Finding small stabilizer for unstable graphs

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([1]Lausanne, [2]Boston, [3]Waterloo, [4]Aachen)

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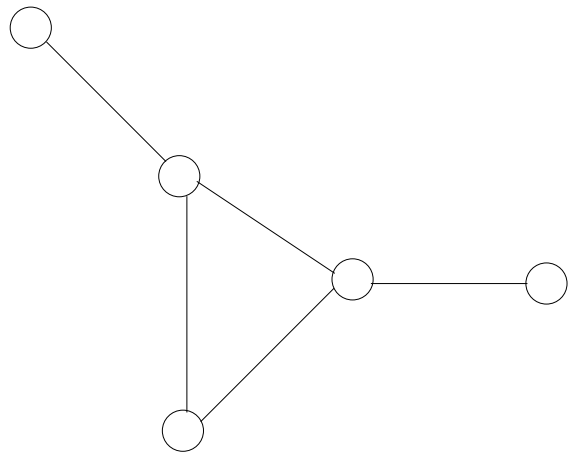

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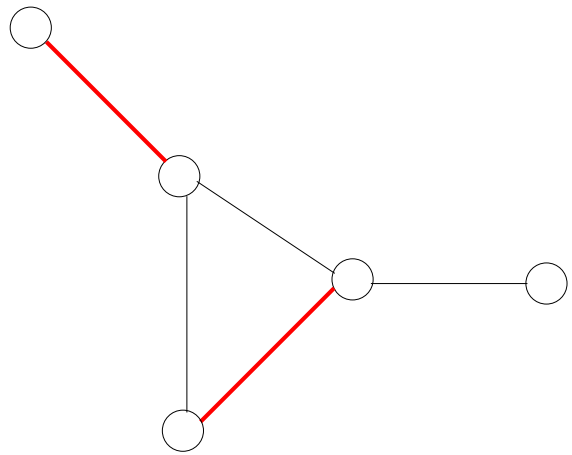

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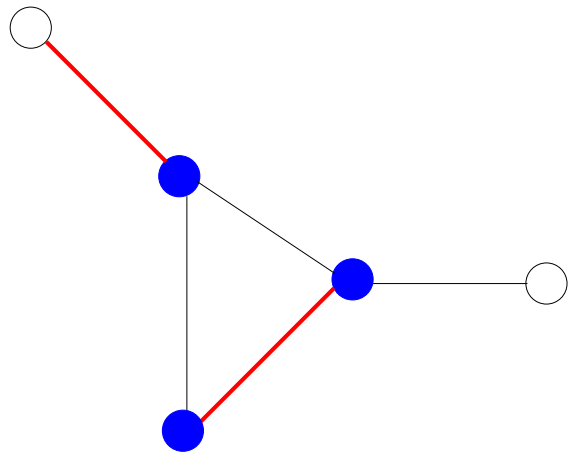

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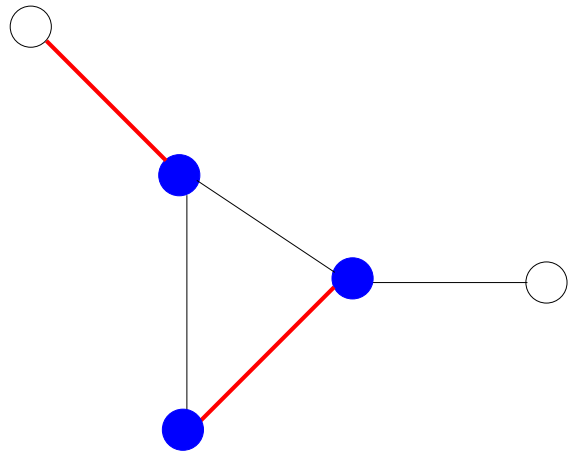

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- Finding a maximum matching is "easy" whereas finding a minimum cover is "hard".

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By duality theory:

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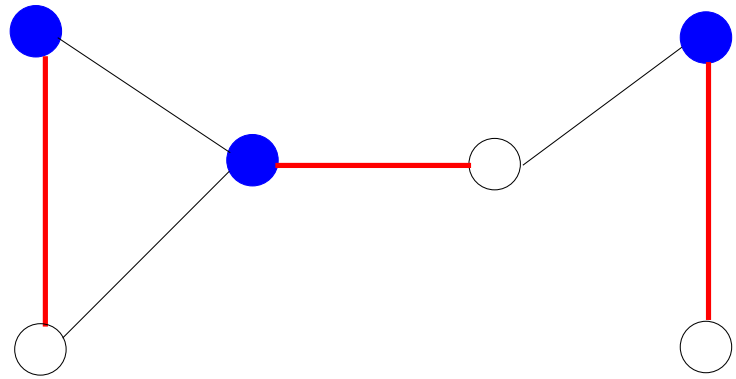

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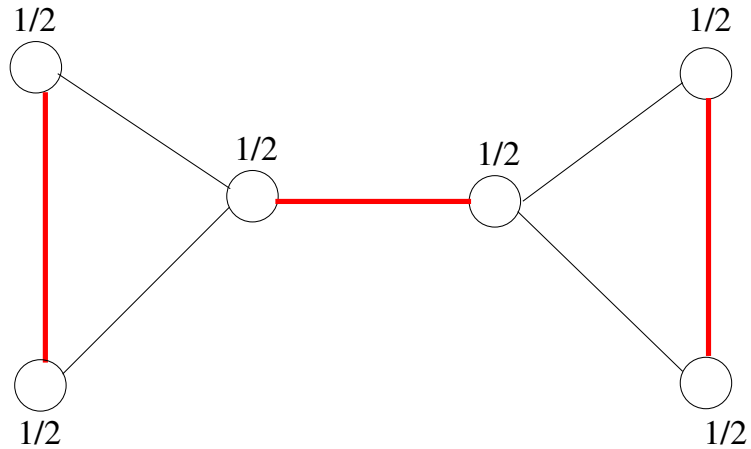

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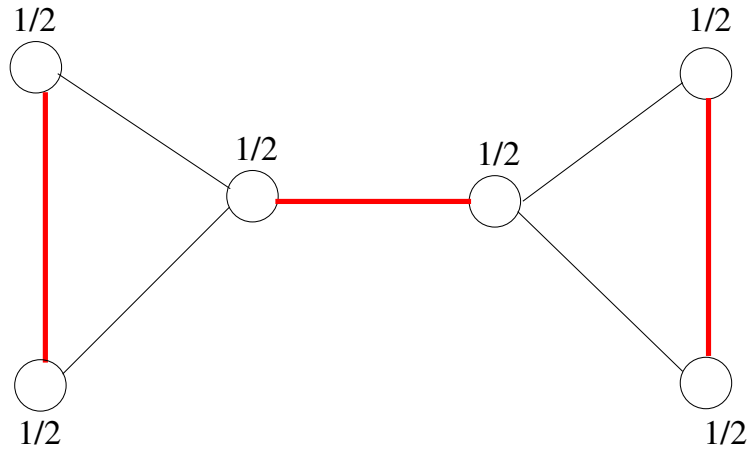

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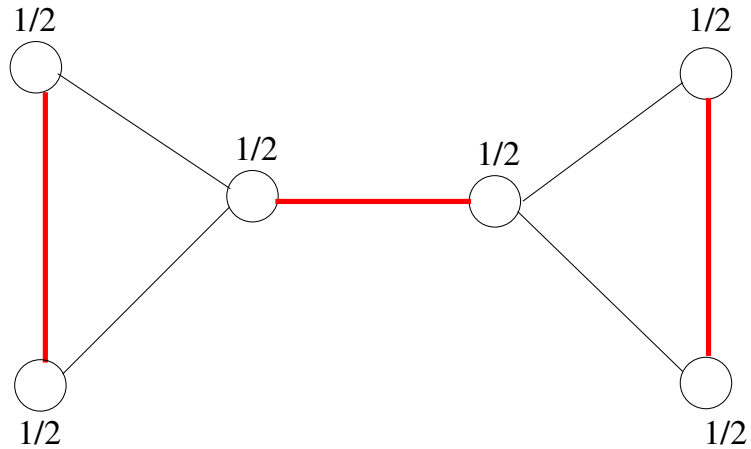

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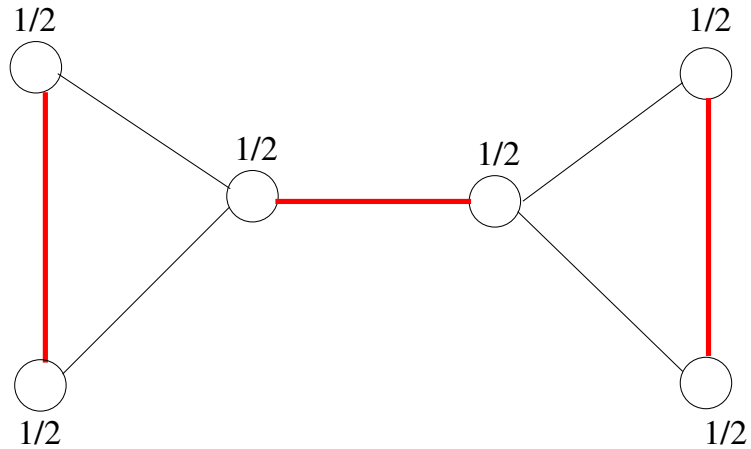

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This talk: Given an unstable graph, how can we find a small stabilizer, i.e., a subset $F \subseteq E$ s.t. $G \setminus F$ is stable.

