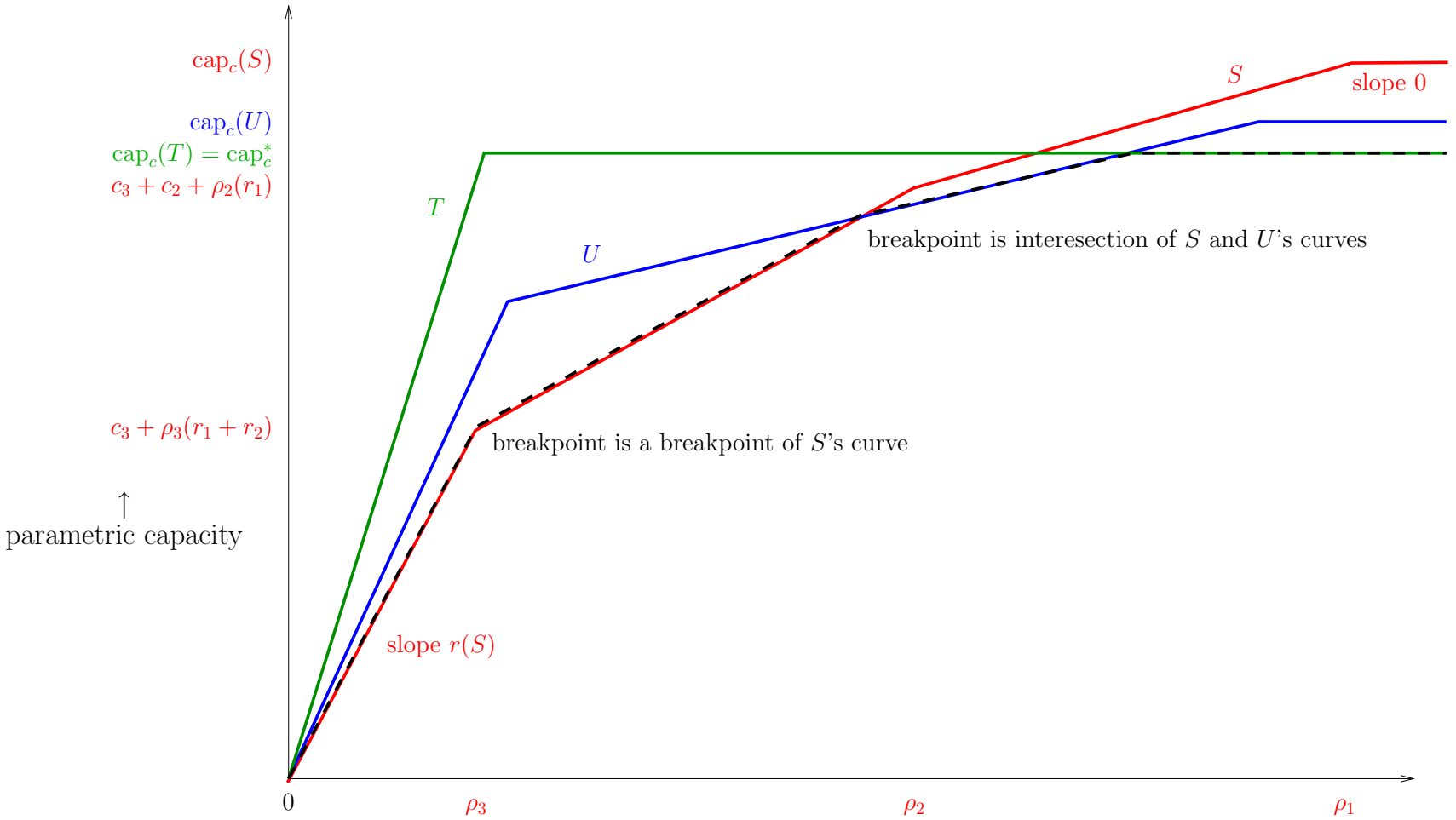
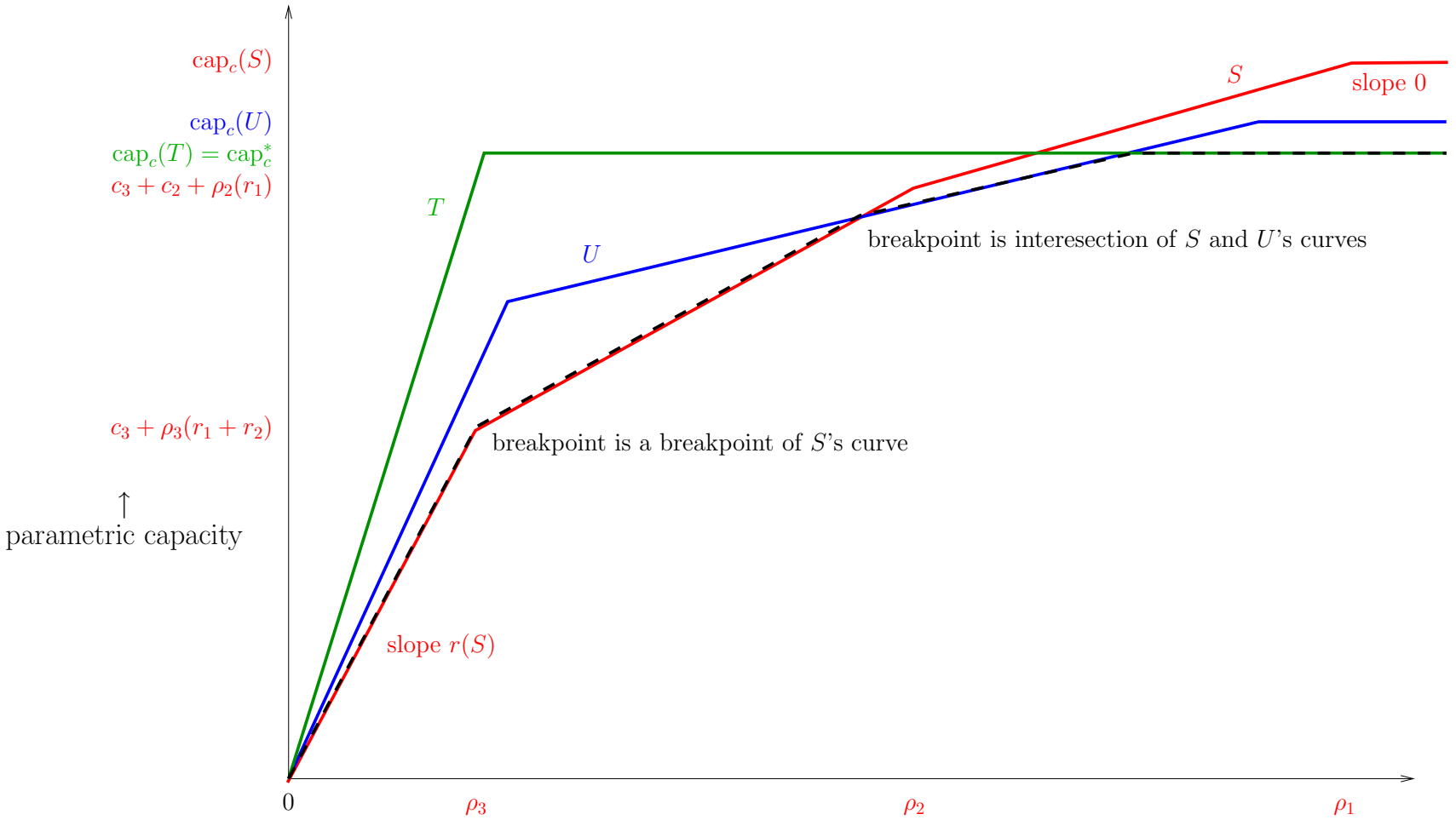


