Colourful linear programming

Complexity, Nash equilibrium and pivot algorithms

Frédéric Meunier and Pauline Sarrabezolles

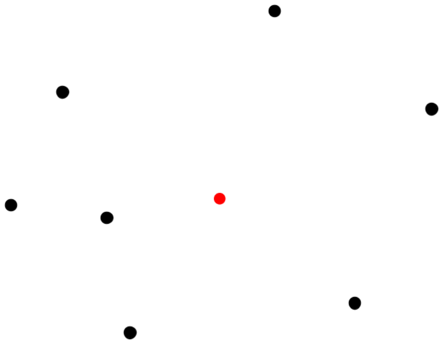
Aussois 2014

Outline of the talk

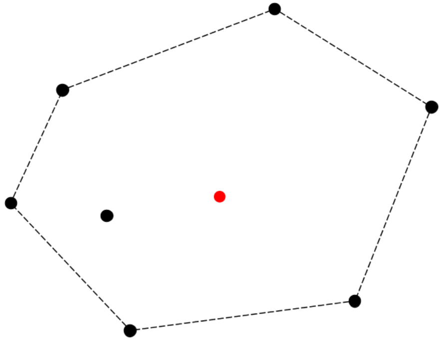
1. Colourful linear programming
2. Complexity and links with Nash equilibrium
3. Pivot algorithms

Colourful linear programming

The Carathéodory Theorem in dimension two



The Carathéodory Theorem in dimension two



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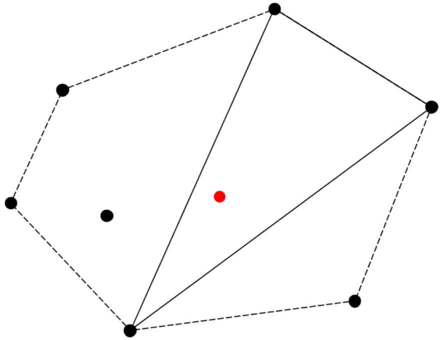


Linear programming

The linear programming problem.

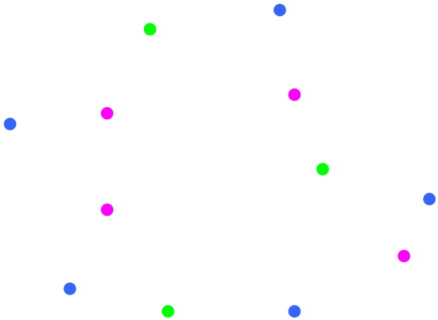
Input : a set $S \subset \mathbb{Q}^d$, a point $p \in \mathbb{Q}^d$.
Output : **Decide** whether there is

$$T \subseteq S,\ |T| \leq d+1, \text{ such that } p \in \operatorname{conv}(T).$$

If "yes", **find** it.

$$\boxed{\text{Carathéodory Theorem} \Longrightarrow \text{If } p \in \operatorname{conv}(S), \text{ there is such a } T.}$$

The colourful Carathéodory Theorem in dimension two



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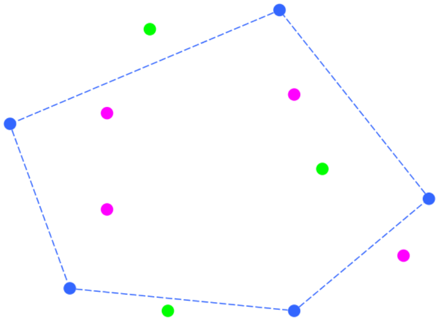

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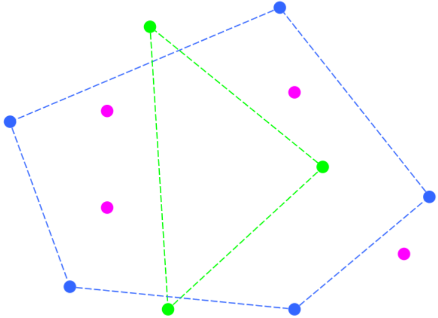

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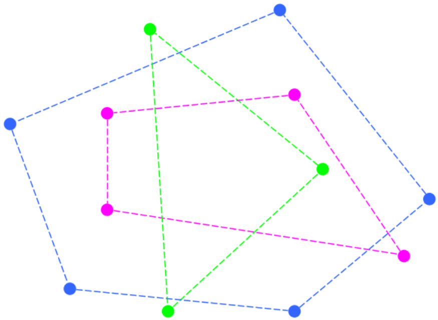

