# Finding small stabilizer for unstable graphs

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([1]Lausanne, [2]Boston, [3]Waterloo, [4]Aachen)

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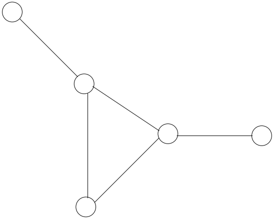

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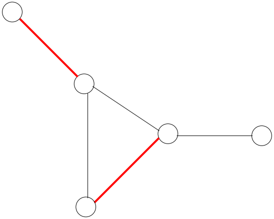

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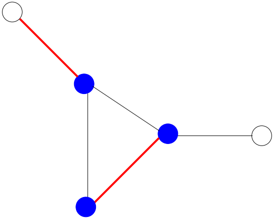

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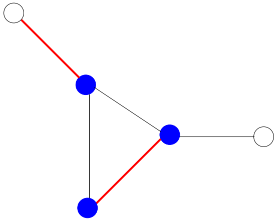

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- Finding a maximum matching is "easy" whereas finding a minimum cover is "hard".

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By duality theory:

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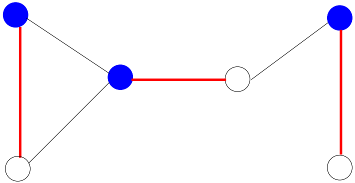

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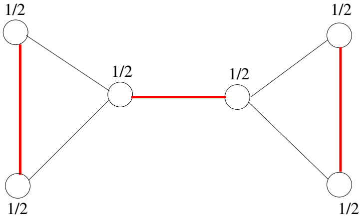


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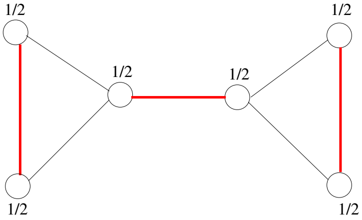


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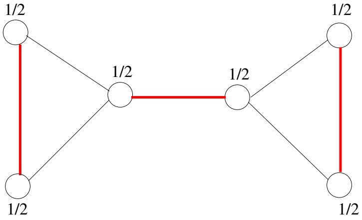


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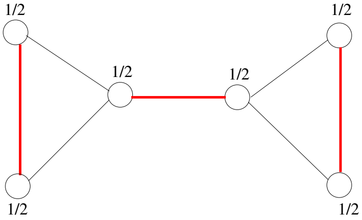


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This talk: Given an unstable graph, how can we find a small stabilizer, i.e., a subset $F \subseteq E$ s.t. $G \setminus F$ is stable.

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Note: Core$(G) \neq \emptyset \iff G$ is stable.

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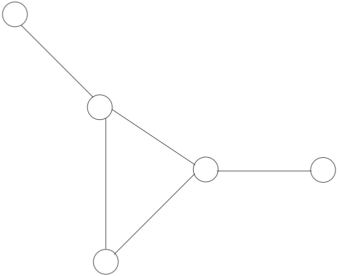

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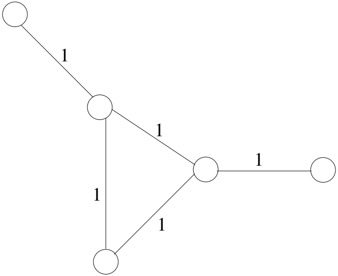


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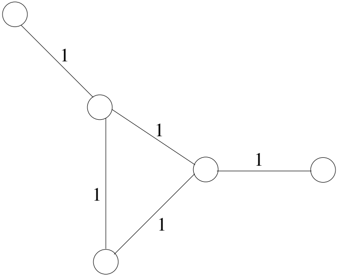


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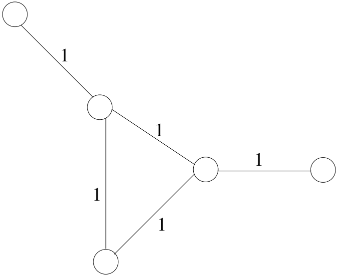


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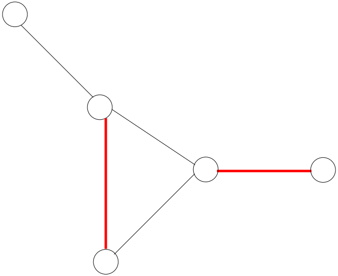

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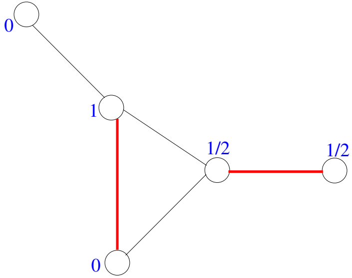