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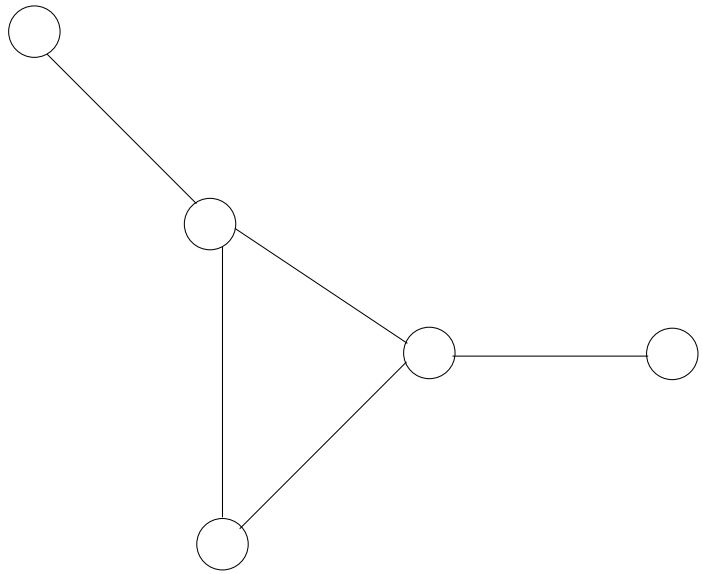

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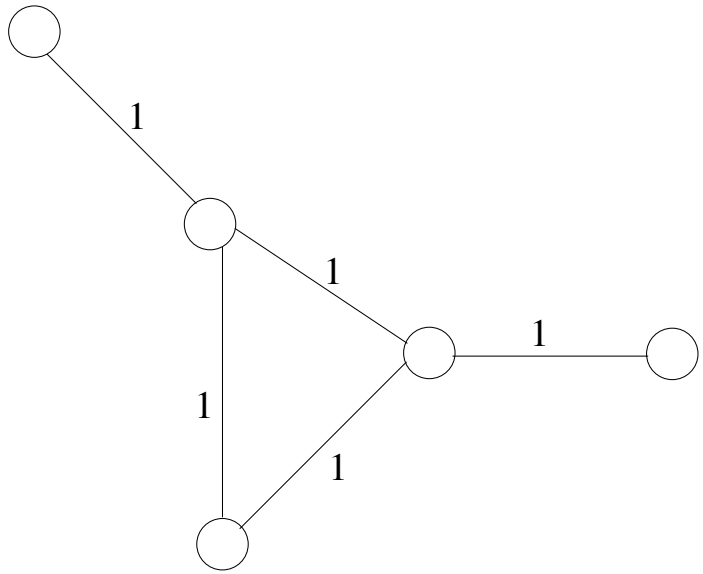

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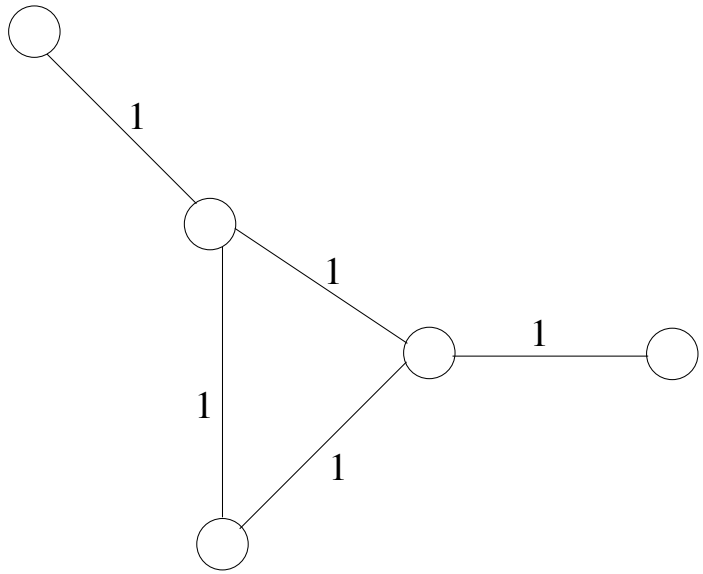

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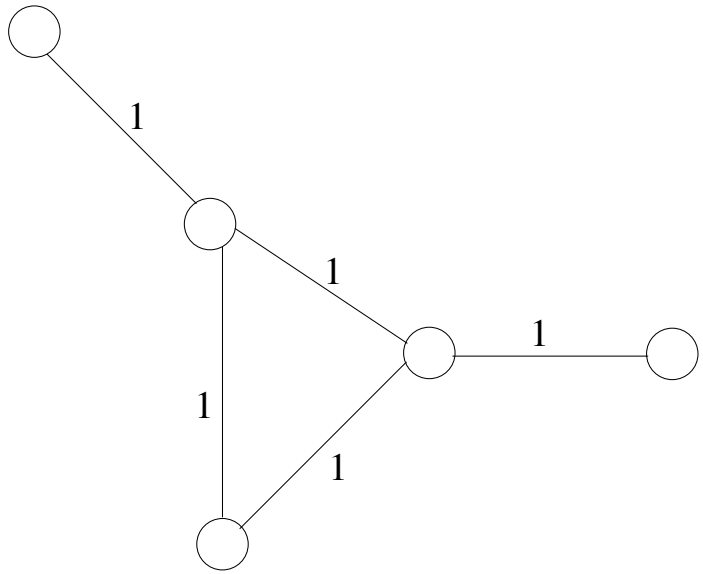

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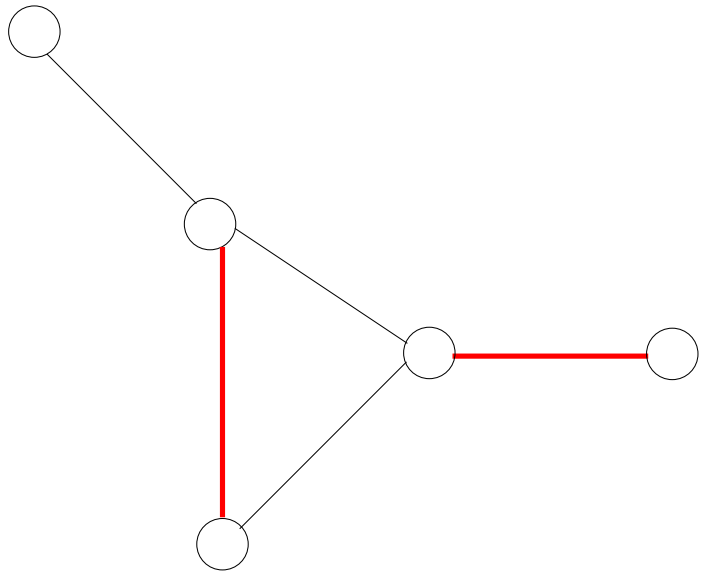

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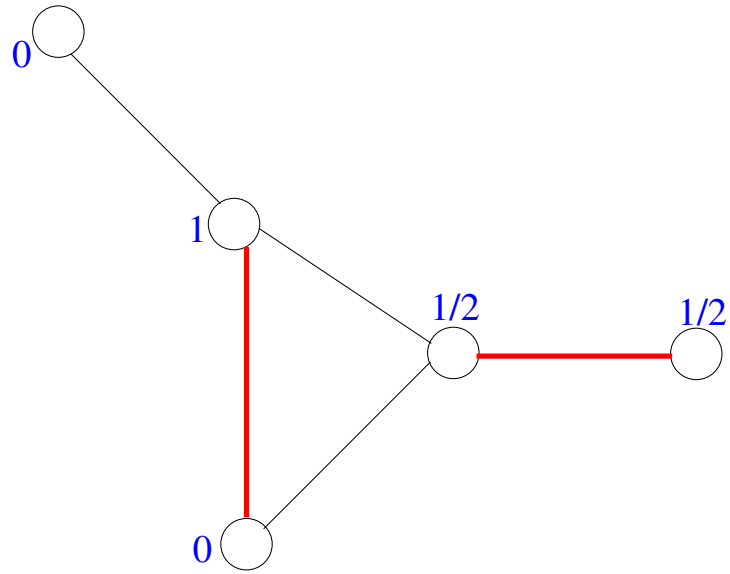