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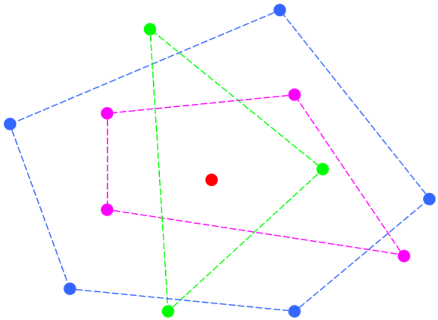

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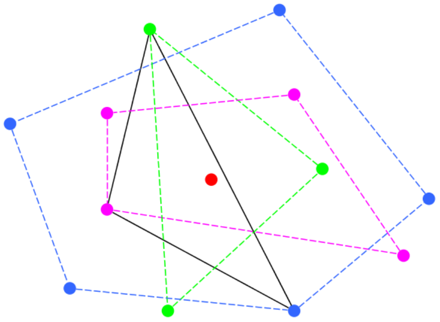

The colourful Carathéodory Theorem [Bárány 1982]

Theorem (CCT)

Let $S_1, \ldots, S_{d+1}$ *be sets of points, and a point* $p \in \mathbb{R}^d$*. If* $p \in \bigcap_{i=1}^{d+1} \mathrm{conv}(S_i)$ *(Bárány's conditions), there is* $T \subseteq \bigcup_{i=1}^{d+1} S_i$ *such that*

$$|T \cap S_i| \leq 1 \textit{ for all } i \textit{ and } p \in \mathrm{conv}(T).$$

$T \subseteq \bigcup_{i=1}^{d+1} S_i$ such that $|T \cap S_i| \leq 1$ for $i = 1, \ldots, d+1$ is **colourful**.

Applications :

- Bound on simplicial depth.
- Proof of Tverberg's Theorem (generalization of Radon's Theorem).

Colourful linear programming [Bárány and Onn in 1997]

The colourful linear programming problem.

Input : k sets, or *colours*, $S_1, \ldots, S_k \subset \mathbb{Q}^d$, a point $p \in \mathbb{Q}^d$.
Output : **Decide** whether there is

$$\text{a colourful } T \text{ such that } p \in \operatorname{conv}(T).$$

If "yes", **find** it.

$$\boxed{\text{CCT} \Longrightarrow \text{under Bárány's conditions, there is such a } T.}$$

Complexity status

Feasibility problem is NP-complete [Bárány and Onn, 1997].

- Details and new results
- Links with Nash equilibrium

Functional problem under Bárány's conditions is unknown.

- Pivot algorithms

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Colourful linear programming, feasibility problem

Input : k sets, or *colours*, $S_1, \ldots, S_k \subset \mathbb{Q}^d$, a point $p \in \mathbb{Q}^d$.
Output : **Decide** whether there is

$$\text{a colourful } T \text{ such that } p \in \mathrm{conv}(T).$$

Complexity status, Bárány and Onn

Proposition (Bárány and Onn, 1997)

When $k = d$, the feasibility problem of CLP is **NP-complete***, even if* $p \in \bigcap_{i=1}^{k} \mathrm{conv}(S_i)$.

$\rightarrow$ Proof by a reduction of PARTITION.

When $k = d + 1$,

- if $p \in \bigcap_{i=1}^{k} \mathrm{conv}(S_i)$, then it is trivial (Colourful Carathéodory Theorem),
- in general, it is stated as an open question by Bárány and Onn (1997)

Complexity status, Bárány and Onn

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- in general, it is **NP-complete**

Complexity status, Bárány and Onn

Proposition (Bárány and Onn, 1997)

When $k = d$, the feasibility problem of CLP is **NP-complete**, *even if* $p \in \bigcap_{i=1}^{k} \mathrm{conv}(S_i)$.

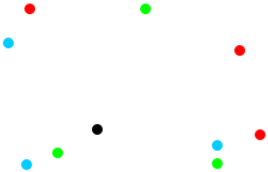
→ Proof by a reduction of PARTITION.

Proposition (Meunier and S.)

When $k = d + 1$, the feasibility problem of colourful linear programming is **NP-complete**.

→ Proof by a reduction of SUBSET SUM, and adapting a proof that LCP is NP-complete.

Consequence of this result



Investigation on conditions insuring the existence of a colourful set.

- Bárány (1982) : $0 \in \text{conv}(S_i)$ for all i.
- Arocha et.al (2009) and Holmsen et. al (2008) : $0 \in \text{conv}(S_i \cup S_j)$ for all $i < j$.
- ...

