Aircraft routing: complexity and compact linear integer program

Axel Parmentier, Frédéric Meunier

CERMICS

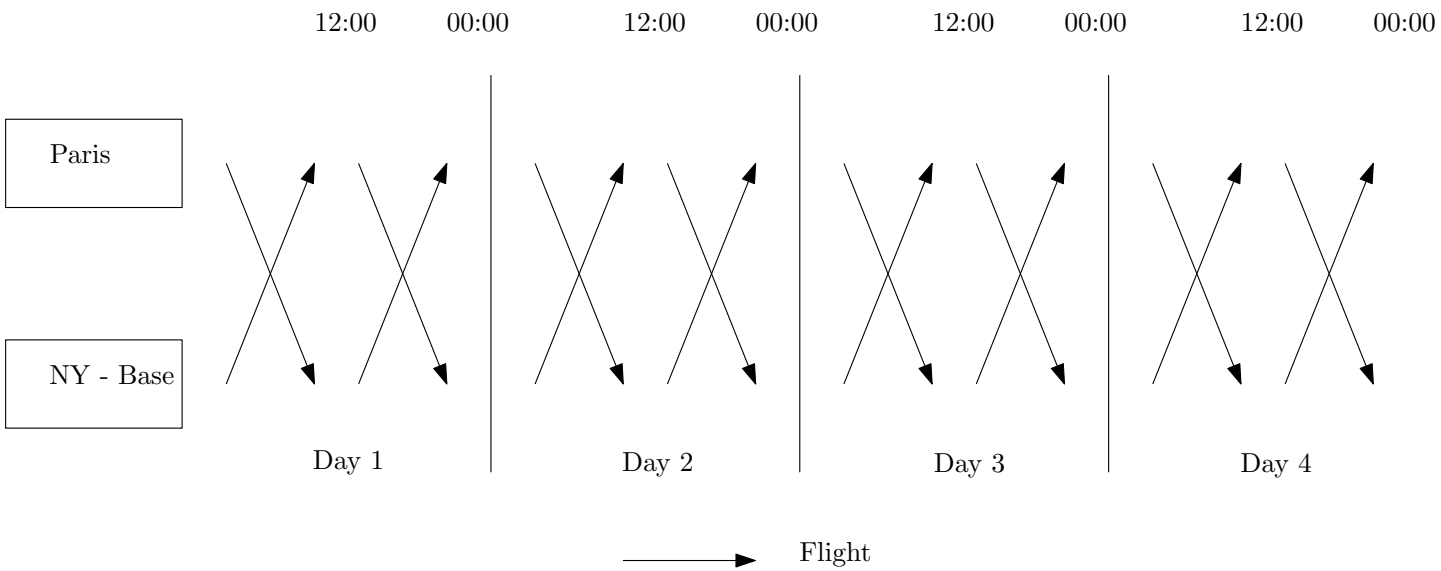
Aussois – January 6, 2014

Table of contents

Aircraft routing problem

Feasible routing

Feasible string

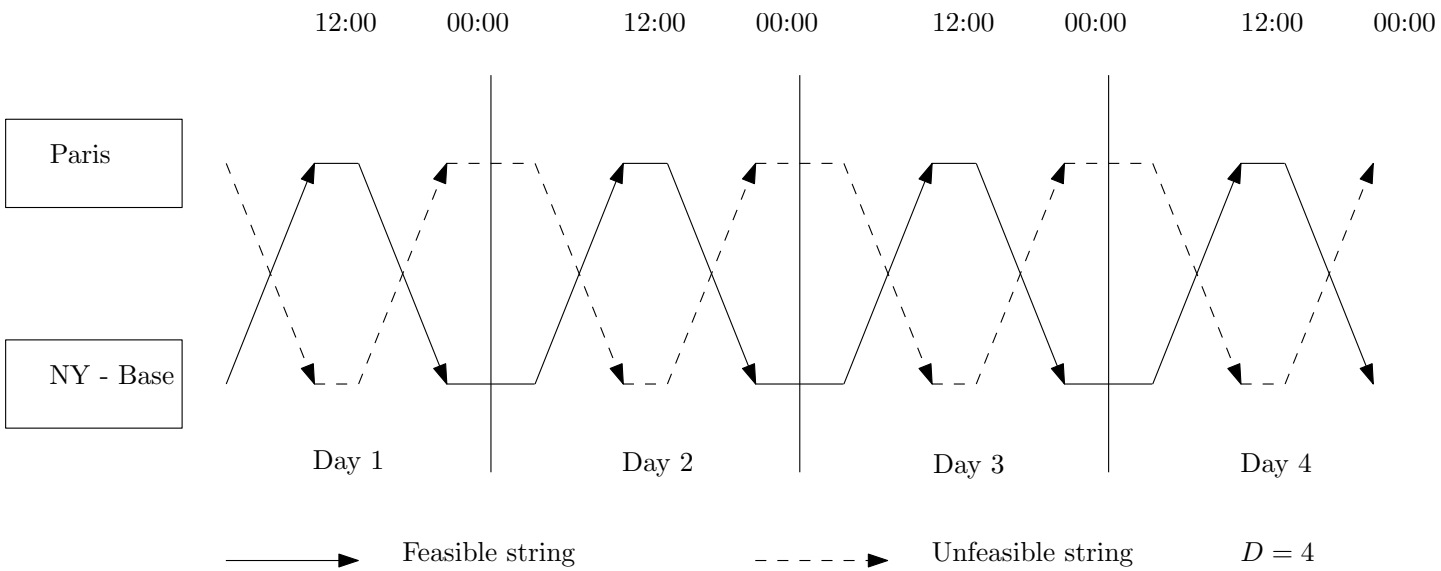


Aircraft routing problem

Feasible routing

- Cover constraint

Feasible string



