

Integer Minimization in Variable Dimension

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$$\min\{f(x, y) : Ax + By \leq b, y \in \mathbb{R}^n, x \in \mathbb{Z}^d\}$$

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Bridge the gap?

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Question: under which conditions on the input is this problem tractable?

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Which conditions one needs to impose on A, b ?

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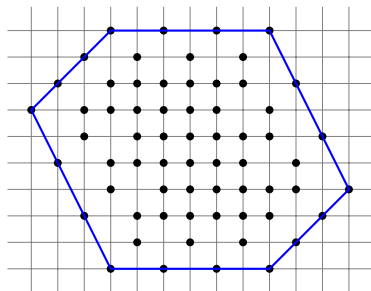
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However, typically

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$$\mathcal{F} = [0, 3]^3 \cap \mathbb{Z}^3$$

$$W = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$



The Main Tool - Diagonal Frobenius Number

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- ▶ M has HNF Identity, and
- ▶ $C(M) = \{M\lambda : \lambda \geq 0\}$ is a pointed cone.

Let $v = M1$. The *diagonal Frobenius number* $F(M)$ is defined as the smallest integer t such that

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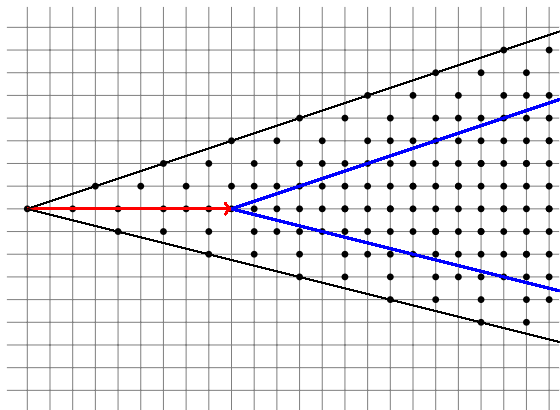
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$$M = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \end{pmatrix}$$

$$v = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

$$F(W) = 1$$



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Theorem (Aliev, Henk 2010).

$$F(M) \leq \frac{(n-d)\sqrt{n}}{2} \sqrt{\det(MM^T)}.$$

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- ▶ For fixed d , the bound is polynomial in the **unary** encoding of M .

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Then there exists $\delta^* = \text{poly}(n, w_{MAX}, F(W))$ such that all points in

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are accessible (points with nonempty fibers).

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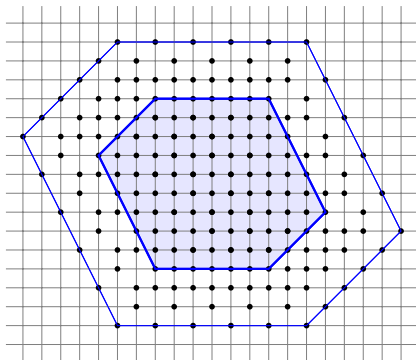
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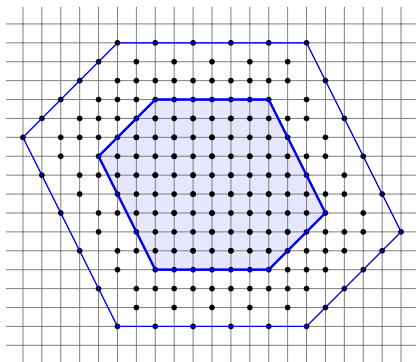
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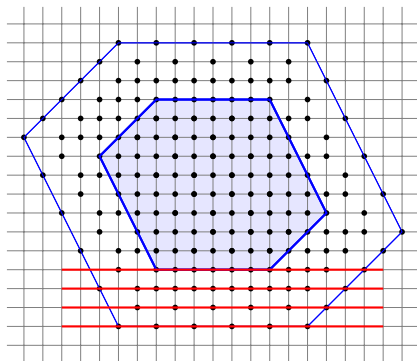
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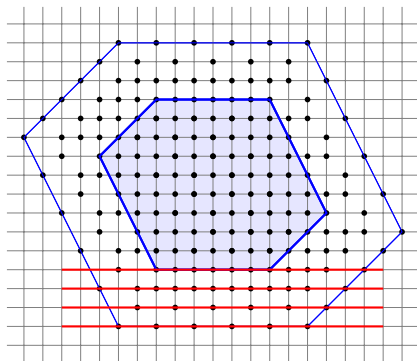


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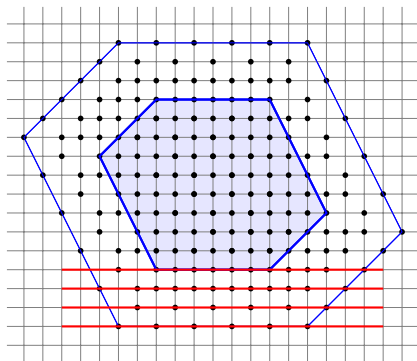
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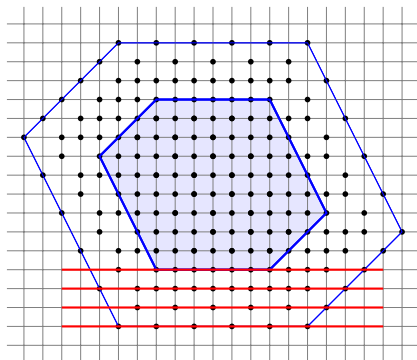
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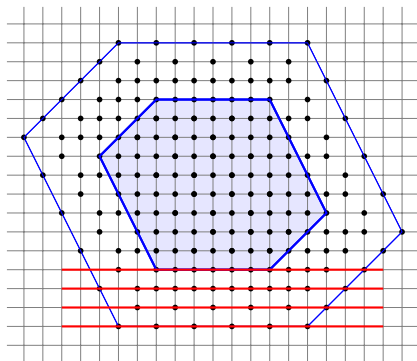


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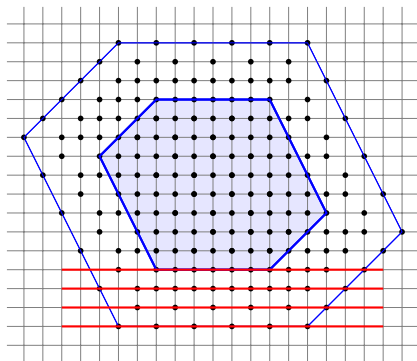
Lemma (line search). The problem restricted to any line can be solved in time polynomial in n , $F(W)$ and w_{MAX} .

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- ▶ Solve $\min\{f(y) : y \in P'\} \Rightarrow y^*$.
- ▶ Obtain $x^* \in P$ such that $Wx^* = y^*$.
- ▶ For each line L , sufficiently close to boundary
 - ▶ Perform line search to find solution to $\min\{f(Wx) : x \in L \cap P\} \Rightarrow x'$.
 - ▶ Replace x^* with x' if its objective value is smaller.
- ▶ Return x^* .

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- ▶ New techniques for dealing with variable-dimension nonlinear programming.

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Theorem (Box Constraints, $d = 2$). The problem

$$\min\{f(Wx) : l \leq x \leq u, x \in \mathbb{Z}^n\}$$

can be solved in time polynomial in n , the unary encoding of $W \in \mathbb{Z}^{2 \times n}$, and the binary encoding of l and u .

- ▶ Arbitrary fixed d .
- ▶ $\mathcal{F} = \{Ax \leq b, x \in \mathbb{Z}^n\}$ and A with small sub-determinants.

Future Work

- ▶ More solvable cases (classes of P and W).
- ▶ New techniques for dealing with variable-dimension nonlinear programming.

Thank You!