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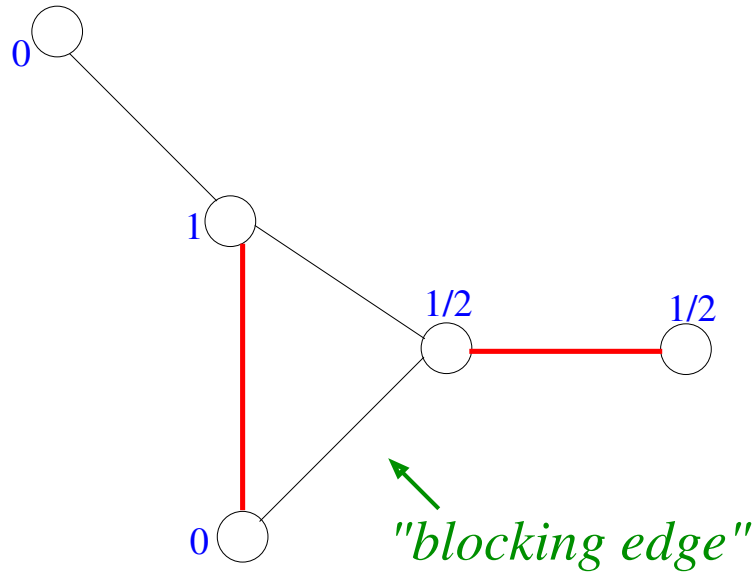

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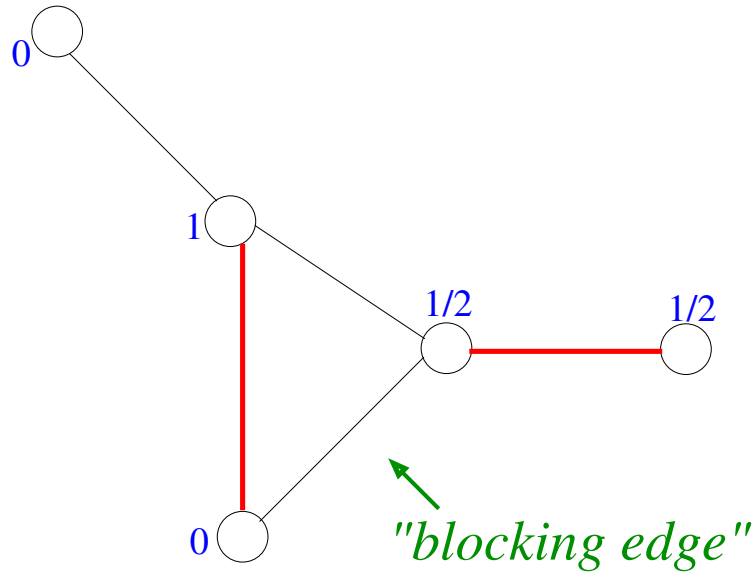

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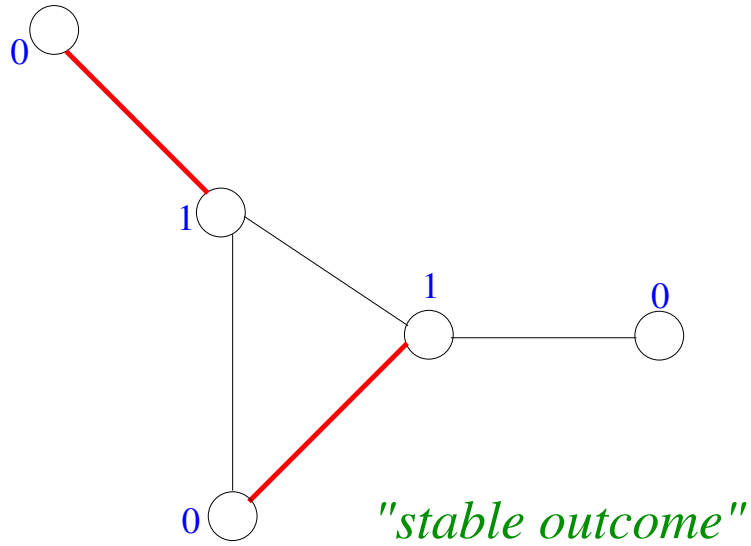

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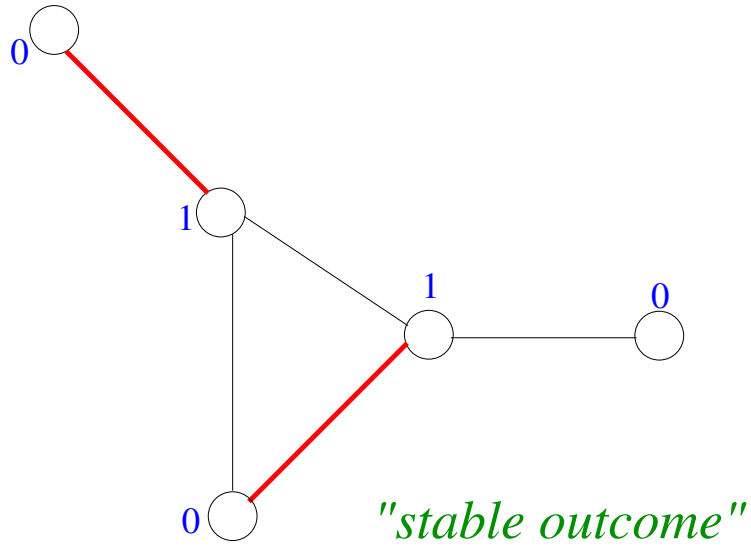

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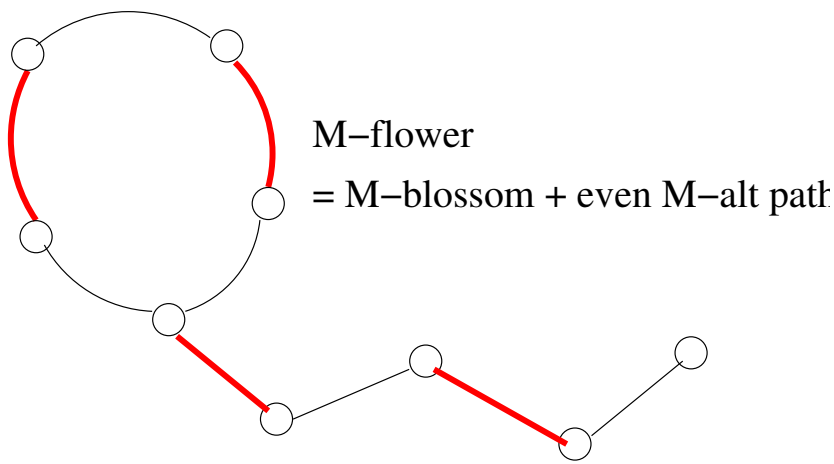

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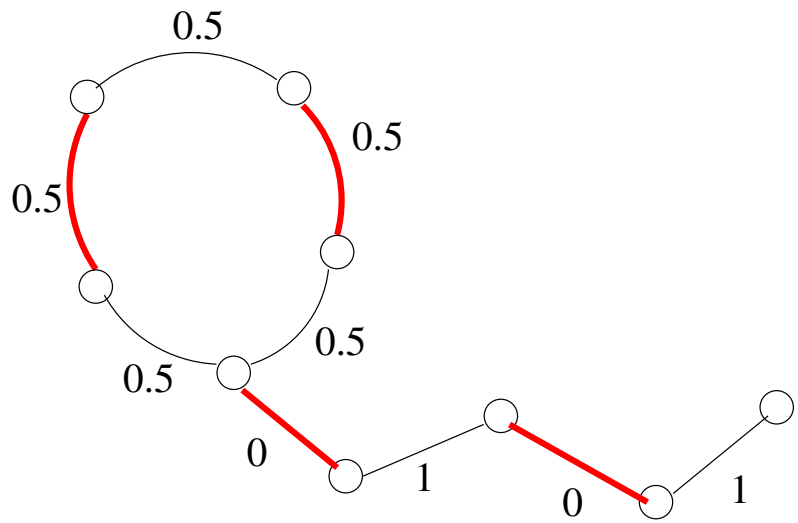