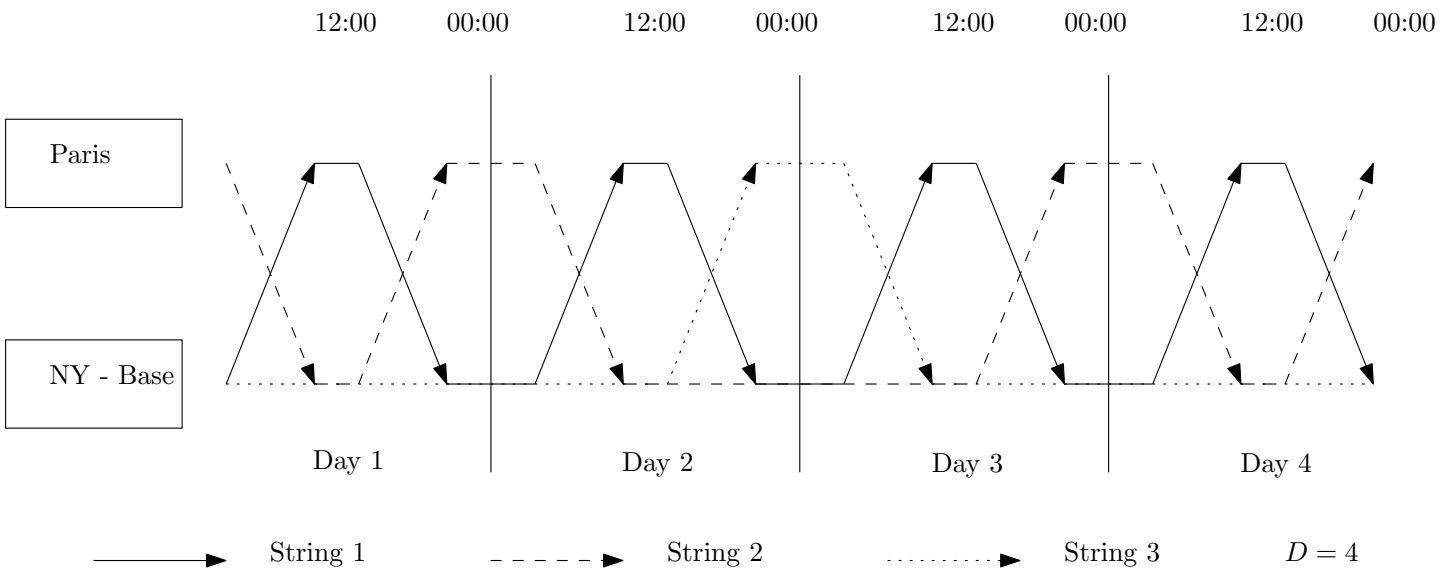
Aircraft routing problem

Feasible routing

- Cover constraint
- Initial and final conditions

Feasible string

- Maintenance constraint



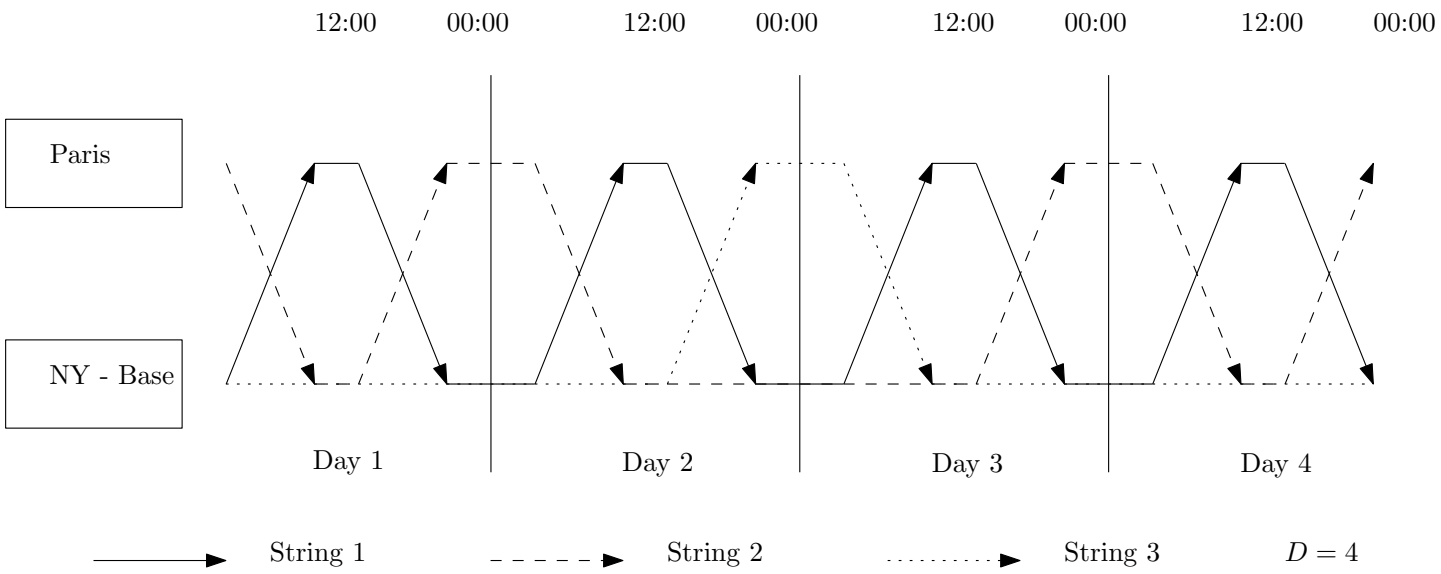
Aircraft routing problem

Feasible routing

- Cover constraint
- Initial and final conditions

Feasible string

- Maintenance constraint



Mathematical formalism: connection graph.