$\boxed{\text{No polynomially checkable ‘iff’ conditions when } k = d + 1.}$

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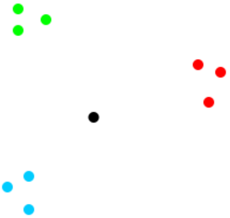

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Links with Nash equilibrium

Bimatrix game

- Two players :
  - the first chooses a probability distribution x on $\{1, \ldots, n\}$,
  - the second chooses a probability distribution y on $\{1, \ldots, m\}$.
- Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ corresponding to the gains.
  Player 1 wins $\sum_{i,j} x_i y_j a_{ij}$*, Player 2 wins* $\sum_{i,j} x_i y_j b_{ij}$*.*

A **Nash equilibrium** is a choice of distributions such that, no player has an interest in changing its distributions.

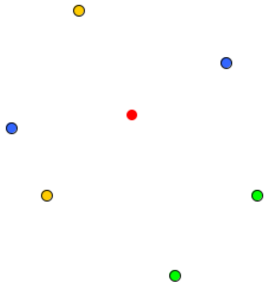
Proposition (Chen, Deng and Teng, 2009)

BIMATRIX *is PPAD-complete.*

Find another colourful simplex [Meunier and Deza, 2011]

'Find another' problem.

Input : $d+1$ pairs of points $S_1, \ldots, S_{d+1}$ in $\mathbb{R}^d$ and a colourful set T such that $0 \in \text{conv}(T)$
Output : **Find** another colourful set T' such that $0 \in \text{conv}(T)$.



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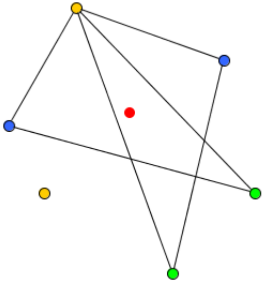

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Proposition (Meunier and S.)

'Find another' is **PPAD-complete**.

$\rightarrow$ Proof by a reduction of BIMATRIX, inspired by the Lemke and Howson algorithm.

Links with colourful linear programming

For theroetical reasons we know that if $P = NP$ then $P = PPAD$. Here is a concrete example of this fact.

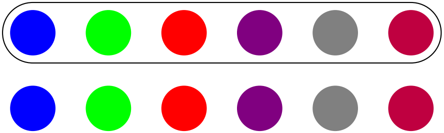

FIGURE: Using colourful linear programming to solve 'find another'

We solve 'find another' by calling $d + 1$ times an algorithm for CLP.

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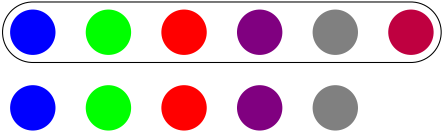

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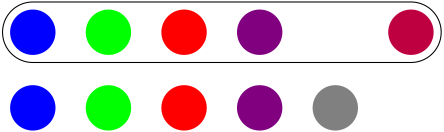

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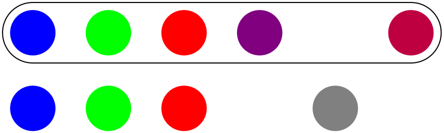

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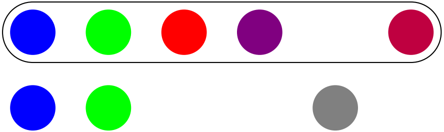

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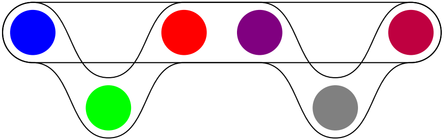


FIGURE: Using colourful linear programming to solve 'find another'

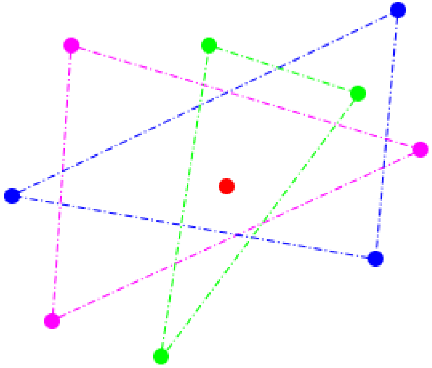
We solve 'find another' by calling $d + 1$ times an algorithm for CLP.

Any algorithm for CLP can be used to solve BIMATRIX in polynomial time as well.

Pivot algorithms

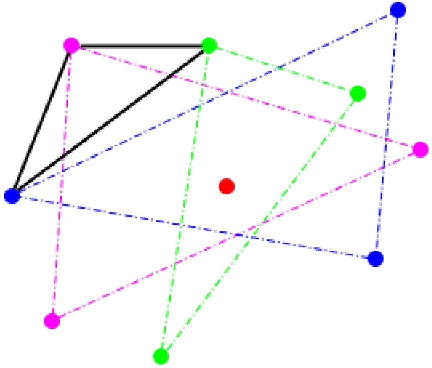
Ideas of Bárány and Onn algorithm

Consider $S_1, \ldots, S_{d+1}$, sets of points, each containing $\mathbf{0}$.