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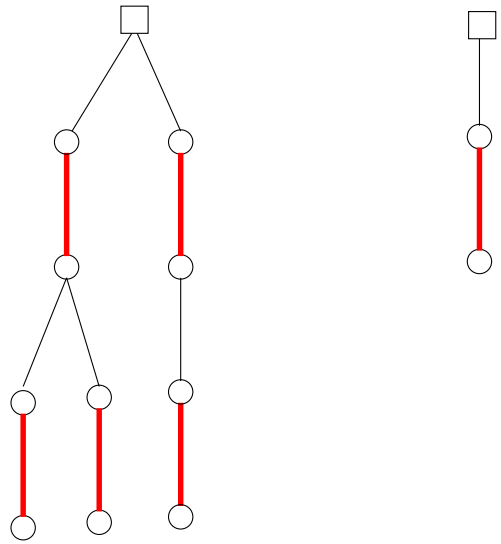

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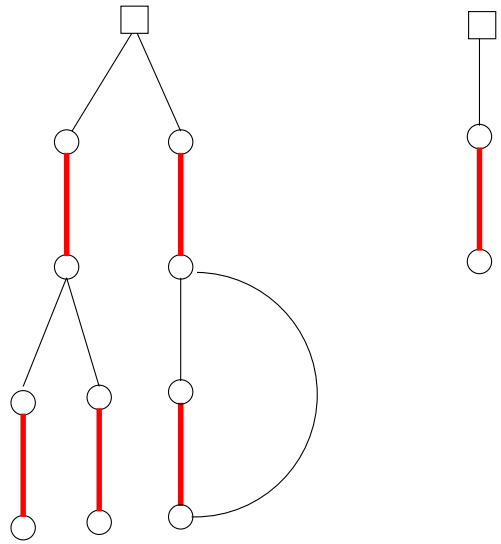

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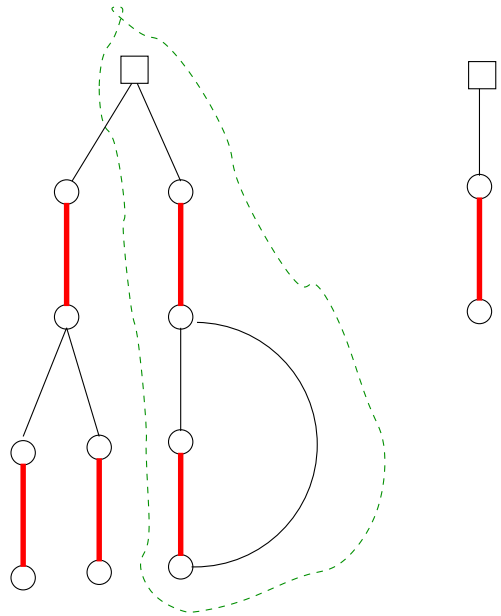

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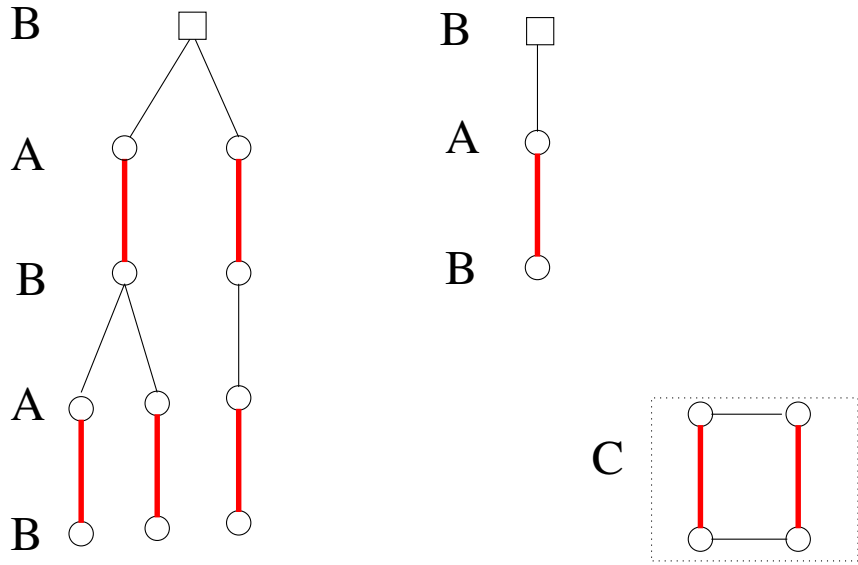

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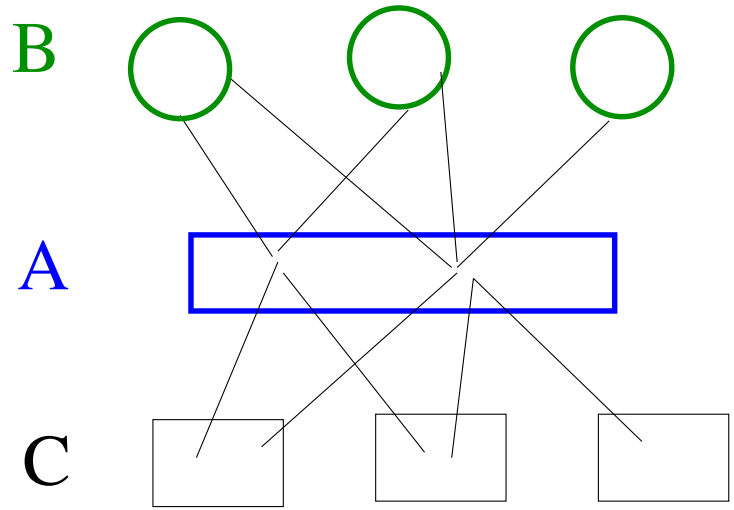

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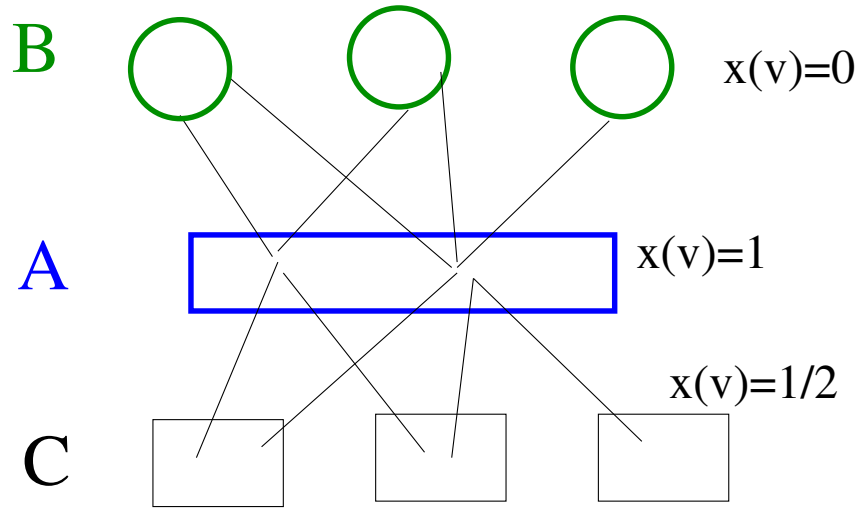

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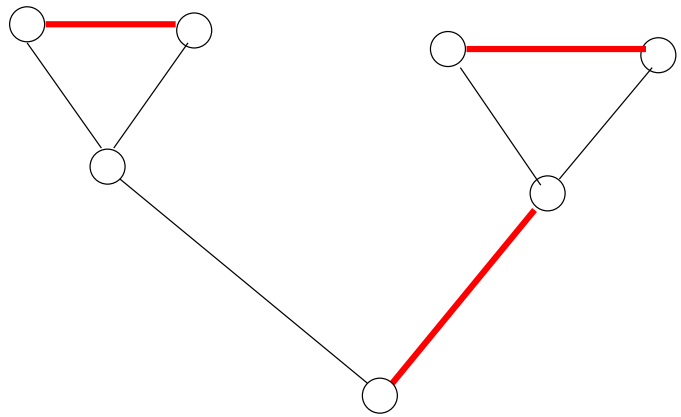

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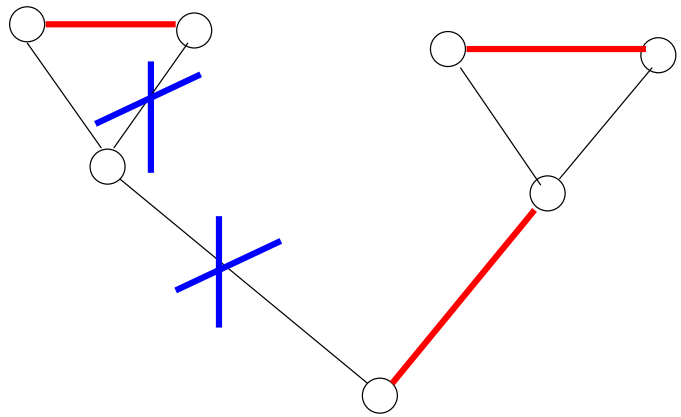