Connection graph

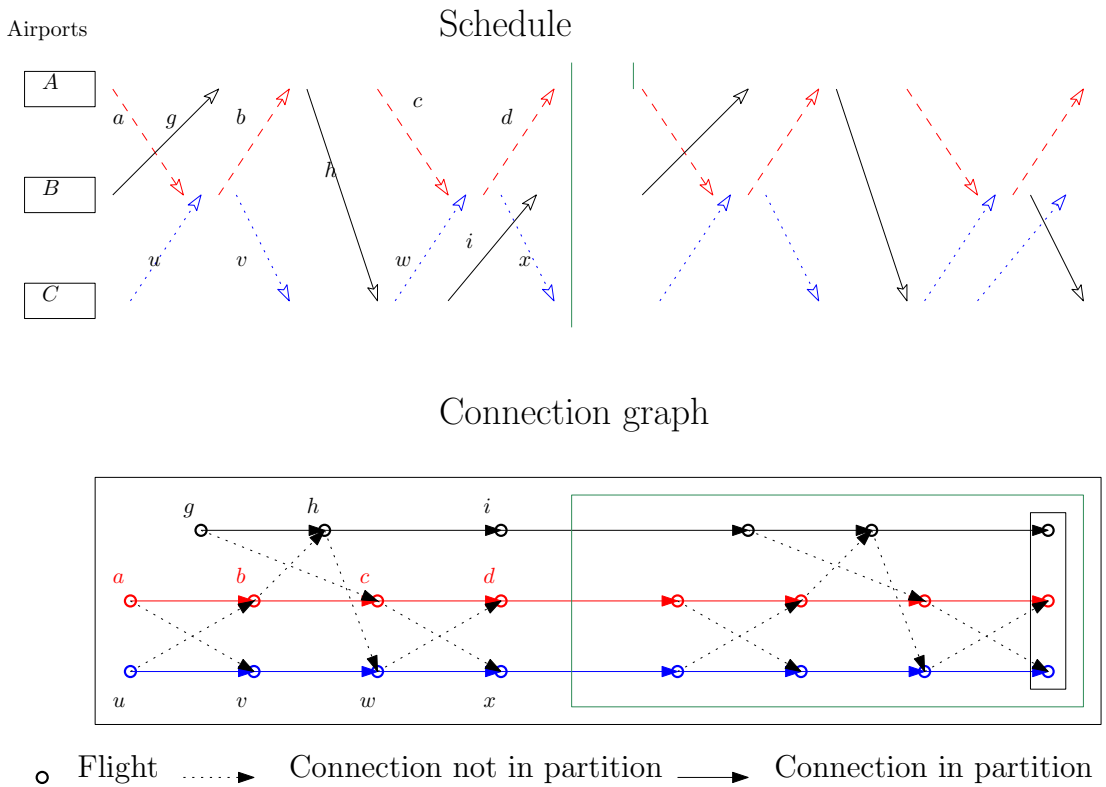
Connection graph $G = (V, A)$:

- Flight = vertex v
- Feasible connection = arc a

Maintenance:

- Night = Directed cut $\delta^-(U)$

A routing in the connection graph is a partition of the vertices of V in ST-paths



Aircraft routing problem

Instance:

- A connection graph $((S, I, T), A)$
- Night sets N_d and maintenance night sets $M_d \subseteq N_d$
- Maintenance constraint D
- Initial and final constraints S_s^o, T_t^o

Question:

- Does a feasible routing exist?

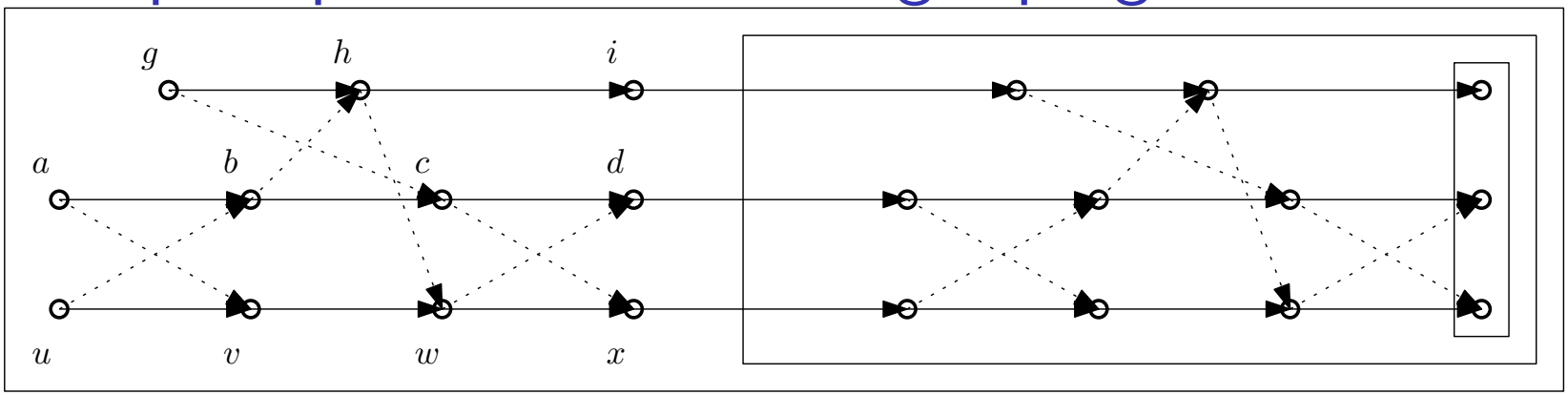
Linear integer program

- $|Feas.\ Connections|D$ variables
- $|Flights|(D+1)$ constraints

Complexity results

- NP-complete
- Polynomial when the number of airplanes is given.

Vertex path partition linear integer program



Flight ..... Connection not in partition ——► Connection in partition

Lemma

A vertex path partition $\mathcal{P}$ is entirely determined by connection arcs $a \in P \in \mathcal{P}$

- $x_a = 1$ if connection a is in the partition

$$\sum_{a \in \delta^-(v)} x_a = \sum_{a \in \delta^+(v)} x_a \quad \forall v \in I$$