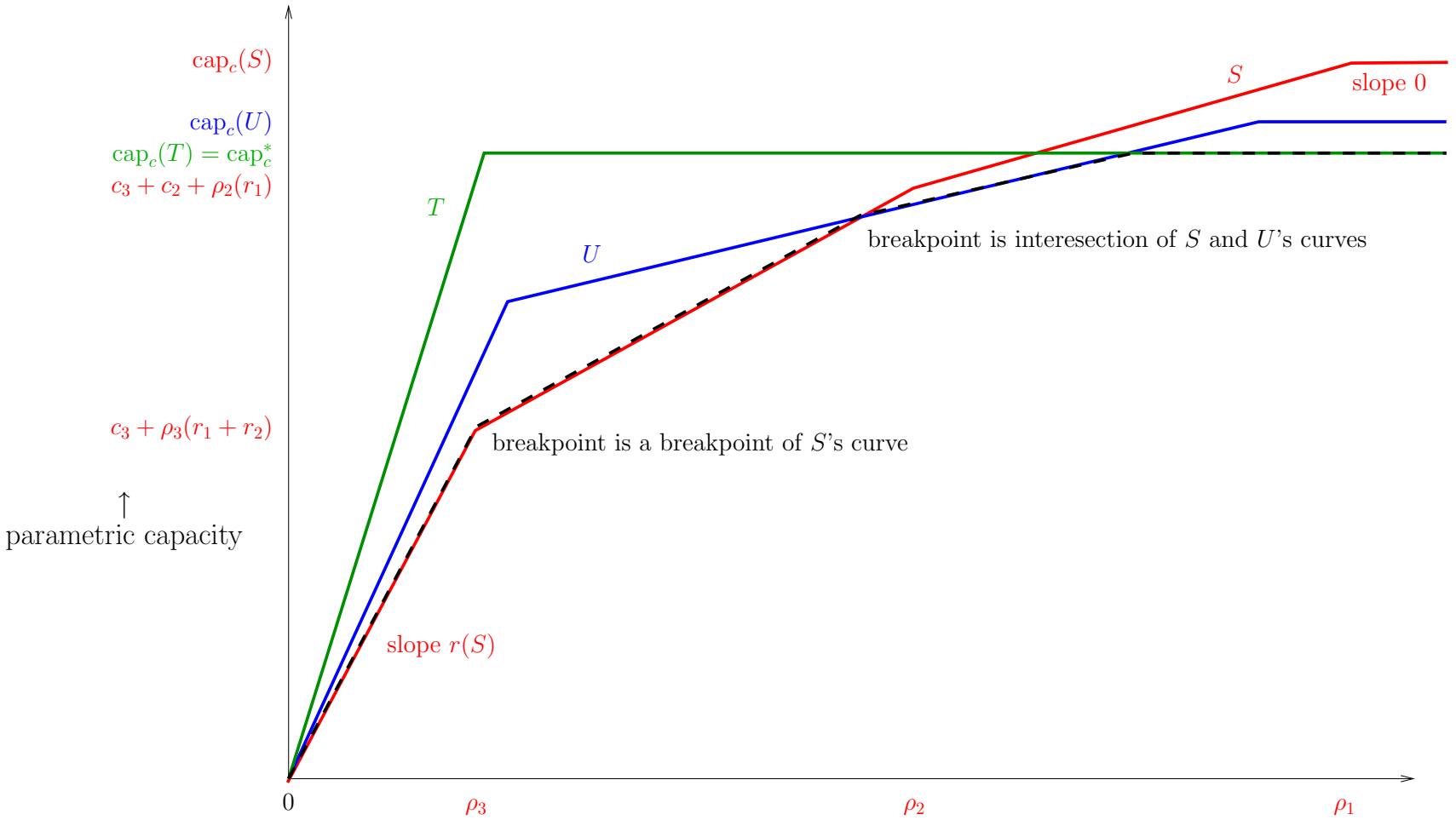
The overall parametric capacity curve: the λ-profile

Since the minimum of a bunch of concave curves is again concave, this time we do not need to linearize. We call this overall parametric capacity curve the λ-profile.



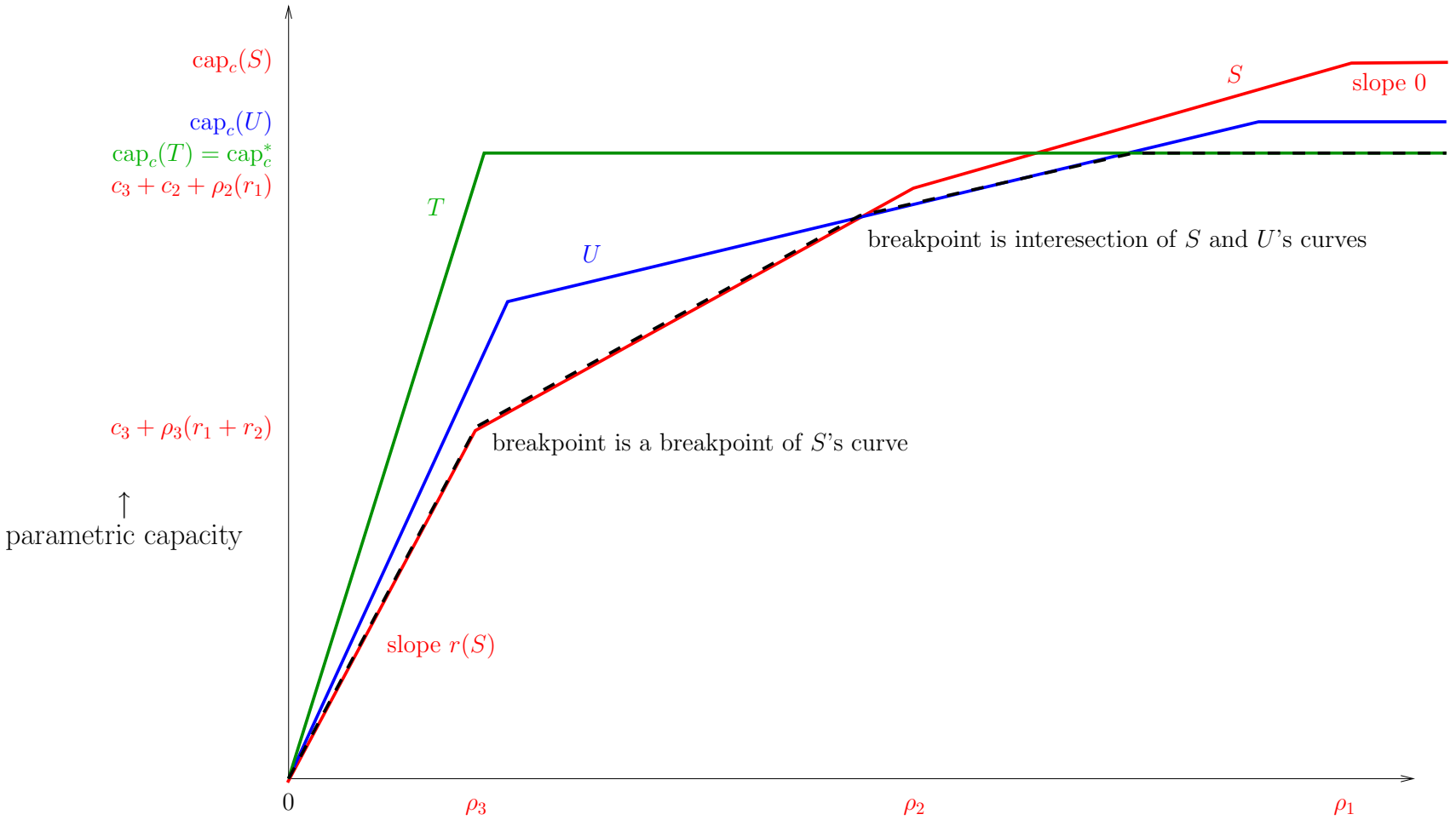
The overall parametric capacity curve: the λ-profile

We can compute things like $\mathrm{cap}^*(\lambda)$ easily using parametric min cut technology.