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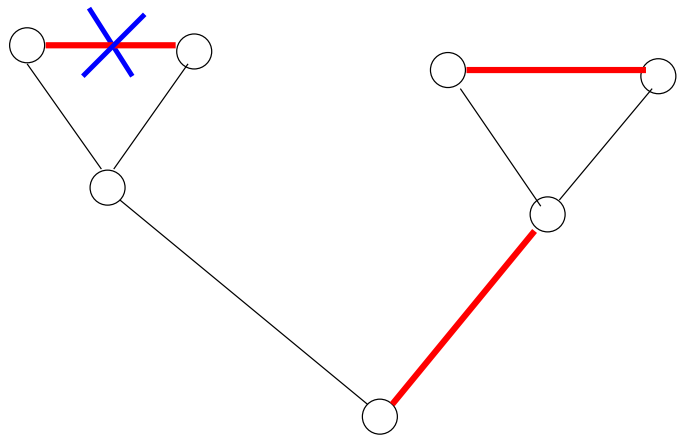

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Min Stabilizer problem: find a stabilizer F of minimum cardinality

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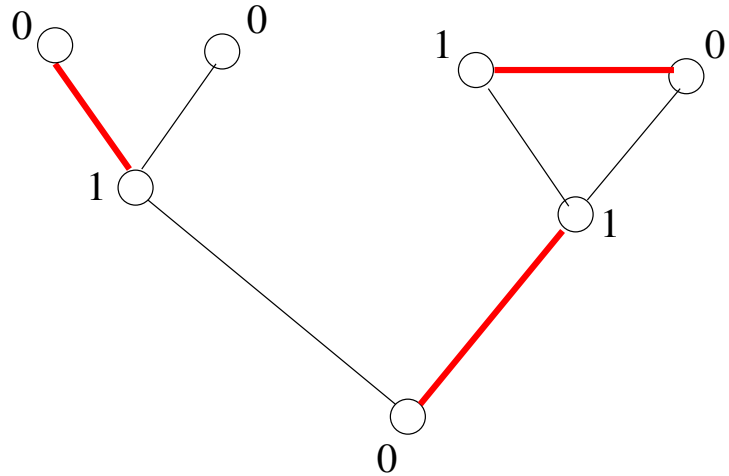

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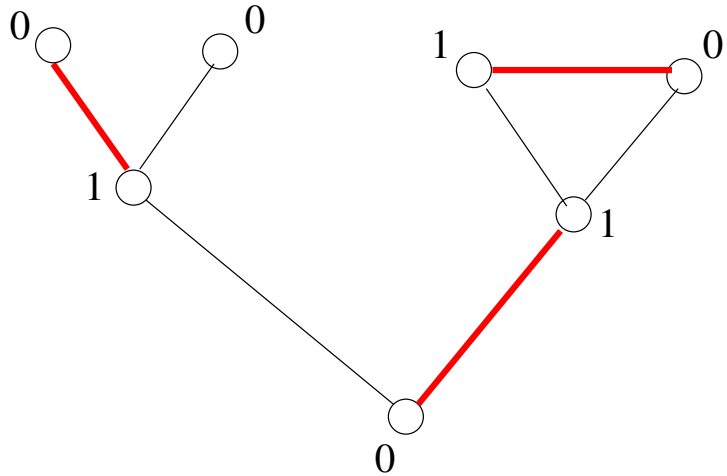

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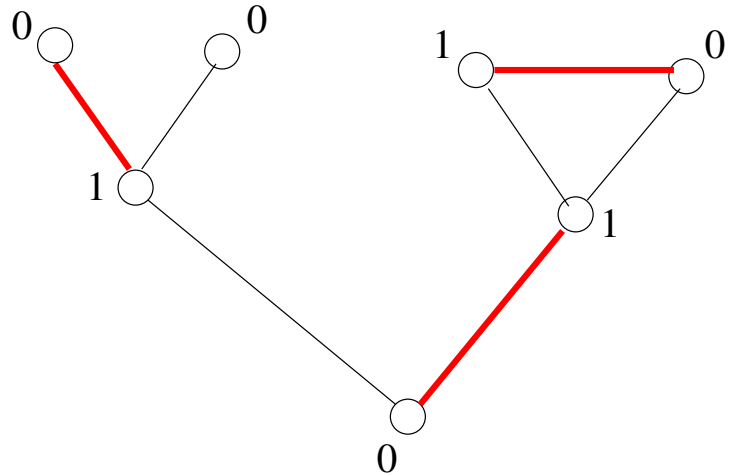

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However, as M is max matching in G, one end of the path must be adjacent to $f \in M \cap F$. It follows that $\hat{M} = M\Delta(P + f)$ is max matching in G with $|\hat{M} \cap F| < |M \cap F|$. Contradiction! $\square$

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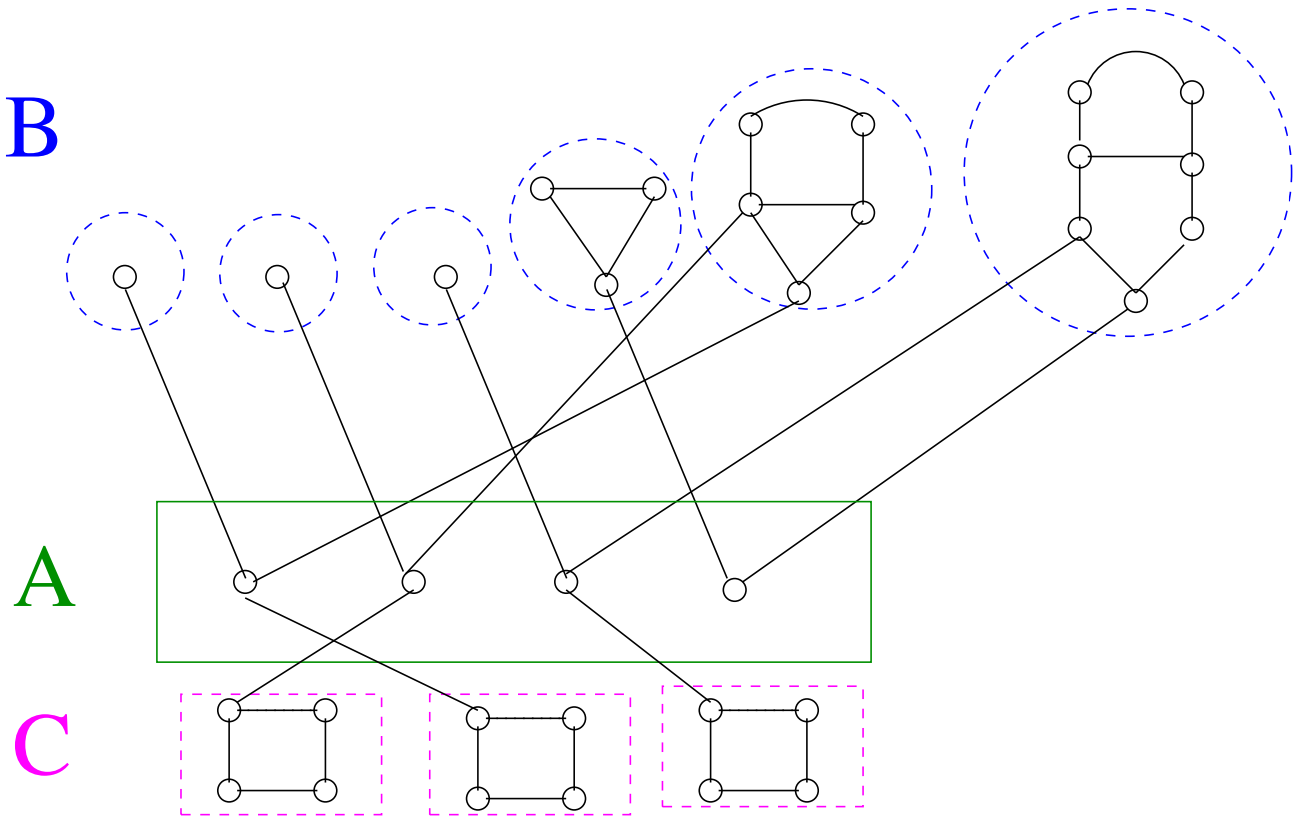
Proof:

Consider the Gallai–Edmonds' Decomposition V=B+A+C



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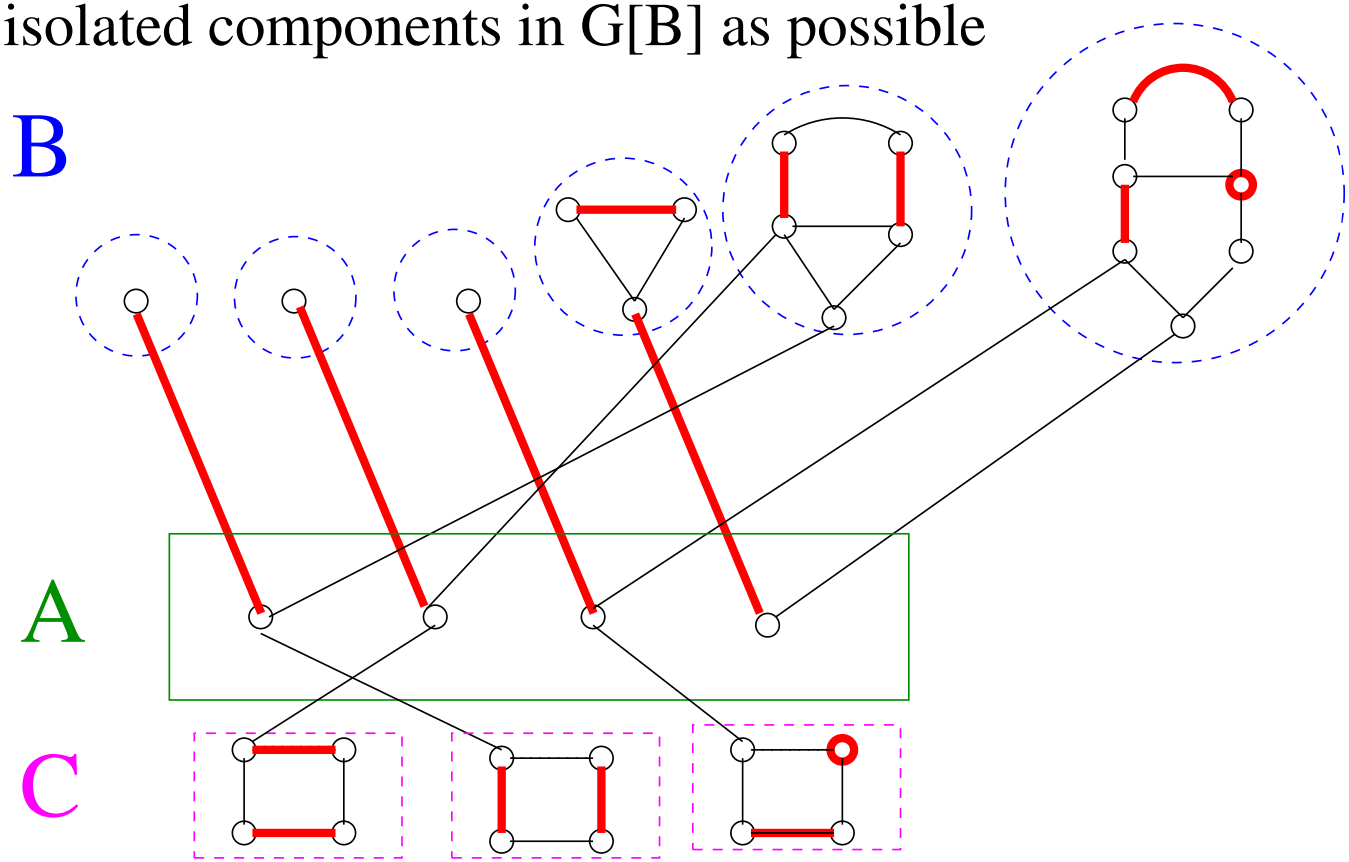
Proof:

Choose max matching M linking as much isolated components in G[B] as possible



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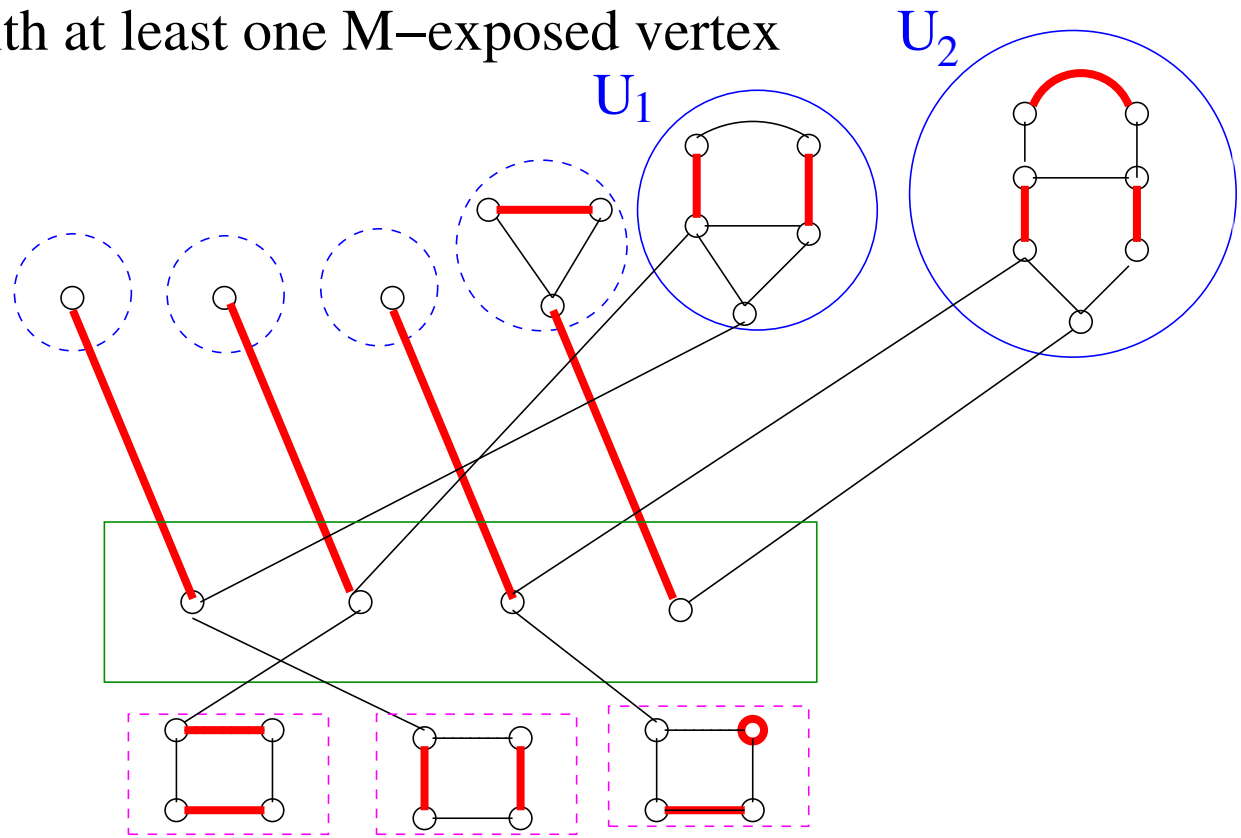
Proof:

Let U_1, ..., U_k denote the non–trivial components in G[B] with at least one M–exposed vertex