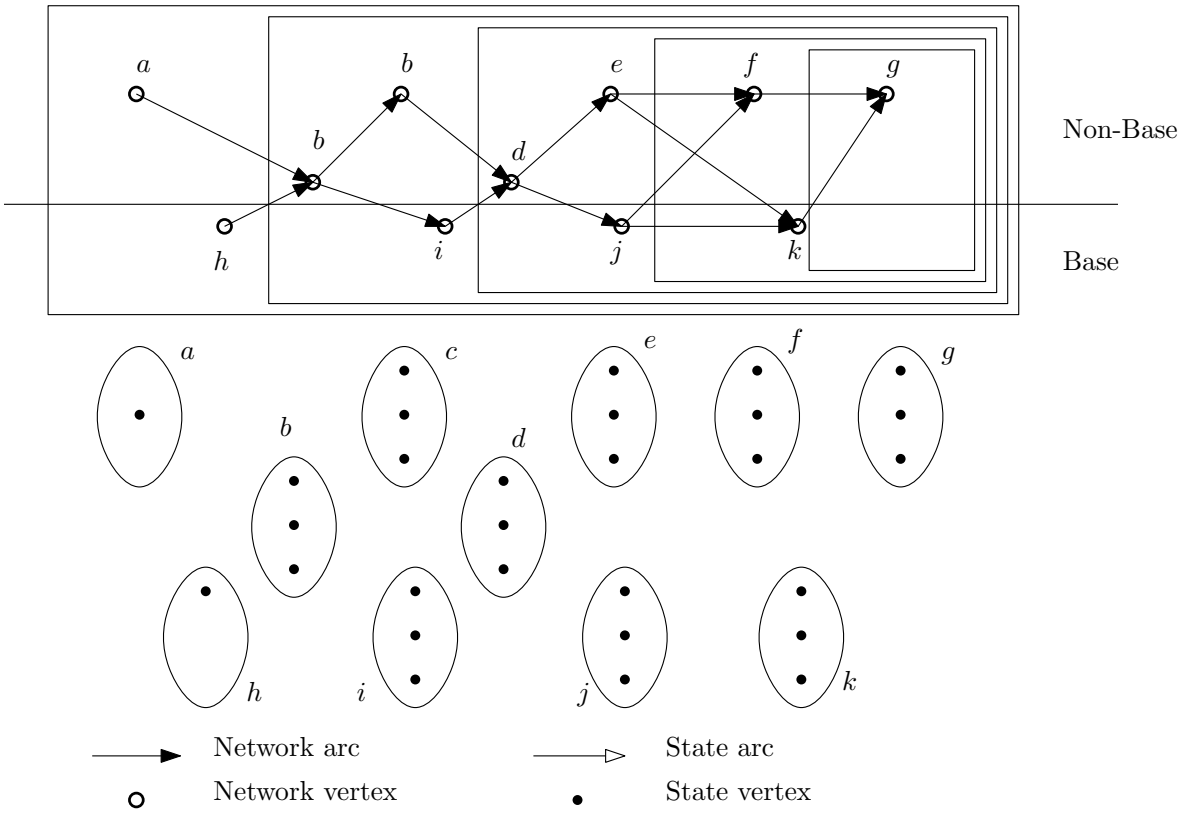
$$\sum_{a \in \delta^-(v)} x_a \geq 1 \quad \forall v \in I \cup T$$

$$\sum_{a \in \delta^+(v)} x_a = 1 \quad \forall v \in S$$

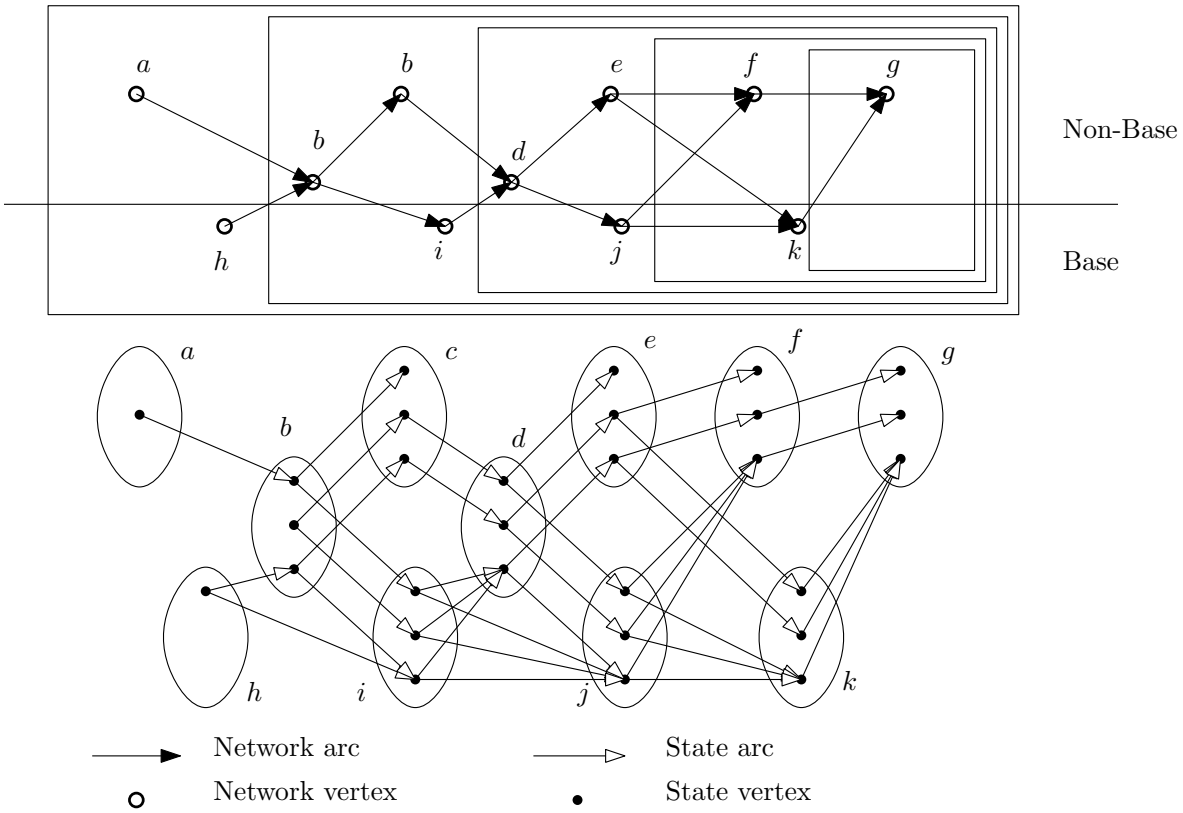
$$x_a \in \{0, 1\} \quad \forall a \in A$$

How can we add constraint in the linear integer program? State graph.