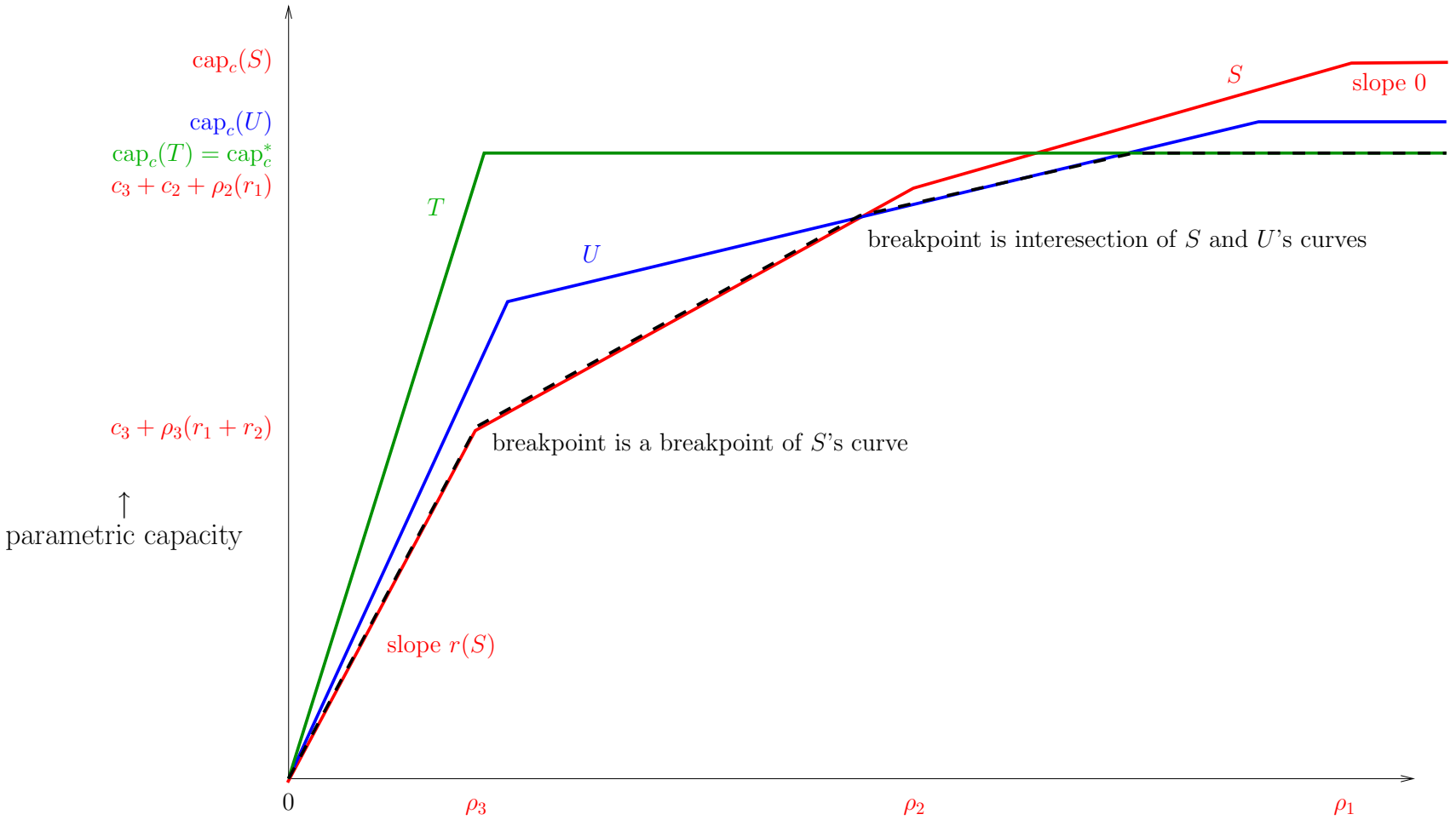
The overall parametric capacity curve: the λ-profile

There is conjugate duality between S's interdiction and parametric capacity curves; convex duality shows this carries over to conjugate duality between the B-profile and the λ-profile.



The overall parametric capacity curve: the λ-profile

The Burch et al. pseudo-approximation framework asks: for a given B, compute the two cuts S_1 and S_2 bracketing B on the B-profile.



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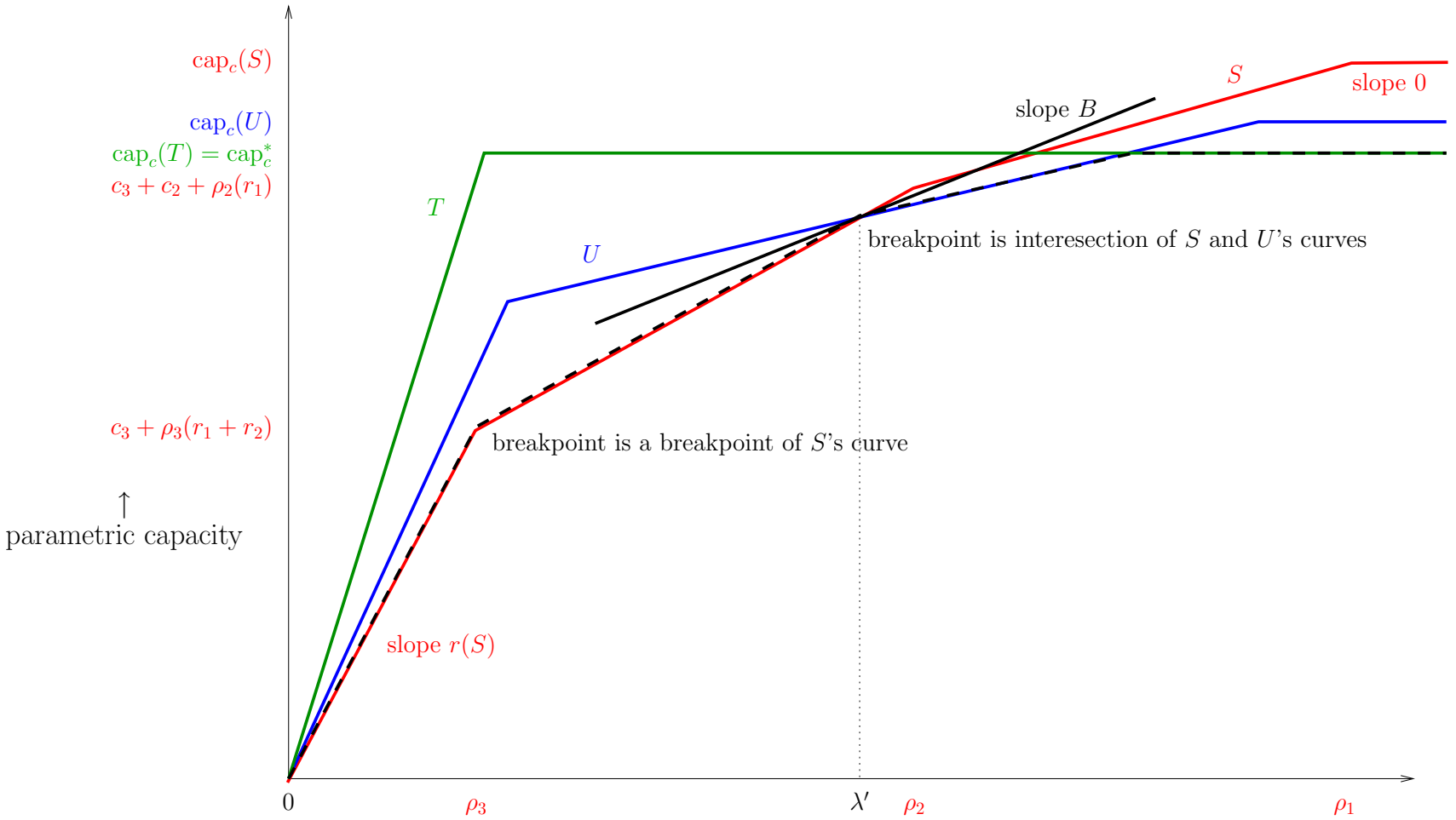
Conjugate duality implies that this is equivalent to finding a breakpoint λ' on the λ-profile whose adjacent slopes bracket B, here S and U; we also get $B_1 = B(S_1, \lambda')$, $C_1 = C(S_1, \lambda')$, $B_2 = B(S_2, \lambda')$, and $C_2 = C(S_2, \lambda')$.



Outline

The key Slope Problem

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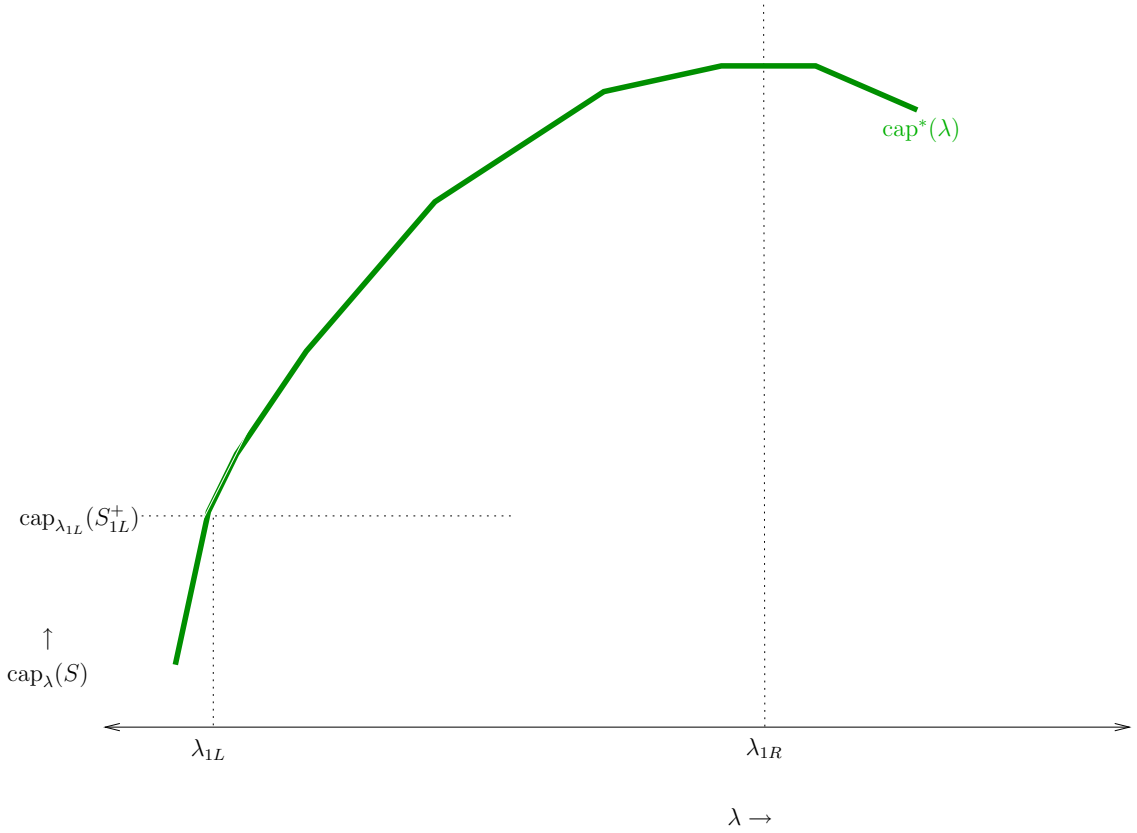
  4. Various scheduling problems with multiple budgets.