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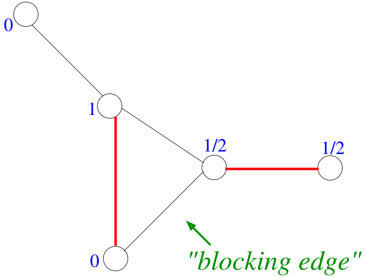


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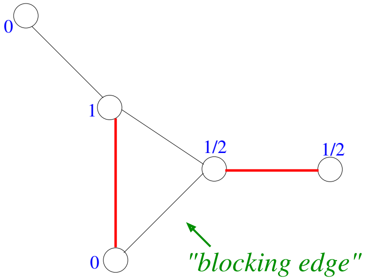


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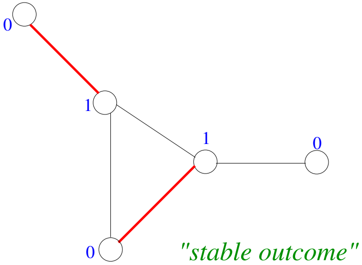


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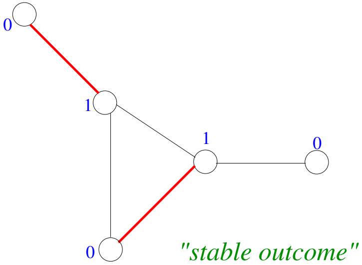


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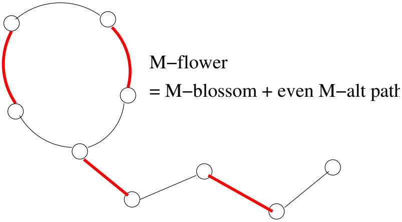

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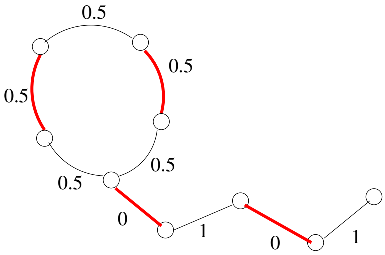

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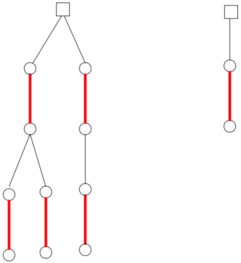

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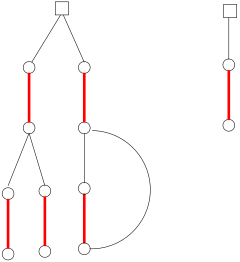

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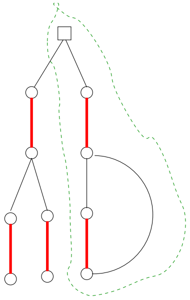

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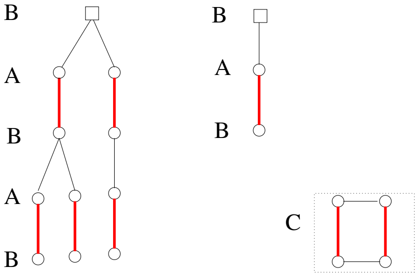


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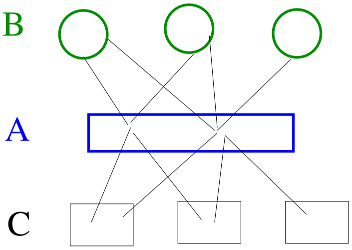


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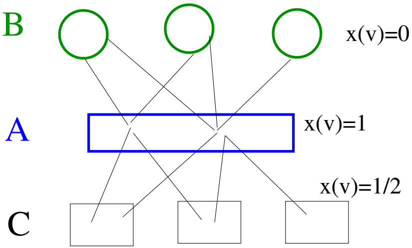


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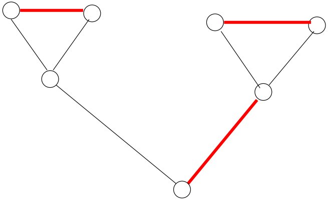

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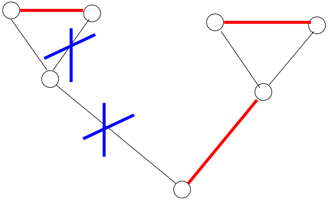

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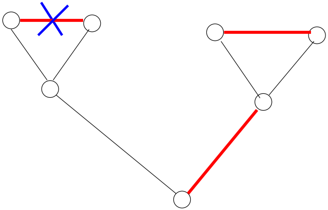

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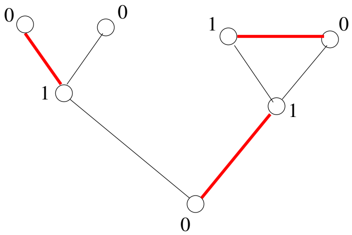