State graph for aircraft routing



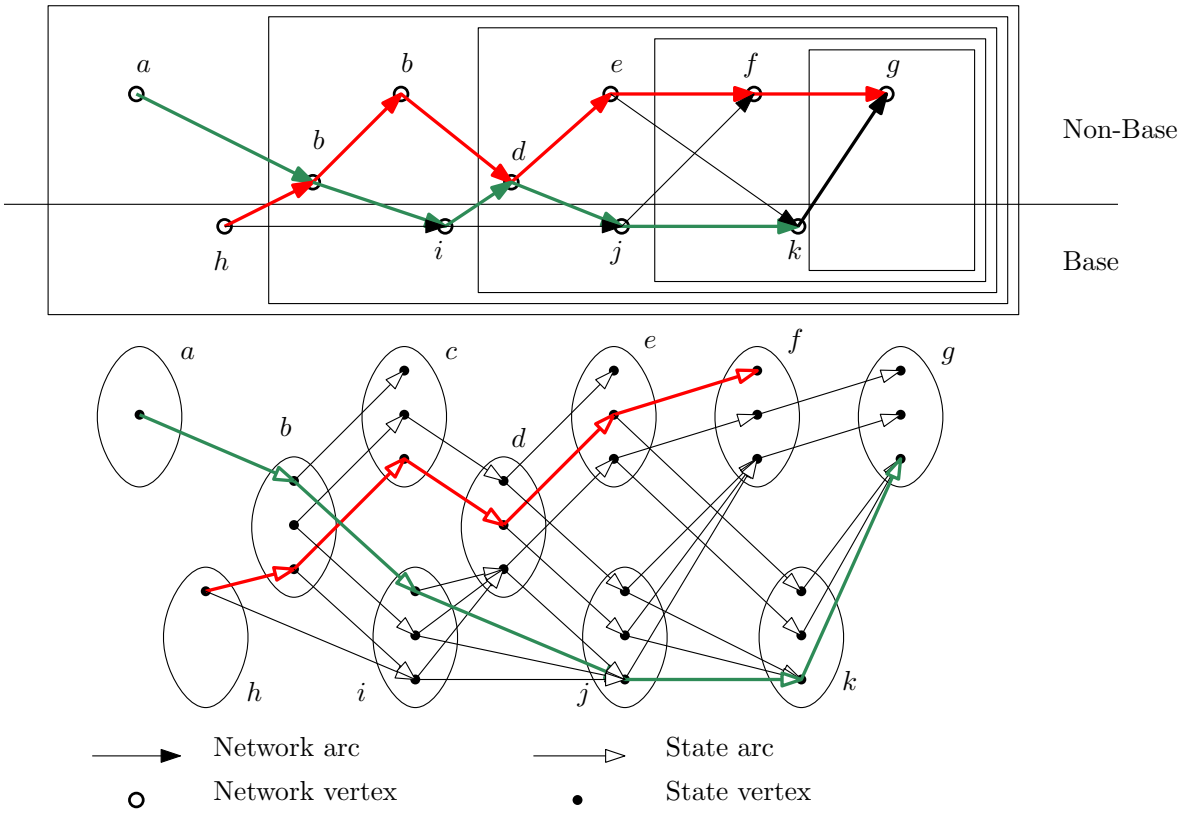
State graph for aircraft routing



D state vertices for each vertex in the connection graph:

- State vertex = number of day since the last night check.
- State arc = day / (maintenance) night

State graph for aircraft routing



D state vertices for each vertex in the connection graph:

- State vertex = number of day since the last night check.
- State arc = day / (maintenance) night

Lemma

A state path induces a feasible path in the connection graph. A partition of the connection graph in state graph paths is a *feasible routing*.

Aircraft routing linear integer program

- $x_\alpha = 1$ if state arc α is in the state path partition.

Theorem (P. & Meunier):

Same linear relaxation as the "classical" column generation approach.

$$\sum_{\alpha \in \delta^-(\vartheta)} x_\alpha = \sum_{\alpha \in \delta^+(\vartheta)} x_\alpha \qquad \forall \vartheta \in \mathcal{V}$$

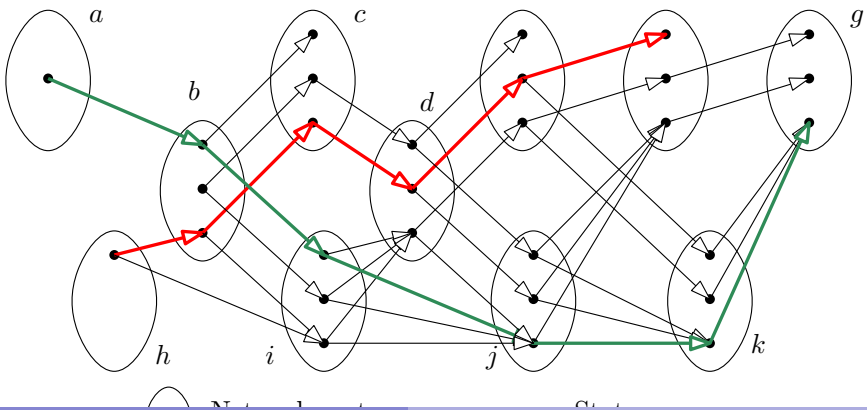
$$\sum_{\substack{\alpha \in \delta^-(\vartheta) \\ \vartheta \in \mathcal{V}_v}} x_\alpha \geq 1 \qquad \forall v \in I \cup T$$

$$\sum_{\alpha \in \delta^-(\vartheta)} x_\alpha = 1 \qquad \forall \vartheta \in \mathcal{S}$$

$$x_\alpha \in \{0, 1\} \qquad \forall \alpha \in \mathcal{A}$$

Optimize passenger connections ($c_\alpha = c_a$)

$$\min \sum_{\alpha \in \mathcal{A}} c_\alpha x_\alpha$$



Complexity

Theorem (P. & Meunier)

Aircraft routing feasibility problem is NP-complete.

Theorem (P. & Meunier)

Aircraft routing feasibility problem can be solved in $O(nD^k)$ where n is the number of flights and k is the number of airplanes.

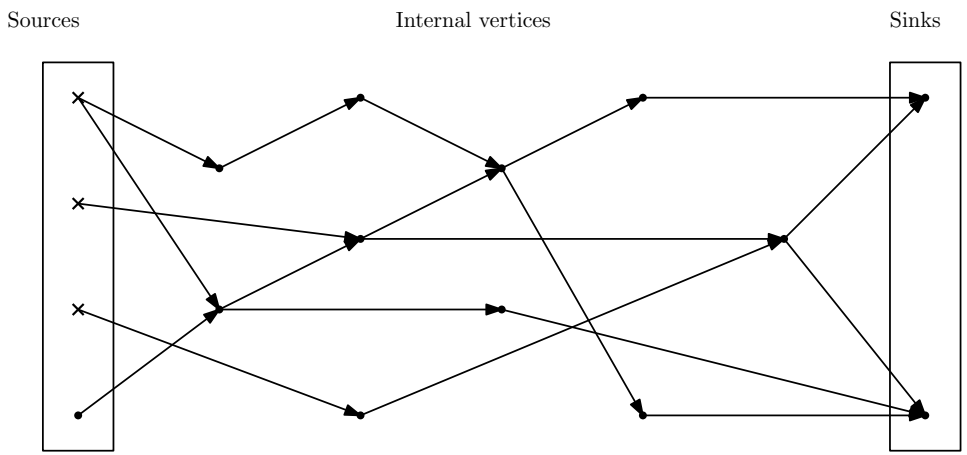
In the connection graph, each airplane is identified: needed for optimization but not for feasibility.