Newton-B gives a better algorithm

Set $\lambda_L = 0$ and $\lambda_R = \text{cap}_r^*$ as in Binary Search. Denote $\text{sl}^+(\lambda_L)$ by sl_L^+ and $\text{sl}^-(\lambda_R)$ by sl_R^-.



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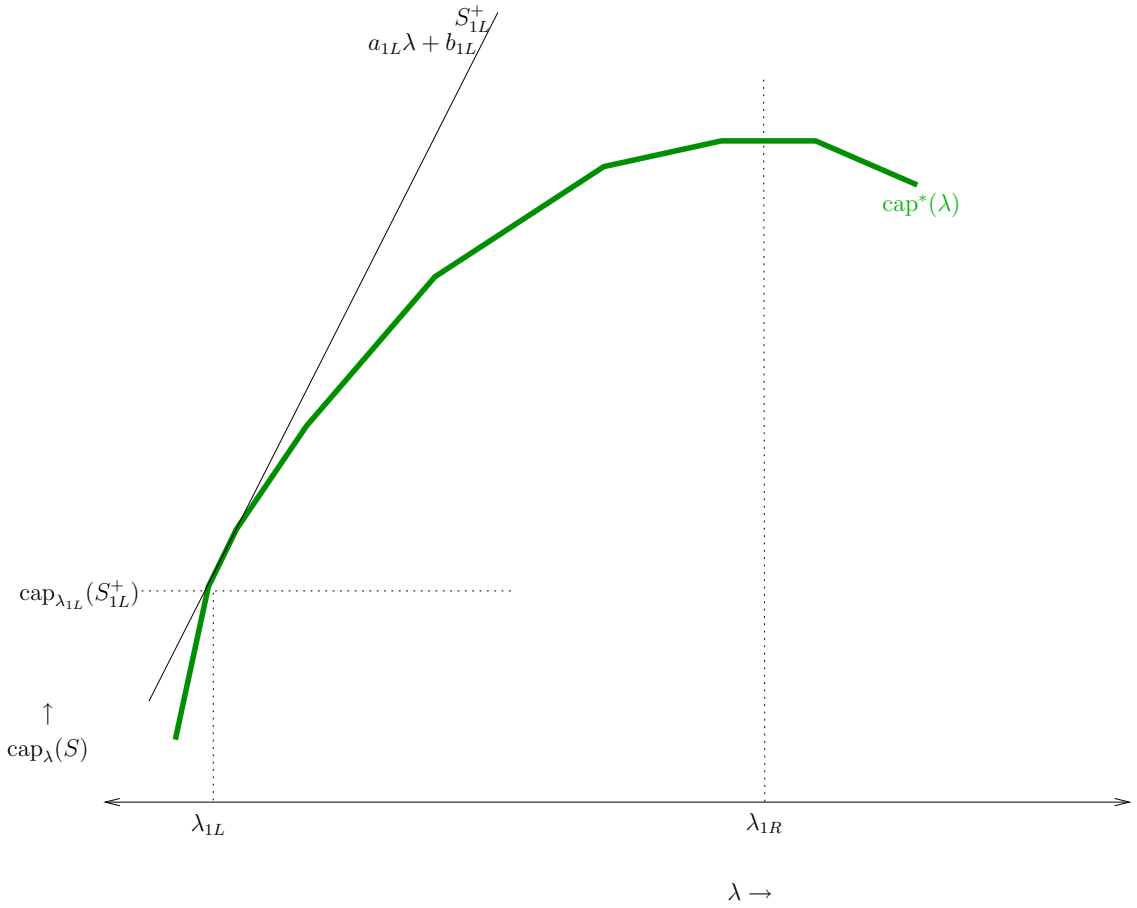

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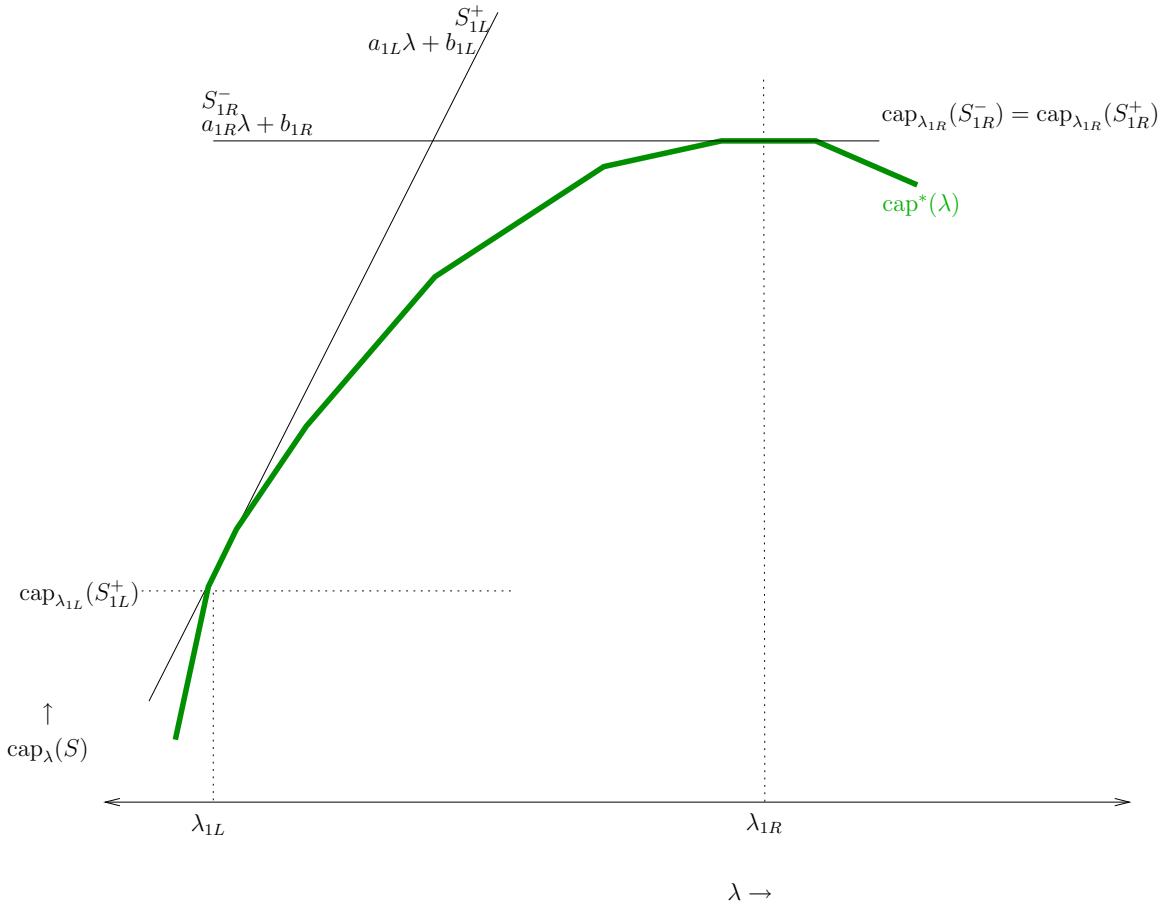