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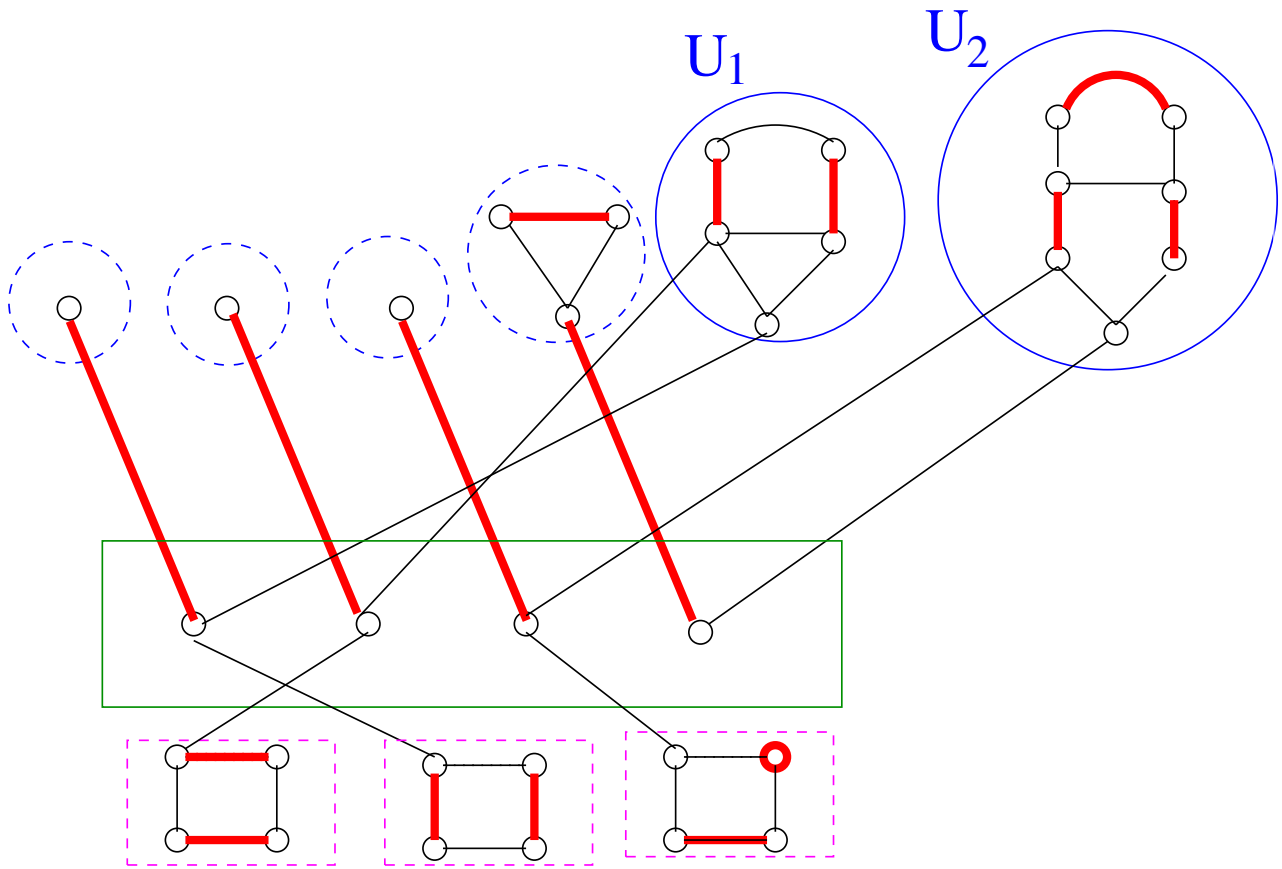
Each U_i has at least one vertex $v_i$essential in G\F



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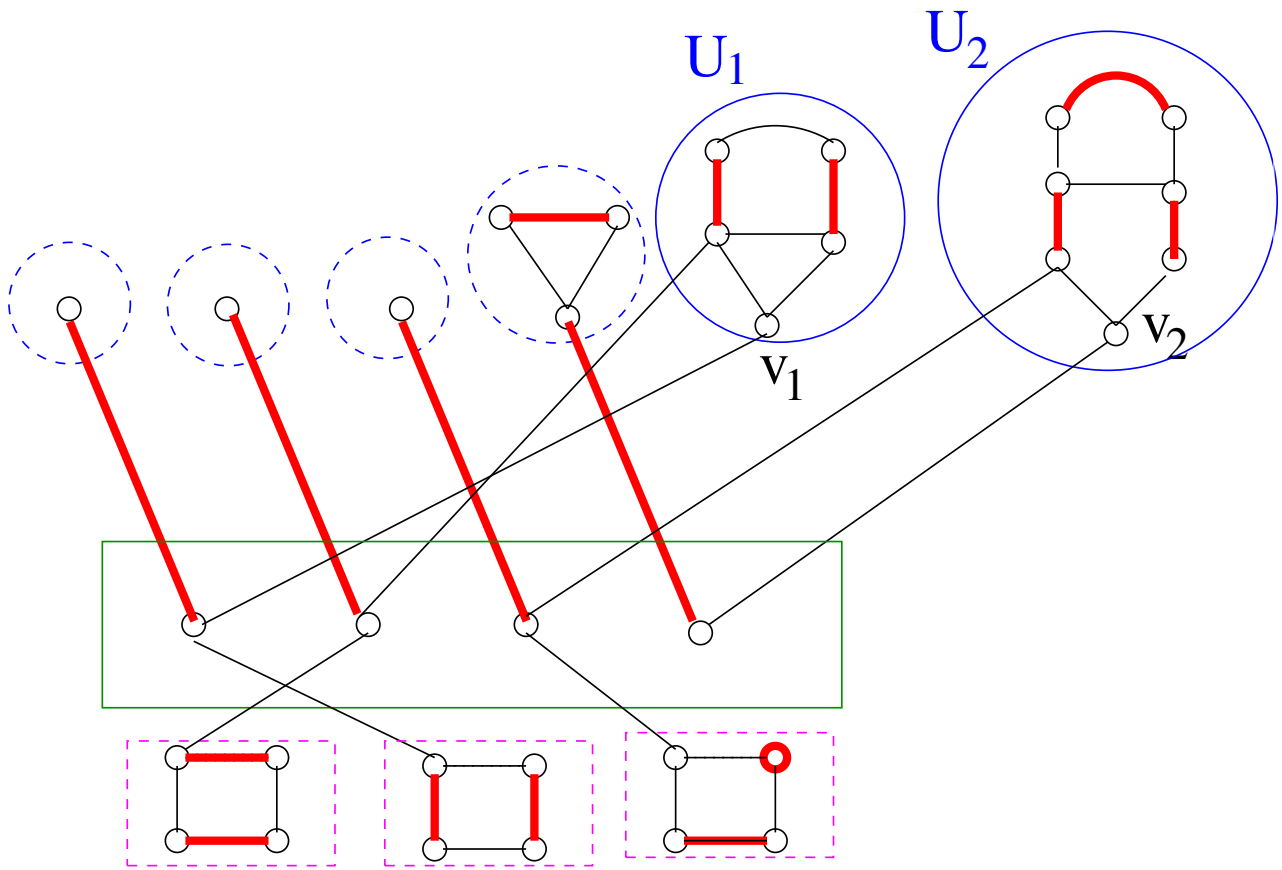

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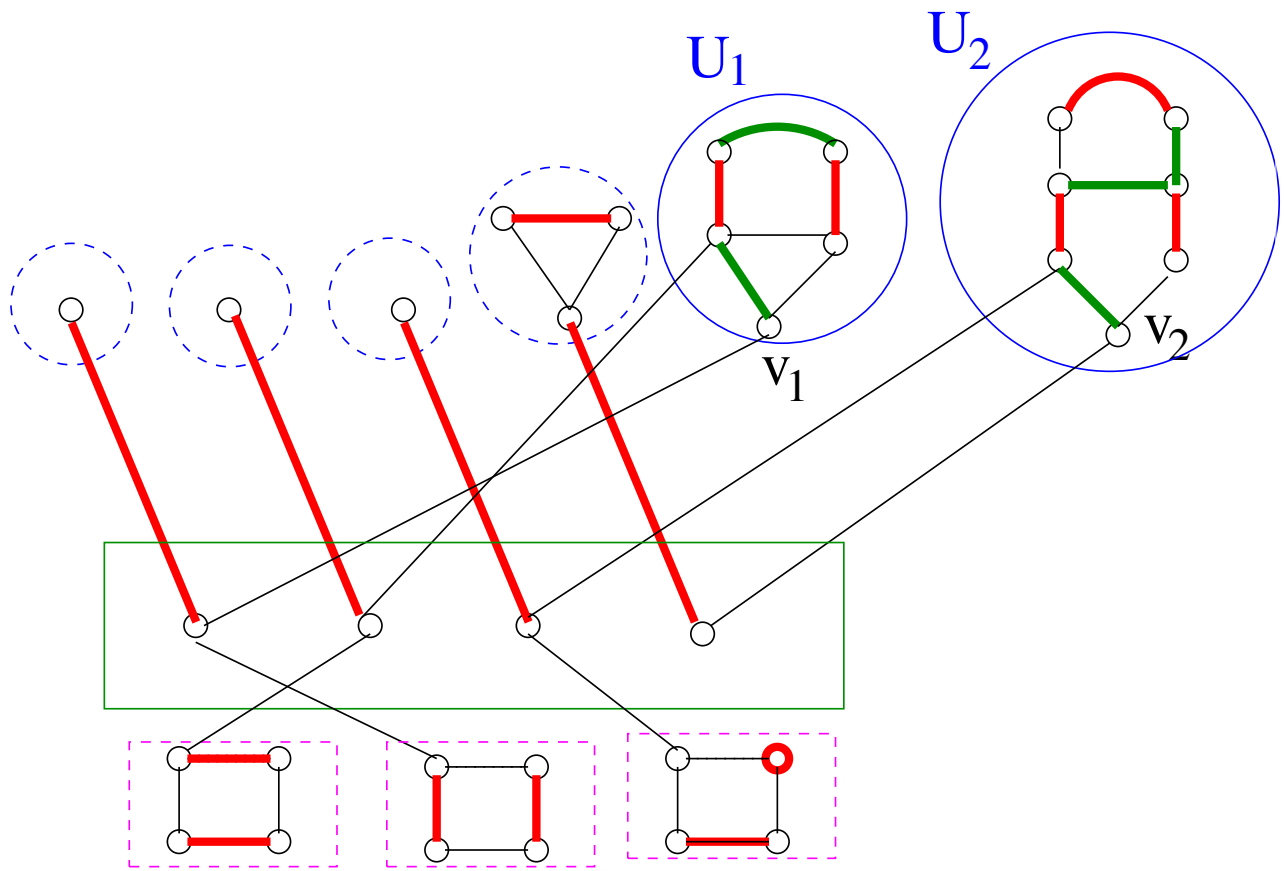
Choose max matching N_1 in G\F