- Introduce the routing equigraph (smaller graph for feasibility only).
- Definition and properties of *equigraph*: guaranty feasibility without identifying aircraft.

Equigraph definition

An acyclic directed graph is an *equigraph* if its vertices can be partitioned in three sets:

- Sources $v \in S$ satisfying $\delta^-(v) = \emptyset$ and $\delta^+(v) \neq \emptyset$.
- Internal vertices $v \in I$ satisfying $\delta^-(v) = \delta^+(v) > 0$.
- Sinks $v \in S$ satisfying $\delta^-(v) \neq \emptyset$ and $\delta^+(v) = \emptyset$



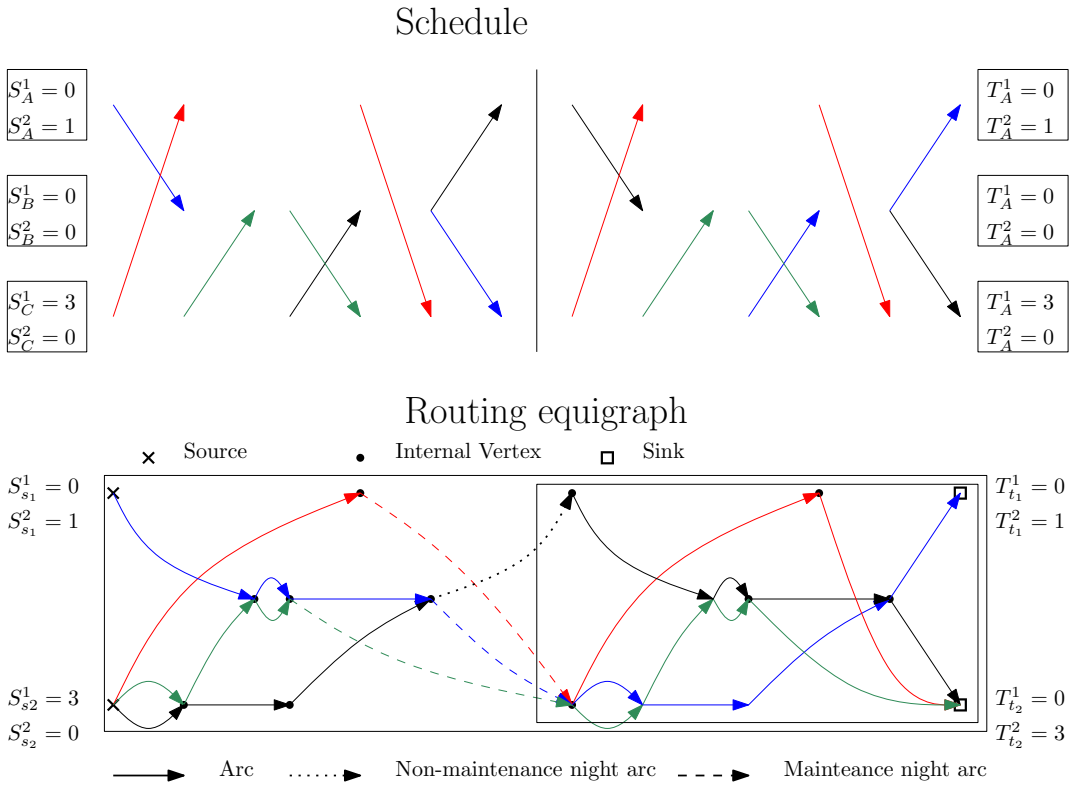
Routing equigraph

Routing equigraph $G = (V, A)$:

- Flight = arc a
- (airport,time) = vertex v

Maintenance:

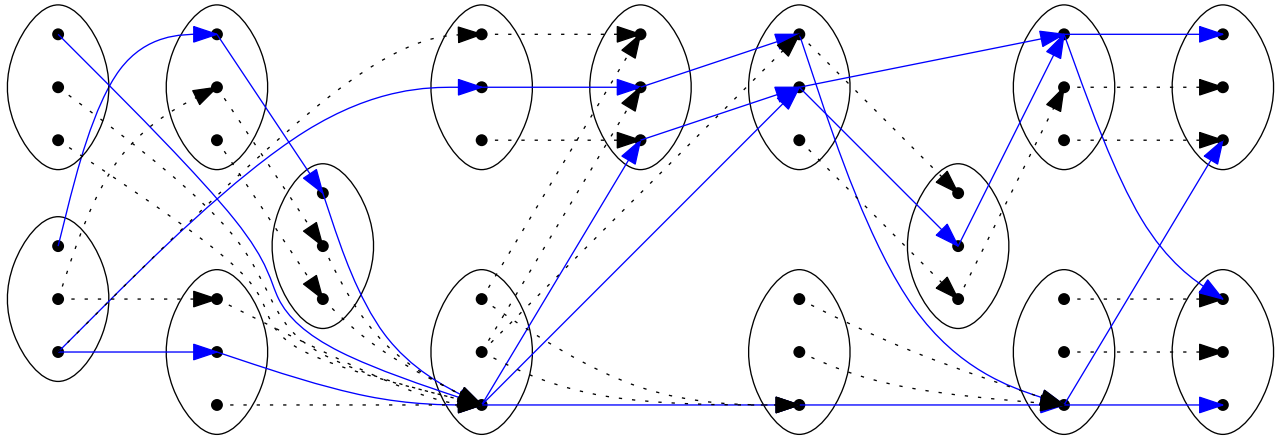
- Night = Directed cut $\delta^-(U)$



A routing in the routing equigraph is a partition of the arcs of A in ST-paths

A digraph is an equigraph if $\delta^-(v) = \delta^+(v)$ for all internal vertices (a vertex v is either a source, a sink or an internal vertex).

State graph on the routing equigraph



$$\sum_{\substack{\alpha \in \delta^-(\vartheta) \\ \vartheta \in \mathcal{V}_v}} x_\alpha = 1, \forall v \in V$$

becomes

$$\sum_{\alpha \in \mathcal{A}_a} x_\alpha = 1, \forall a \in A$$

State path partition gives an equigraph in the state graph.

