Integer Minimization in Variable Dimension

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Motivation - Nonlinear Integer Optimization

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Many new results in fixed dimension d

$$\min\{f(x,y) : Ax + By \leq b, y \in \mathbb{R}^n, x \in \mathbb{Z}^d\}$$

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Bridge the gap?

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Question: under which conditions on the input is this problem tractable?

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- Partition problem. $w_1, \cdots, w_n \in \mathbb{Z}_+$, $D = \frac{1}{2}\sum_{i=1}^n w_i$.
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Which conditions one needs to impose on A, b?

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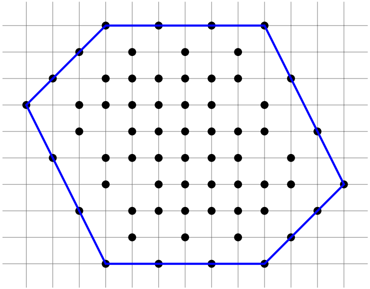
$$\mathcal{F} = [0,3]^3 \cap \mathbb{Z}^3$$

$$W = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 0 & 1 \end{pmatrix}$$



The Main Tool - Diagonal Frobenius Number

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- M has HNF Identity, and
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Let $v = M1$. The *diagonal Frobenius number* $F(M)$ is defined as the smallest integer t such that

$$(tv + C(M)) \cap \mathbb{Z}^d \subset \{Mx \ : \ x \in \mathbb{Z}_+^m\}.$$

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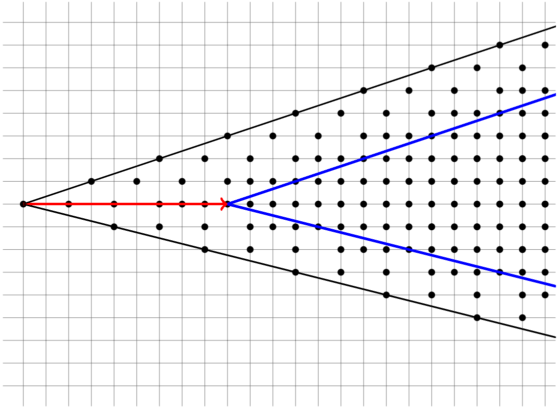
$$M = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \end{pmatrix}$$

$$v = \begin{pmatrix} 9 \\ 0 \end{pmatrix}$$

$$F(W) = 1$$



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Theorem (Aliev, Henk 2010).

$$F(M) \leq \frac{(n-d)\sqrt{n}}{2}\sqrt{\det(MM^T)}.$$

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- For fixed d, the bound is polynomial in the unary encoding of M.

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Lemma (Inner region). Let $P = \{x \in \mathbb{R}^n \ : \ Ax \leq b\}$. Let

$$P_\delta = \{x \in \mathbb{R}^n \ : \ x + B(\delta) \subset P\}.$$

Then there exists $\delta^* = poly(n, w_{MAX}, F(W))$ such that all points in

$$Q_{\delta^*} = \{Wx : x \in P_{\delta^*}\} \cap \mathbb{Z}^d$$

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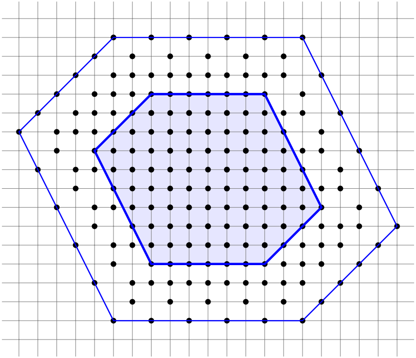
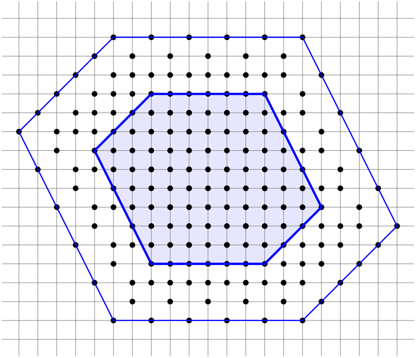