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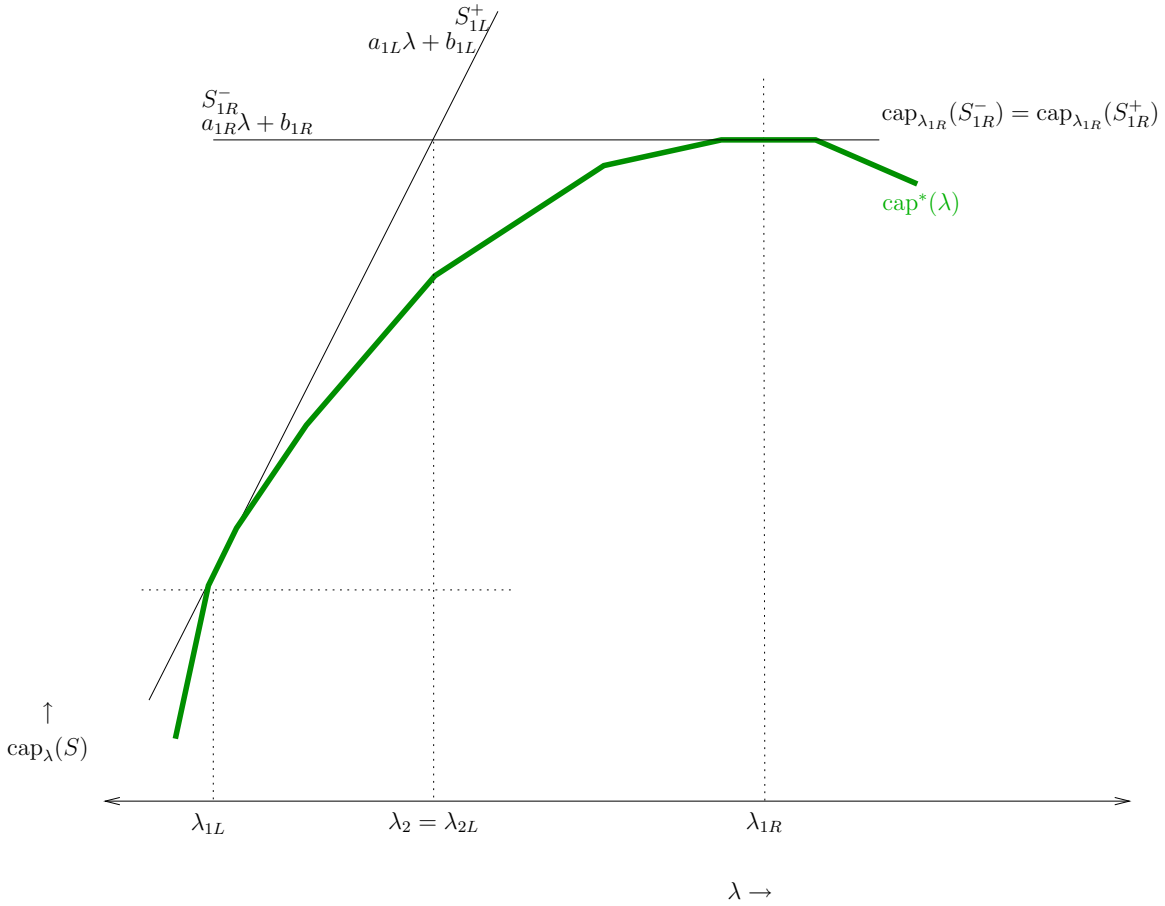
Compute $\hat{\lambda}$ as the intersection of the line of slope sl_L^+ through $(\lambda_L, \mathrm{cap}^*(\lambda_L))$, and the line of slope sl_R^- through $(\lambda_R, \mathrm{cap}^*(\lambda_R))$, a max flow w.r.t. $\hat{\lambda}$, and $\mathrm{sl}^-(\hat{\lambda})$ and $\mathrm{sl}^+(\hat{\lambda})$.



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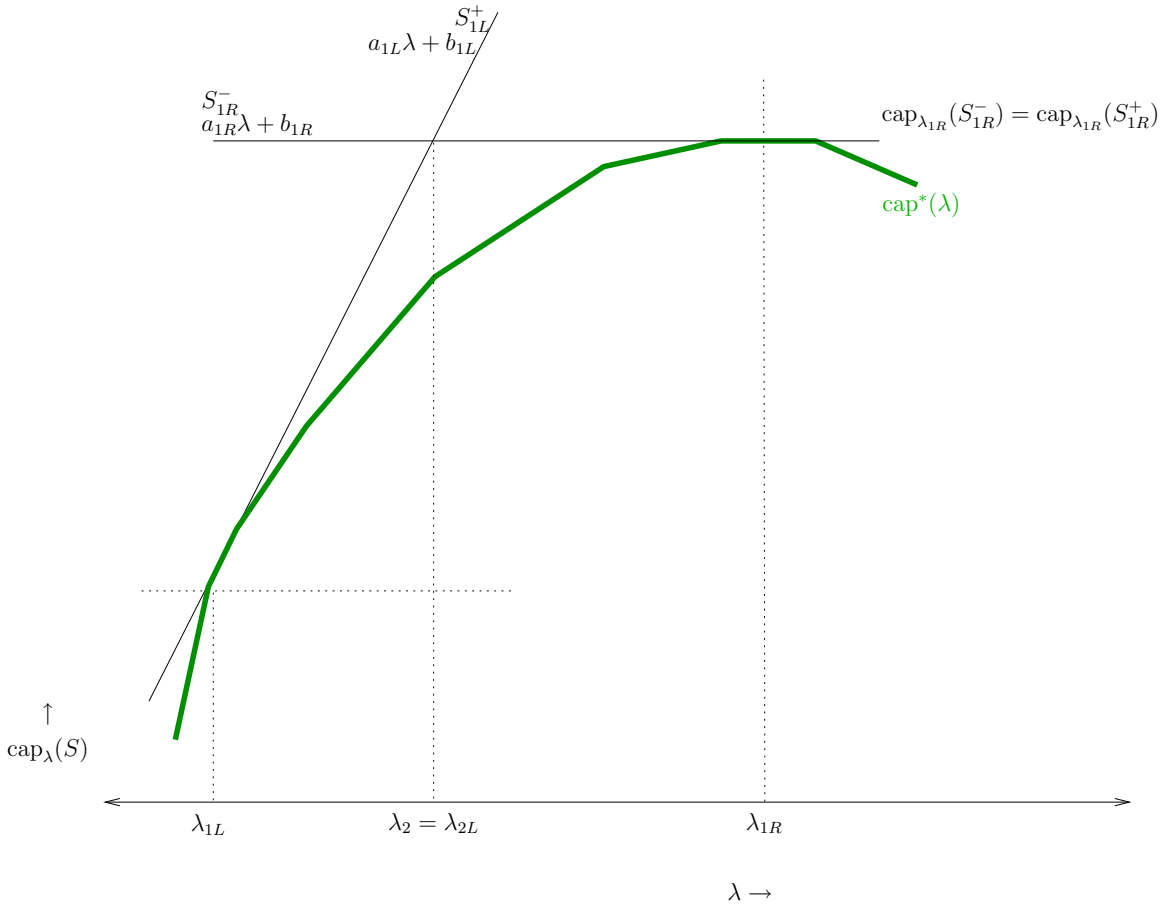
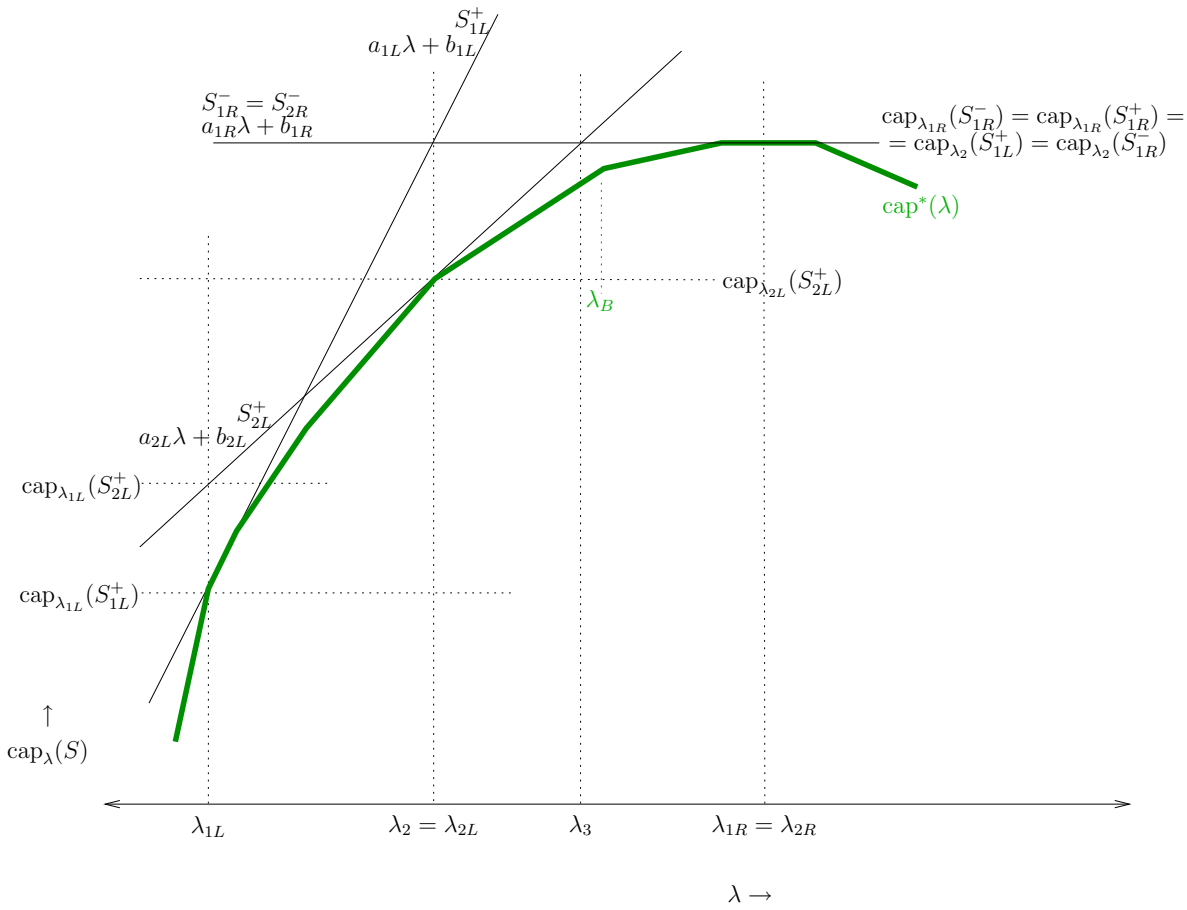
If $B \in [\mathrm{sl}^+(\hat{\lambda}), \mathrm{sl}^-(\hat{\lambda})]$, then $\lambda_B = \hat{\lambda}$ and we can stop.