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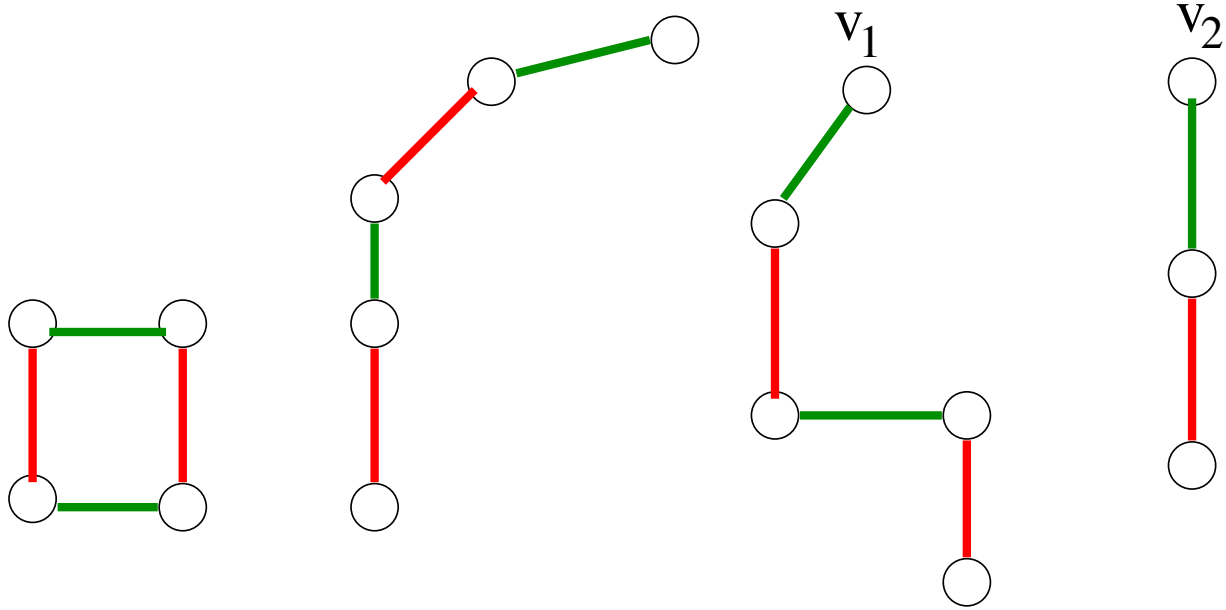
F stabilizer of G $\implies |F| \geq 2(\nu_f(G) - \nu(G))$

Proof:

N$\triangle$M is disjoint union of even cycles and even paths



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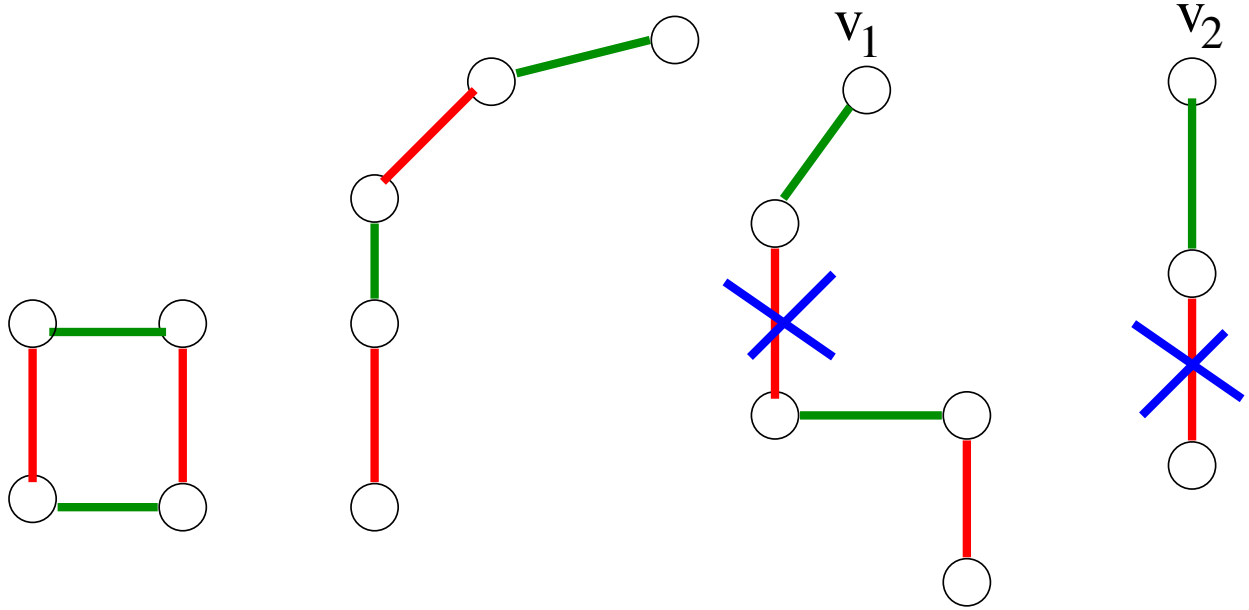
Note: each of the k paths starting at v_i has at least one edge in F.



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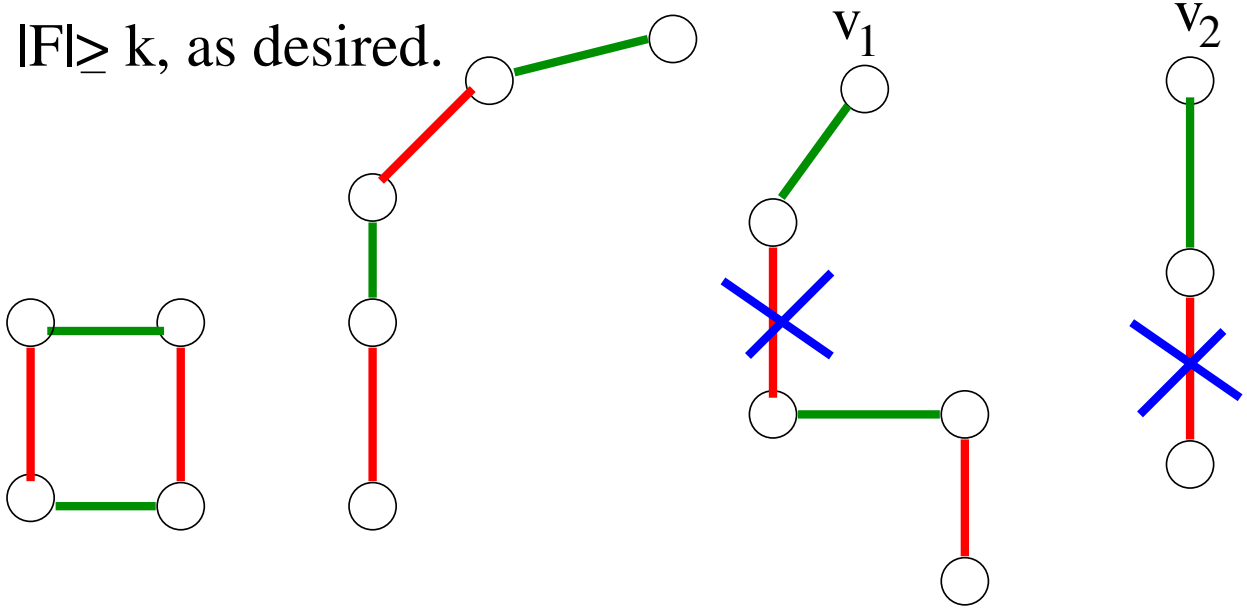
Thus, |F|≥ k, as desired.



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