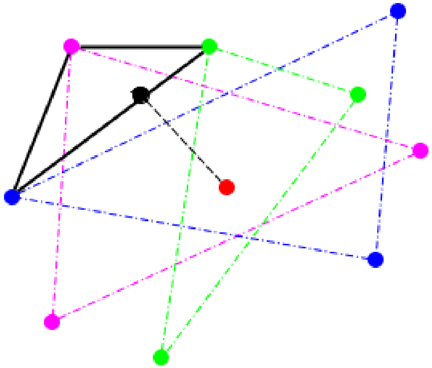
Ideas of Bárány and Onn algorithm

Consider a colourful simplex.



Ideas of Bárány and Onn algorithm

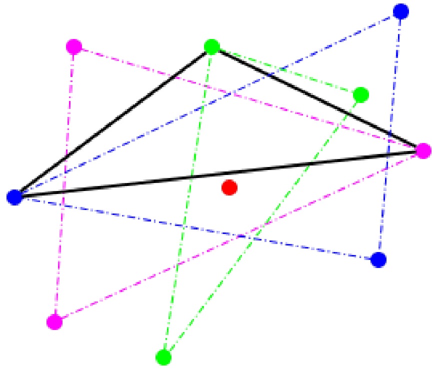
Consider the closest point to the $\mathbf{0}$ in this simplex.



This point lies on a facet of the colourful simplex. A colour i is missing on this facet.

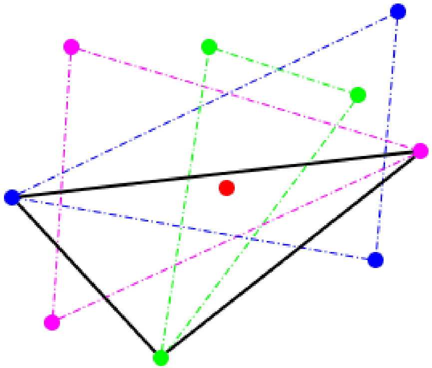
Ideas of Bárány and Onn algorithm

Replace the vertex of colour i with another vertex of the same colour, getting a point closer to $\mathbf{0}$



Ideas of Bárány and Onn algorithm

Iterate...



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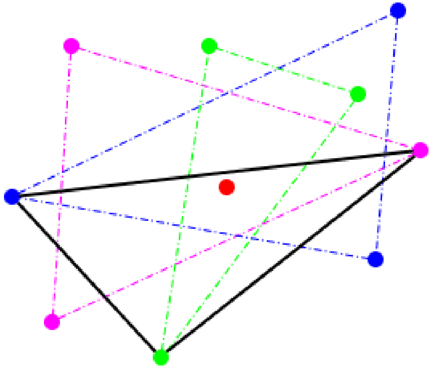
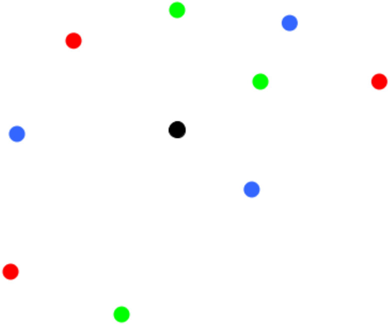


This algorithm uses a distance computation.

A combinatorialization of Bárány's algorithm

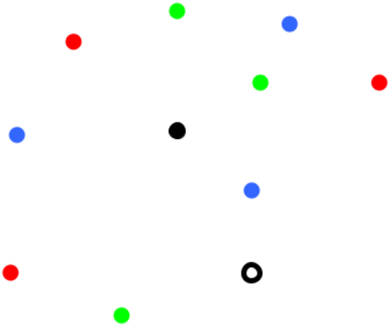


Our problem is to find a **colourful feasible basis** of

$$\begin{pmatrix} A_1 & A_2 & \cdots & A_{d+1} \\ 1 & 1 & \cdots & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$s.t. \quad x \geq 0$$

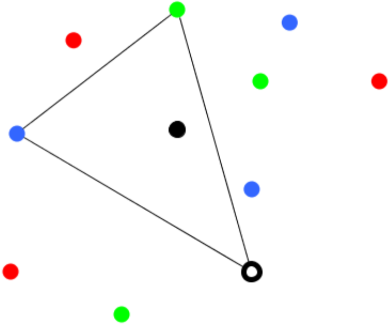
A combinatorialization of Bárány's algorithm