Newton-B gives a better algorithm

Otherwise, if $B < \mathrm{sl}^+(\hat{\lambda})$ then replace λ_L by $\hat{\lambda}$; else ($B > \mathrm{sl}^-(\hat{\lambda})$) replace λ_R by $\hat{\lambda}$ and go to 2.



The key inequality

- The analysis flows from a key inequality that says that each iteration cuts down the gap between our current slope and our target slope by a factor of at least 2, or the gap between our current vertical position and an upper bound on the vertical position of the optimal breakpoint by a factor of at least 2. Thus Newton-B is never worse than Binary Search.

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- Indeed, this Newton-B algorithm and its analysis works for any concave (or convex) function, even continuous ones.

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  - Choose a path-flow $f \in \mathcal{F}$ of maximum robust value $\mathrm{Rval}(f) := \mathrm{val}(f) - z(f)$, where $z(f) := \max_{\mu\in X}(\sum_{P\in\mathcal{P}} f_P \cdot \max_{e\in P} \mu_e)$ is the maximum amount of f the interdictor can delete using a budget of B

The Maximum Robust Flow problem

- MRF has been studied with uniform removal cost ($r_e = 1 \; \forall e \in E$) and integral interdiction, i.e., when the set of strategies for the interdictor is $\mathcal{X} = \{\mu_e \in \{0,1\}^E \mid \sum_{e \in E} \mu \leq B\}$. In this case:

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- Our contributions:

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  - Algorithm: compute any path decomposition of the optimal solution to the Slope Problem

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and let $\bar{f} = \{\bar{f}_P, P \in \mathcal{P}\}$ be a path-decomposition of $\bar{x}$. Also, let f^* be a flow with maximum robust value $\mathrm{Rval}(f^*)$. Then the following hold:

where $\rho = 1 - (1 - \frac{1}{k})^k$ and $k = \max_{P \in \mathcal{P}} |E(P)|$.

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(ii) $\mathrm{Rval}(f^*) - \mathrm{Rval}(\bar{f}) \leq \frac{1-\rho}{\rho}(\mathrm{val}(f^*) - \mathrm{Rval}(f^*))$

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Outline

Multiple budgets equals multiple parameters

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- Now we'd be trying to find a point on the parametric surface whose local derivatives bracket the given budgets in the coordinate directions.
- As before we could solve this via LP, but we'd prefer a combinatorial algorithm.
- Interdiction already gets complicated with two parameters, so let's consider a simpler multiple parameter scheduling problem instead.

Chen's '94 scheduling problem

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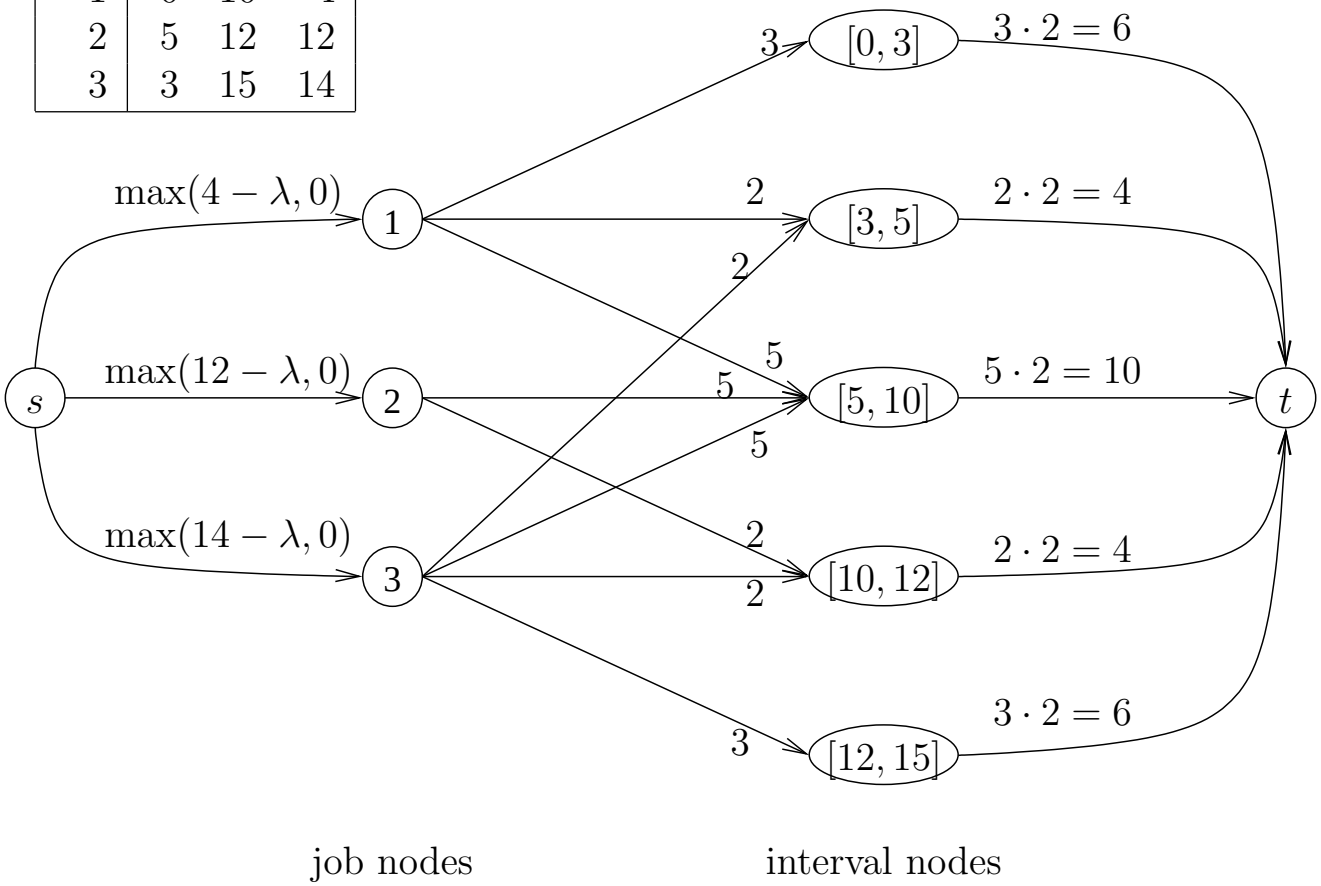
- Now we want to minimize λ such that there exists a flow saturating all residual job arcs.