An Outline of the Algorithms



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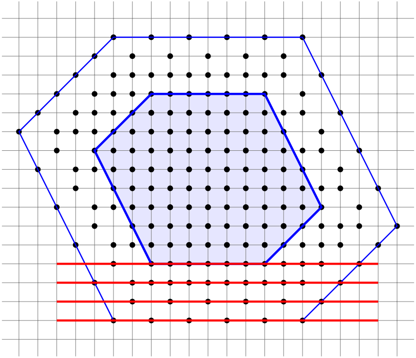

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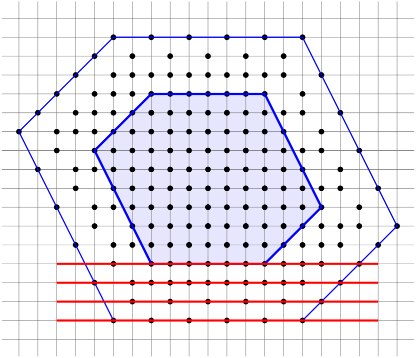

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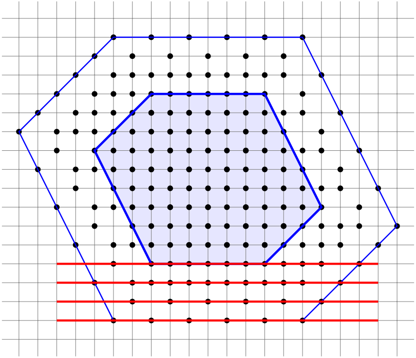

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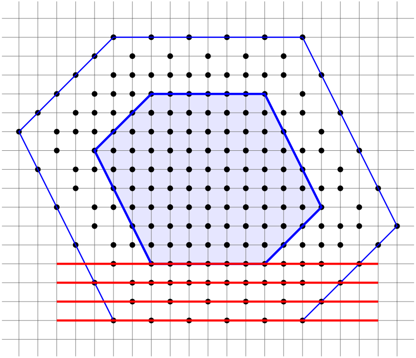

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Lemma (line search). The probelm restricted to any line can be solved in time polynomial in $n, F(W)$ and w_{MAX}.

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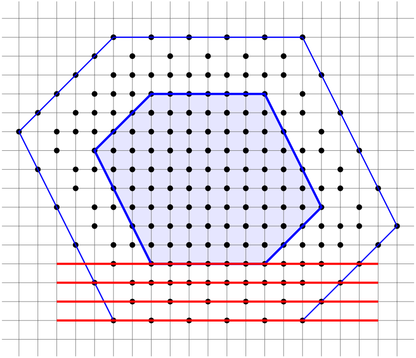

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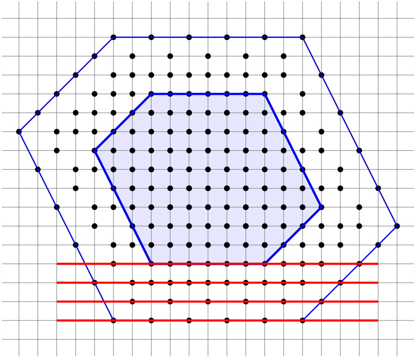

- Solve $\min\{f(y) \ : \ y \in P'\} \quad \Rightarrow y^*$.
- Obtain $x^* \in P$ such that $Wx^* = y^*$.
- For each line L, sufficiently close to boundary
  - Perform line search to find solution to $\min\{f(Wx) \ : \ x \in L \cap P\} \quad \Rightarrow x'$.
  - Replace x^* with x' if its objective value is smaller.
- Return x^*.

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- New techniques for dealing with variable-dimension nonlinear programming.

Summary

Theorem (Box Constraints, $d = 2$). The problem

$$\min\{f(Wx) \ : \ l \leq x \leq u, \ x \in \mathbb{Z}^n\}$$

can be solved in time polynomial in n, the unary encoding of $W \in \mathbb{Z}^{2\times n}$, and the binary encoding of l and u.

- Arbitrary fixed d.
- $\mathcal{F} = \{Ax \leq b, x \in \mathbb{Z}^n\}$ and A with small sub-determinants.

Future Work

- More solvable cases (classes of P and W).
- New techniques for dealing with variable-dimension nonlinear programming.

Thank You!