We introduce a dummy point v, and solve

$$\begin{aligned} \min \quad & z \\ s.t. \quad & Ax + zv = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\ & x, z \geq 0 \end{aligned}$$

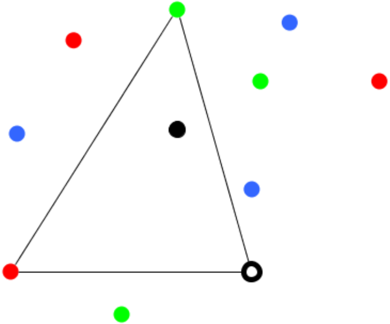
A combinatorialization of Bárány's algorithm



We proceed by **simplex pivots**.

- Start from a colourful feasible basis (color i is missing),
- choose a point t of **negative reduced cost** in S_i,
- enter t in the current basis by simplex pivot.
- until v leaves the basis.

A combinatorialization of Bárány’s algorithm



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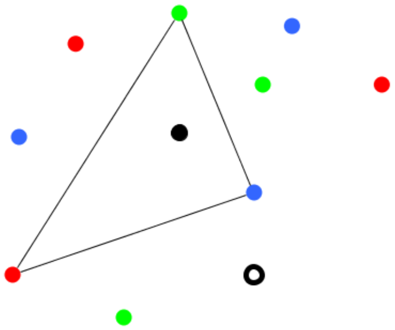

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This algorithm is combinatorial. It is similar to a 'Phase I' algorithm

Counting colourful simplices
Joint work with Antoine Deza

Colourful simplicial depth

Theorem (Colourful Carathéodory Theorem)

Let $S_1, \ldots, S_{d+1}$ *be sets of points. If* $0 \in \bigcap_{i=1}^{d+1} \mathrm{conv}(S_i)$ *and* $|S_i| = d+1$ *for* $i = 1, \ldots, d+1$*, there is a colourful simplex* T *such that* $0 \in \mathrm{conv}(T)$.

Colourful simplicial depth

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How many colourful simplices at least ?

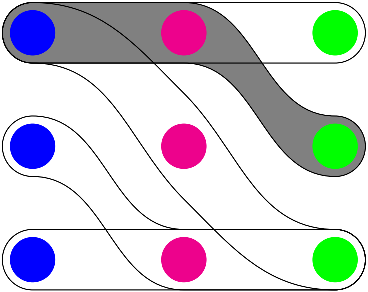
The successive improvements

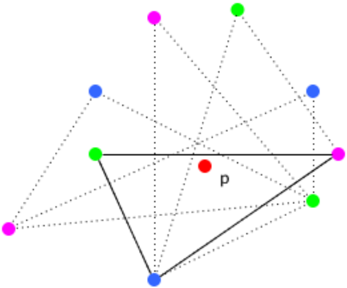
Conjecture

There are at least $\mathbf{d^2+1}$ *colourful simplices containing* 0 *in their convex hull.*

	Lower bound for $\mu(d)$	Conjecture true for d up to
Bárány, 1982	$d+1$	1
Deza et al., 2006	$2d$	2
Bárány and Matoušek, 2007	$\max(3d, \frac{1}{5}d^2+\frac{1}{5}d)$	3
Stephen and Thomas, 2008	$\frac{1}{4}d^2+d+1$	$\emptyset$
Deza, Stephen, and Xie, 2011	$\frac{1}{2}d^2+d+\frac{1}{2}$	$\emptyset$
Deza, Meunier, and S., 2012	$\frac{1}{2}d^2+\frac{7}{2}d-8$	4

Octahedral systems





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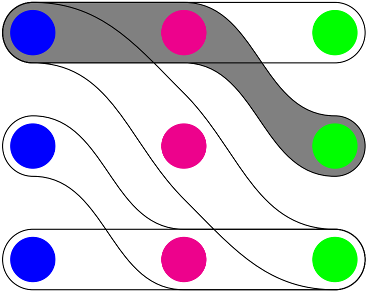

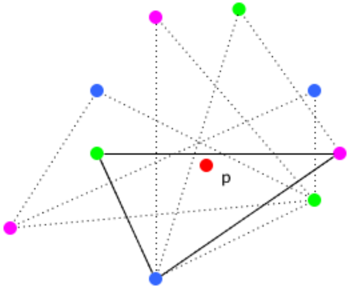

We improve the bound with this approach.

Open questions

- What is the complexity of CLP ?
- How efficient is the pivot algorithm without distance computation ?