Chen's scheduling problem: example

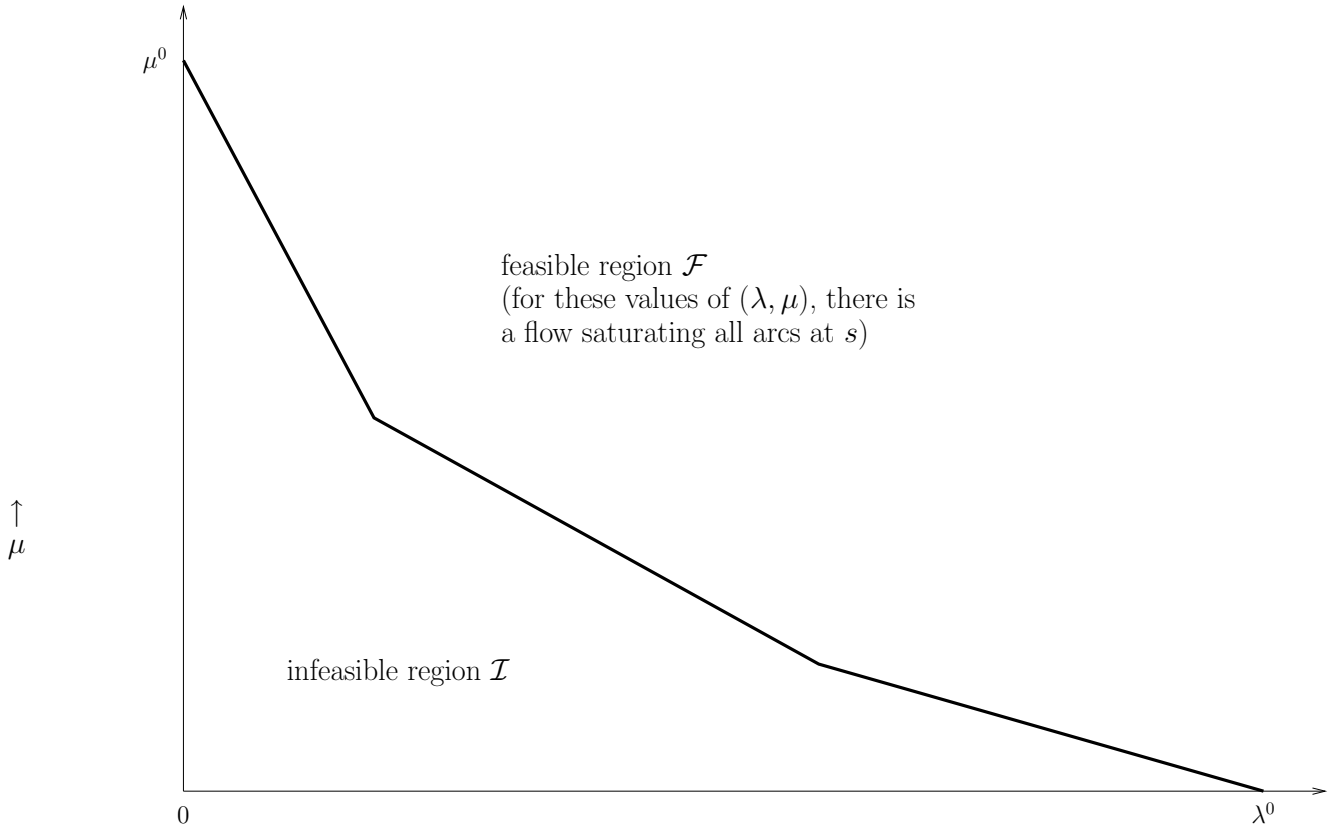
This single-parameter version can be solved using Gallo-Grigoriadis-Tarjan (GGT) '89 parametric min cut in $O(1)$ max flow time.

job	r_j	d_j	p_j
1	0	10	4
2	5	12	12
3	3	15	14



Two-parameter Chen

Suppose now that there are two ways to outsource, λ and μ such that if we pay $\$\lambda + \$\mu$, we reduce p_j to $\max(0, p_j - a_j\lambda - b_j\mu)$.
In the (λ, μ) plane there is a piecewise linear convex curve separating feasible points from infeasible ones.



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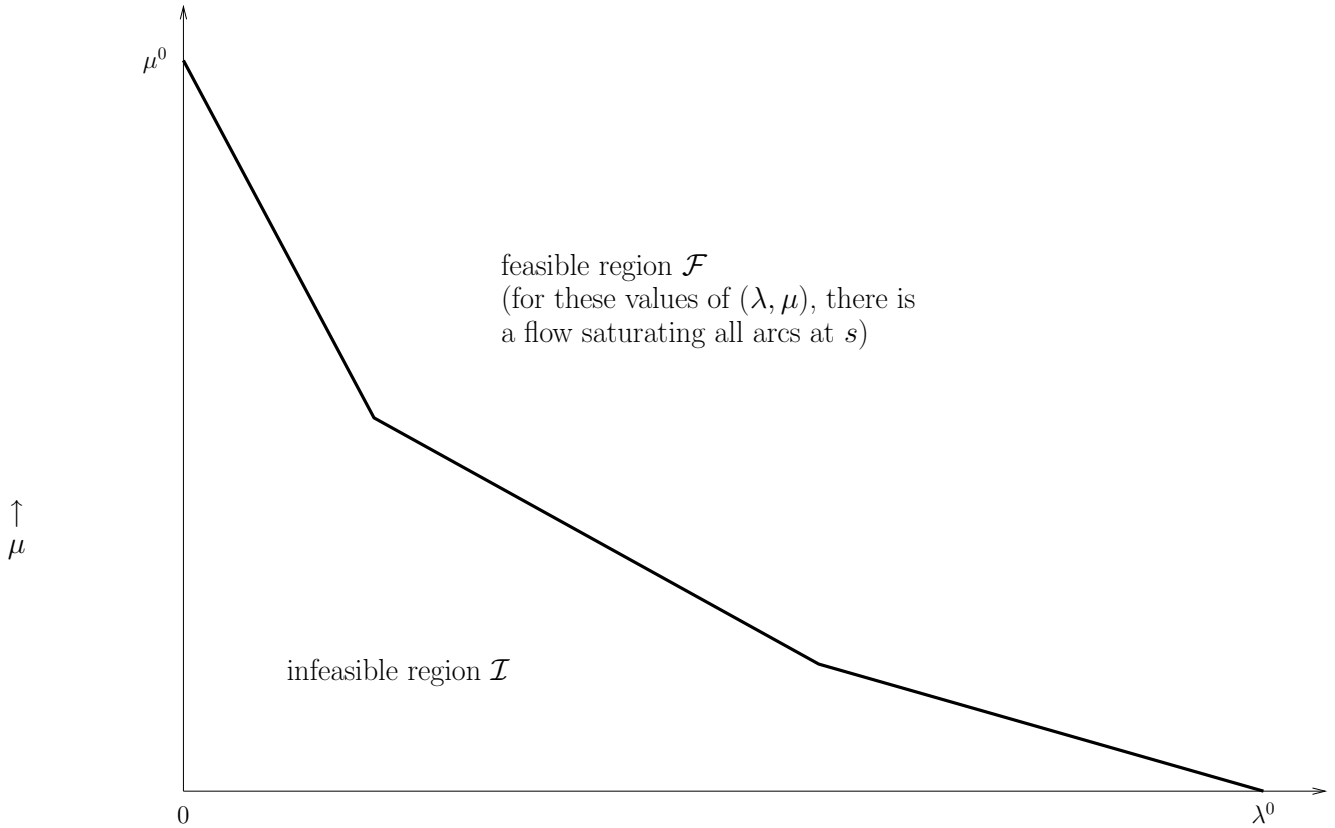
For node subset S with $D \subseteq \delta^-(S) \cap \delta^+(\{s\})$, the constraints defining this region have the form $\lambda a(D) + \mu b(D) \geq p(D) - c(\delta^+(S))$.



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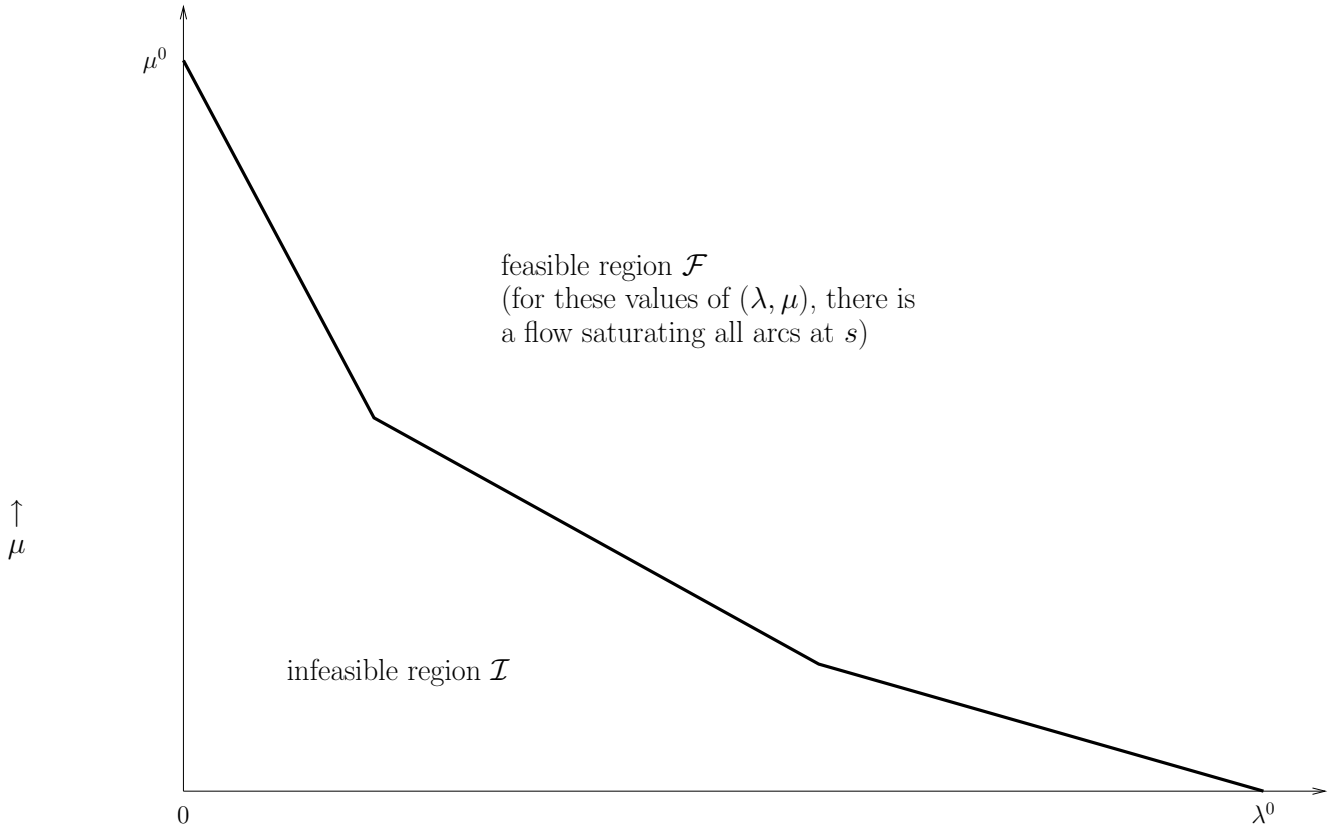