- How to obtain a path partition of the arcs of an equigraph?
- Searching a feasible equigraph instead of a path partition: faster deterministic algorithm.

Equigraph partition problem and greedy algorithm

Instance:

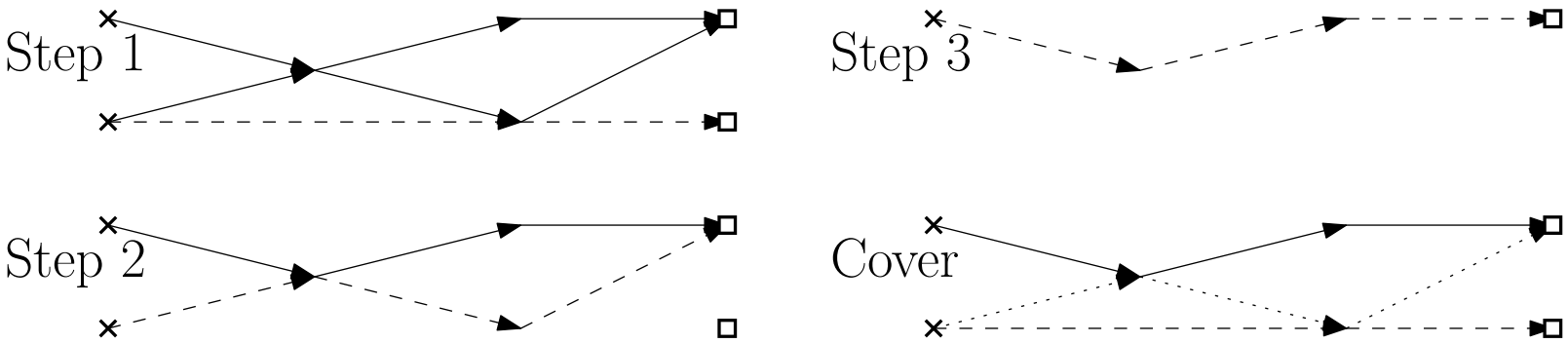
- An equigraph $((S, I, T), A)$

Solution:

- A partition of the arcs of G in $S - T$ paths.

Greedy algorithm lemma

If P is a path from S to T, then $G \backslash P$ is still an equigraph



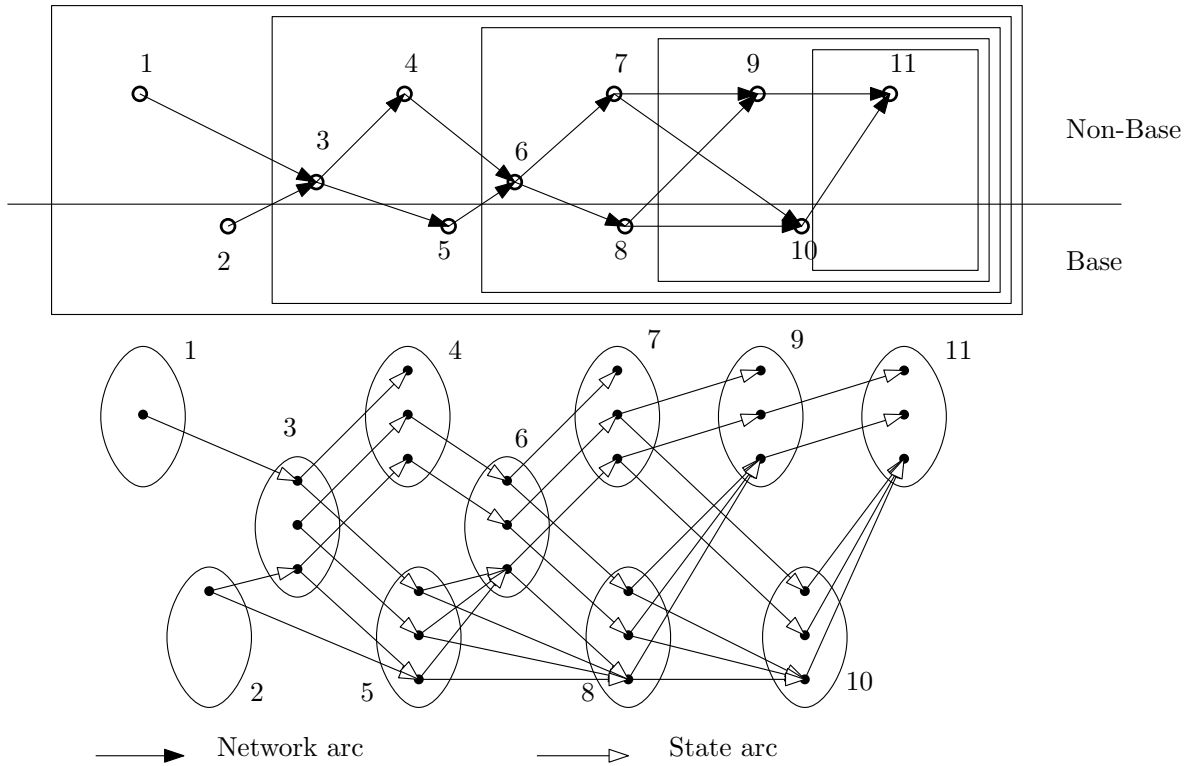
Fixed parameter tractability

Theorem (P. & Meunier)

Aircraft routing is fixed parameter tractable in D^k. It can be solved in $O(nD^k)$.

The following algorithm is inspired from Fortune, Hopcroft, and Wyllie, *the directed subgraph homeomorphism problem*, 1980.

- Build a state graph on the routing equigraph
- Order the vertices of the equigraph in a topological ordering



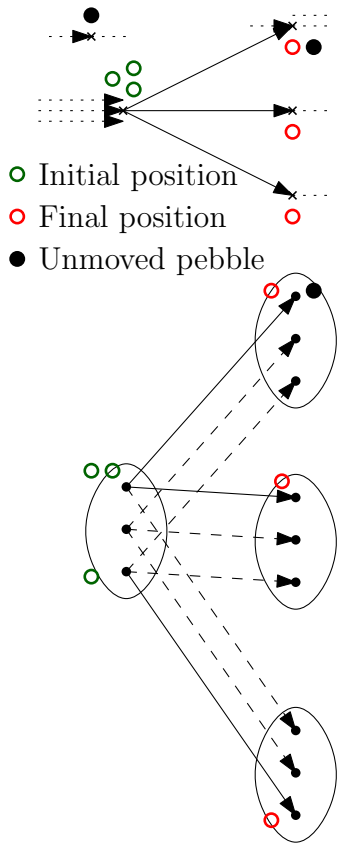
Pebbling game

Game goal:

Move k identical pebbles from the source configuration to the sink configurations set using legal moves.