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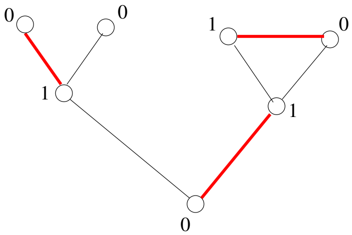


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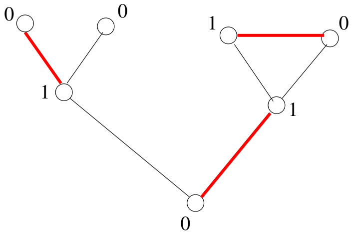


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### Theorem (Key)

*If $F$ is a minimum stabilizer, then $\nu(G \setminus F) = \nu(G)$.*

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We use this structural insight to show:

### Theorem

*The* MINIMUM STABILIZER PROBLEM *is NP-hard (vertex cover).*

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However, as $M$ is max matching in $G$, one end of the path must be adjacent to $f \in M \cap F$. It follows that $\hat{M} = M\Delta(P + f)$ is max matching in $G$ with $|\hat{M} \cap F| < |M \cap F|$. Contradiction! $\square$

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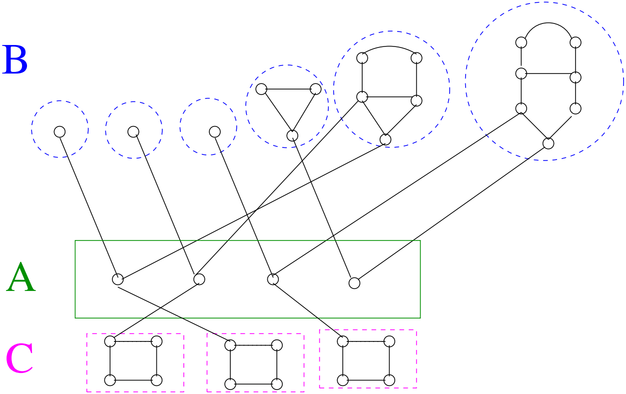
Proof:

Consider the Gallai–Edmonds' Decomposition V=B+A+C



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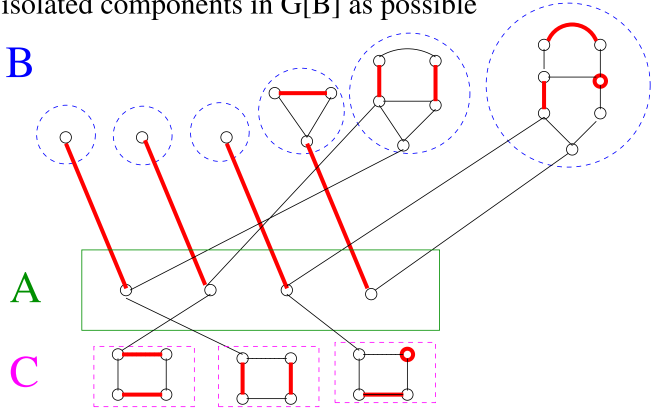
Proof:

Choose max matching M linking as much isolated components in G[B] as possible



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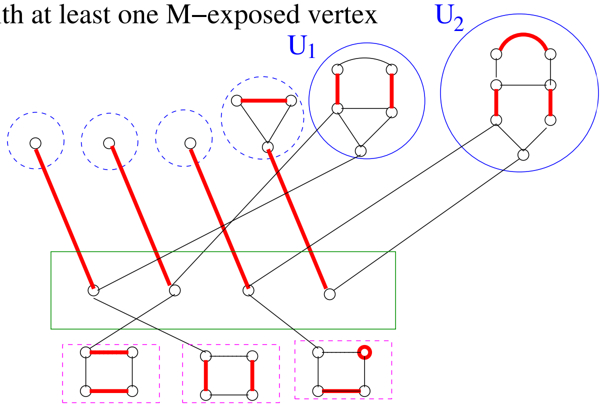
Let $U_1$, ..., $U_k$ denote the non–trivial components in G[B] with at least one M–exposed vertex



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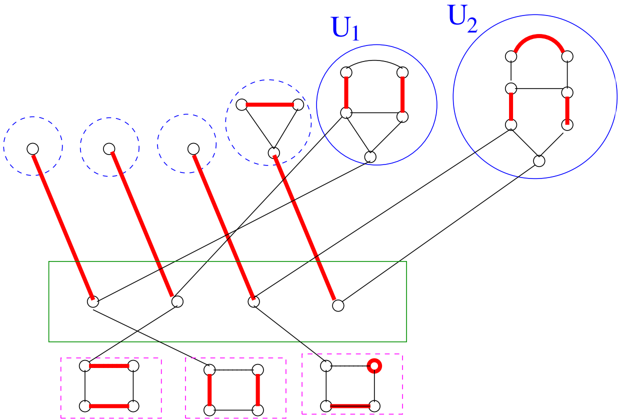
Each $U_i$ has at least one vertex $v_i$ essential in G\F