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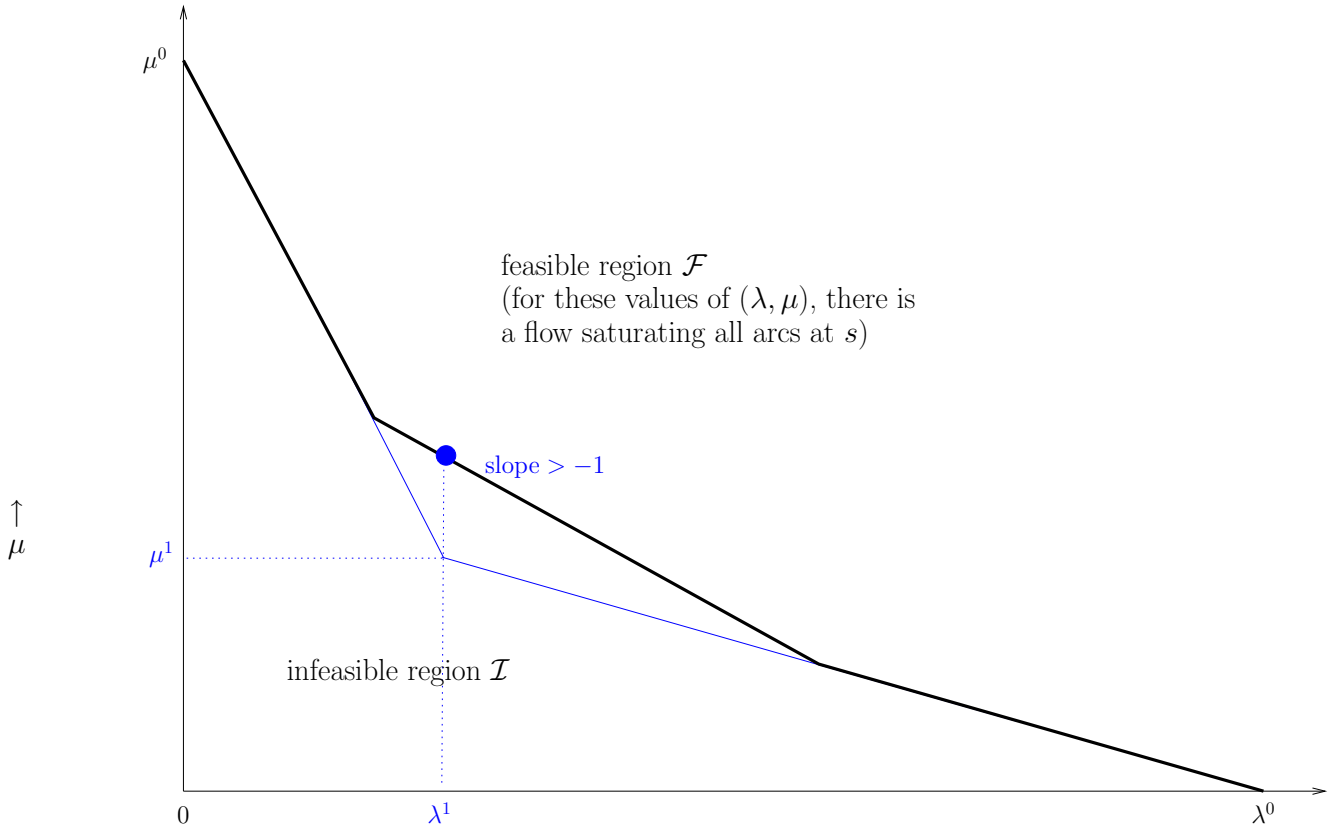

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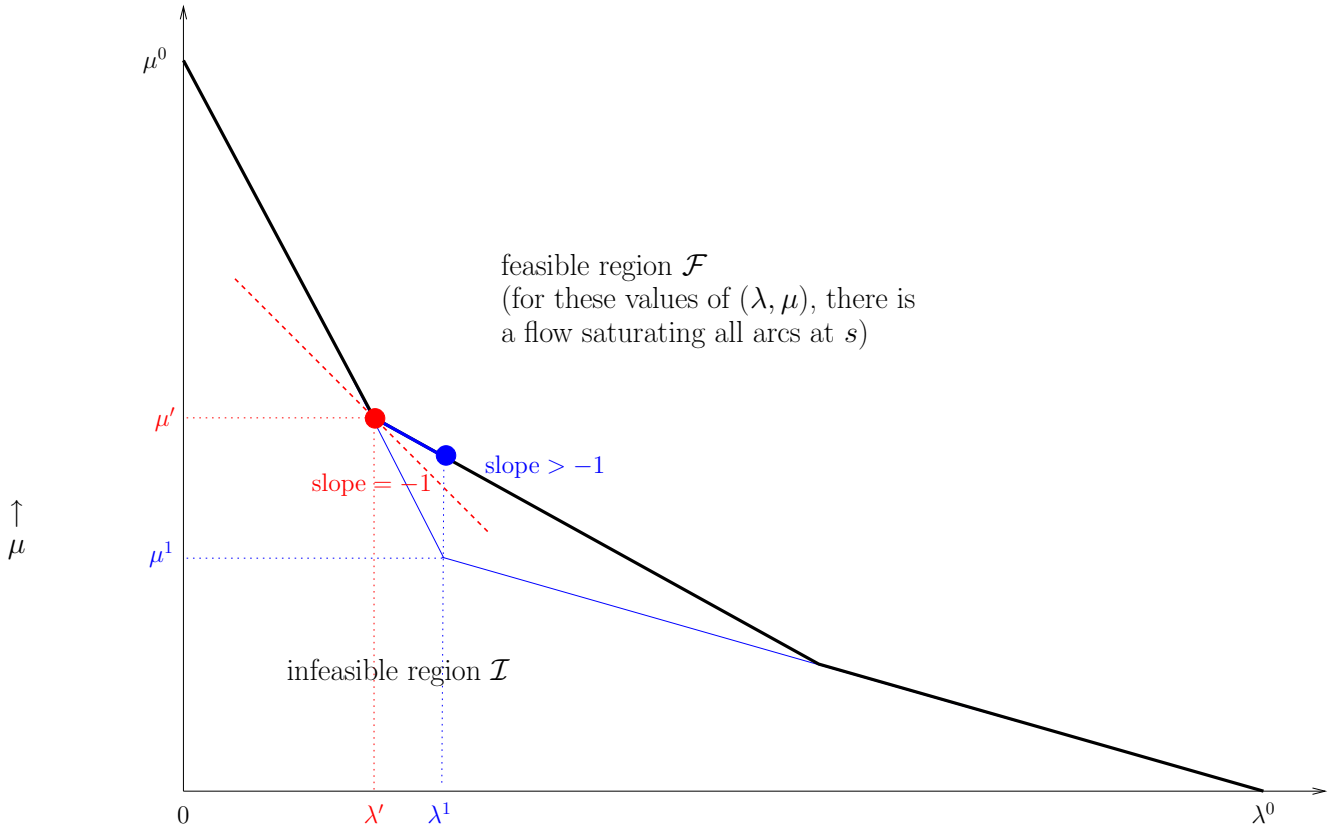

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- This generalizes to any fixed number of parameters.
- Open Question: LP is polynomial even when the number of parameters is not fixed. Can we get a combinatorial algorithm then?

Any questions?

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Comments?

Thank you

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