Vertices are enumerated using a topological ordering.
For each vertex u, one *legal move* is done:

1. Pebble initially on ϑ_1 can be moved to ϑ_2 if $(\vartheta_1, \vartheta_2)$ is in $\mathcal{G}$
2. Exactly one state arc $\alpha \in \mathcal{A}_a$ is traversed by a pebble for each $a \in \delta^-(v)$.

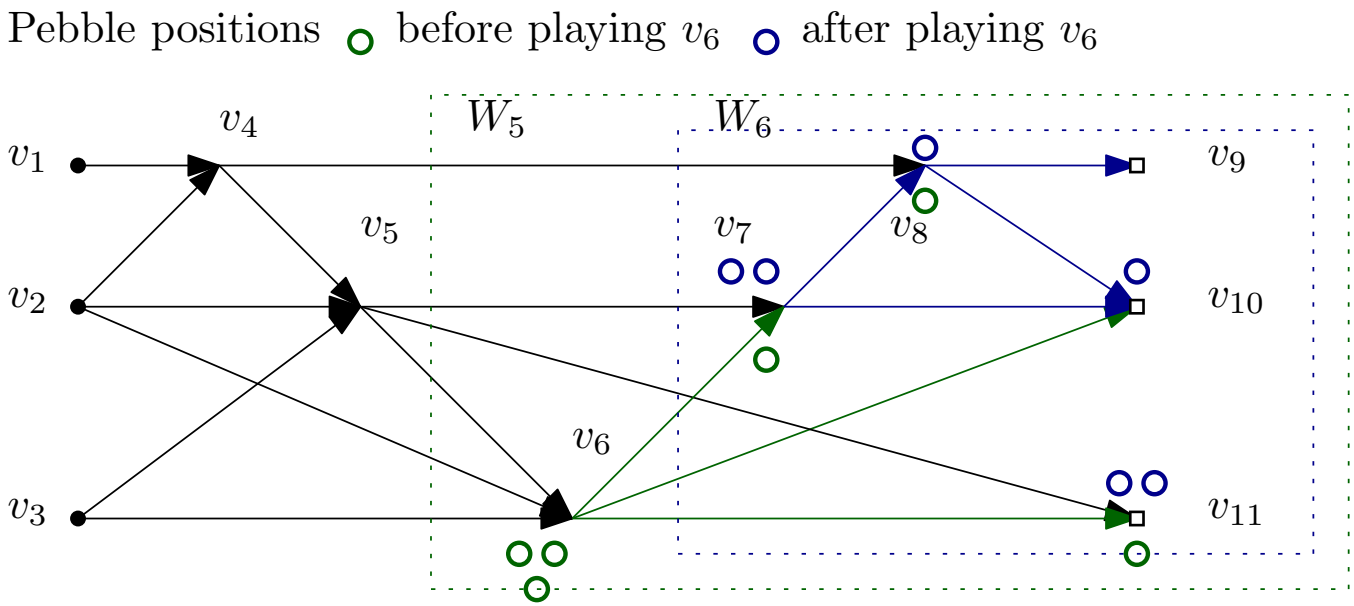


Lemma

Pebbling game can be won $\Leftrightarrow$ State graph can be partitioned in paths

Pebble configurations

Topological ordering: a set of induced subgraphs $G_i = (V_i, A_i)$ where $V_i = \{v_j | j \geq i\}$



Number of pebbles on v

- Before playing v_i: $|\delta^-_G(v)| - |\delta^-_{G_i}(v)|$
- After playing v_i: $|\delta^-_G(v)| - |\delta^-_{G_{i+1}}(v)|$

Algorithm complexity

The configuration graph G_C is defined as follows:

- Vertices V_C: Pebble configurations
- Arcs A_C: Legal moves. $|A_C| \leq |V| \cdot D^k$

A *path from the source configuration to the sink configurations set* gives a solution to the pebbling game

Theorem (P. & Meunier)

Aircraft routing problem is polynomial when k is fixed. Pebbling game algorithm gives a solution in $O(nD^k)$

Proof.

A path finding algorithm is linear in the number of arcs in an acyclic directed graph. □

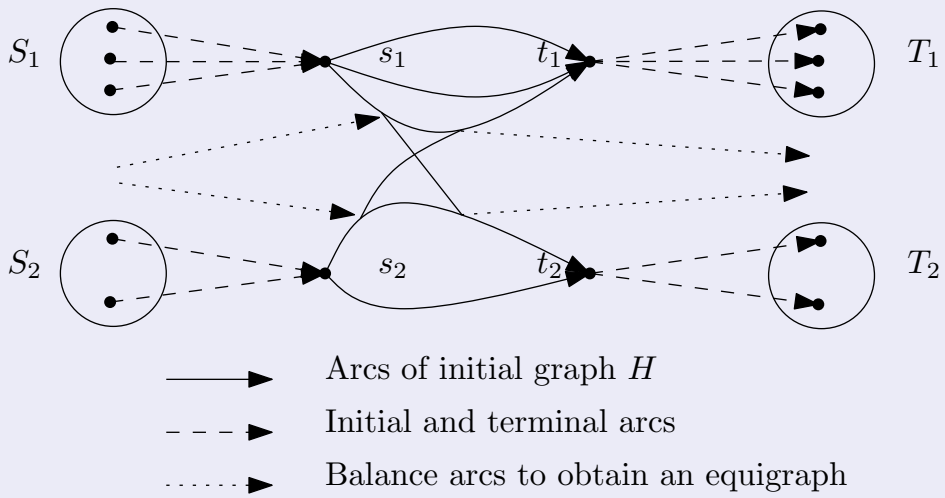
NP-completeness

Theorem (P. & Meunier)

Aircraft routing is NP-complete

Proof.

Reduction from the two commodity arc disjoint paths on acyclic directed graph.



Questions?