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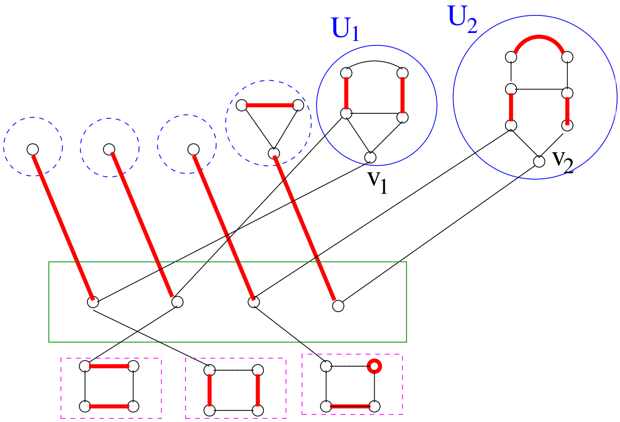

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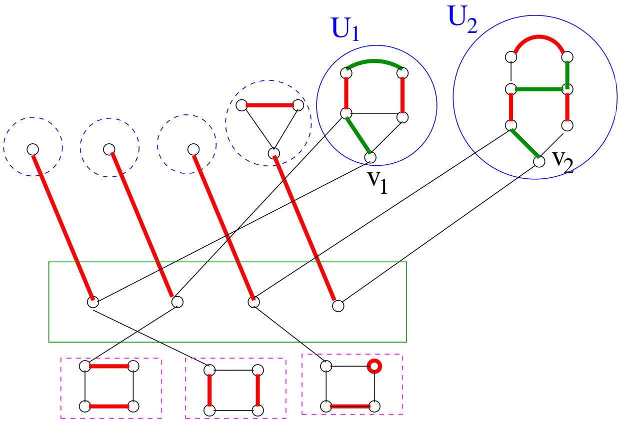
Choose max matching N in G\F



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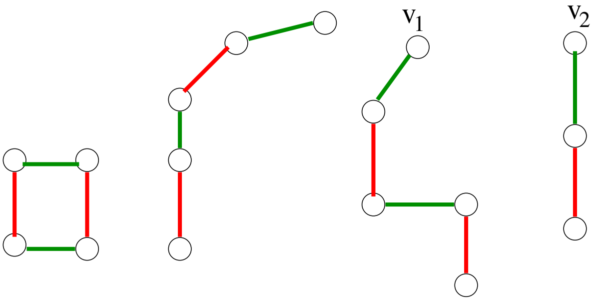
Proof:

N△M is disjoint union of even cycles and even paths



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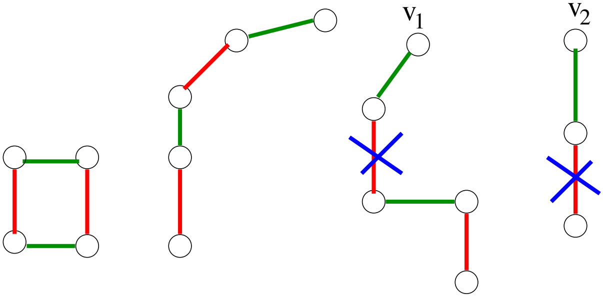
Note: each of the k paths starting at $v_i$ has at least one edge in F.



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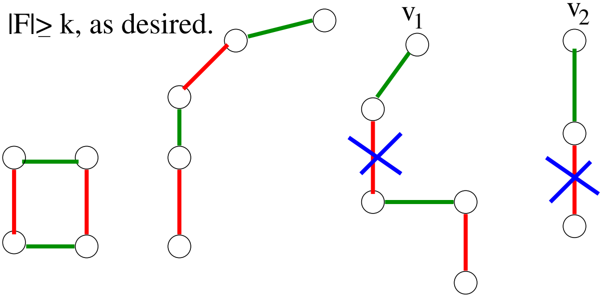
Note: each of the k paths starting at v has at least one edge in F.

Thus, |F|≥ k, as desired.



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*If $\nu(G) < \nu_f(G)$, can find in poly-time a set $L \subseteq E$ with $|L| \leq 4\omega$, $\nu(G) = \nu(G \setminus L)$, and $\nu_f(G \setminus L) = \nu_f(G) - \frac{1}{2}$.*

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At most $2(\nu_f(G) - \nu(G)) \leq$OPT iterations.

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