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The Online Replacement Path Problem

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With David Adjiashvili (ETH) and Gianpaolo Oriolo (University of Rome “Tor Vergata”)

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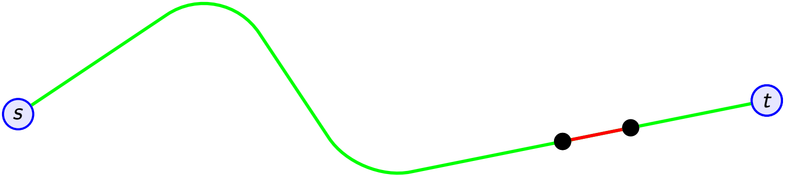

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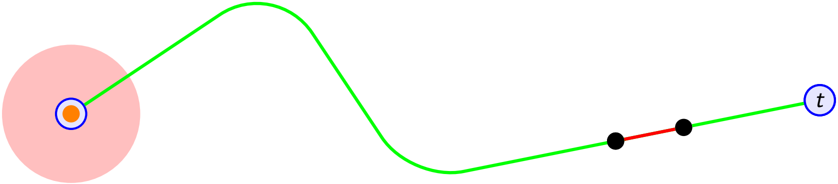

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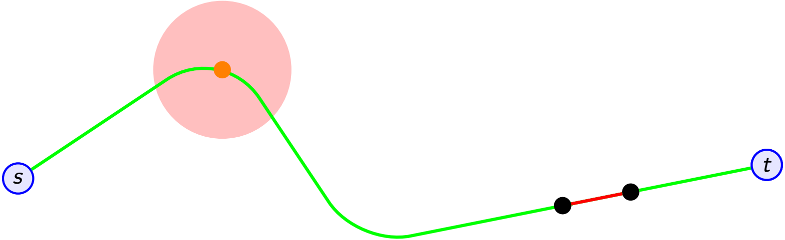

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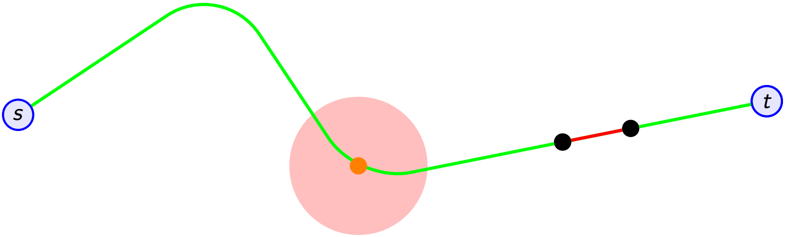

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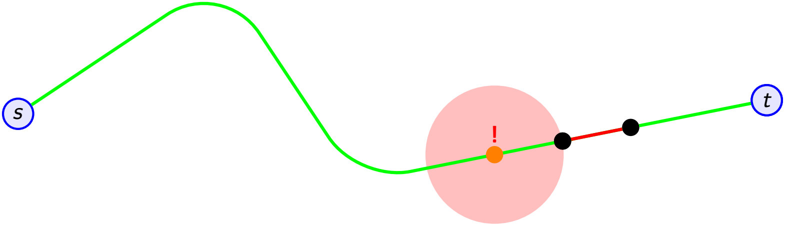

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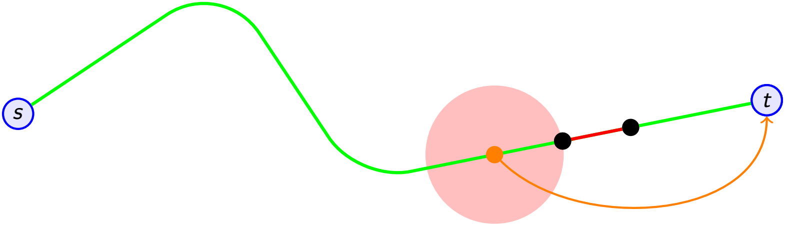

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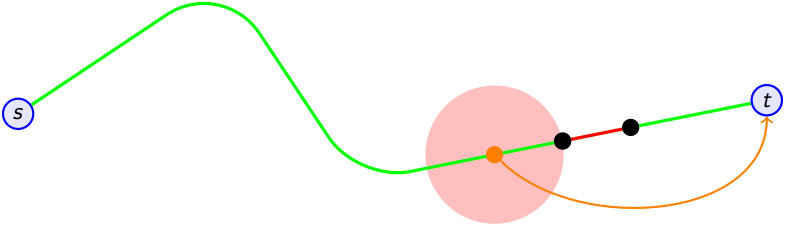

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- **Goal:** Provide an s-t path to the RM, such that the worst-case total travel time is minimized.

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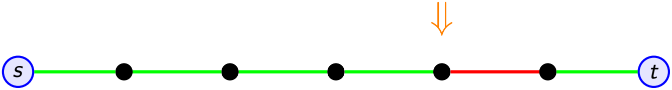

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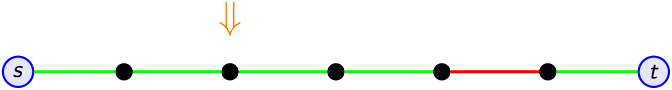

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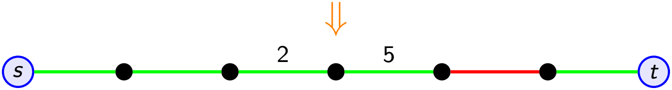

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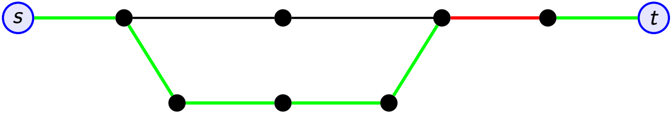

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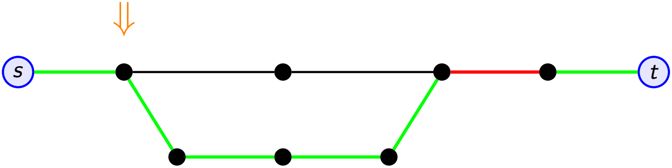

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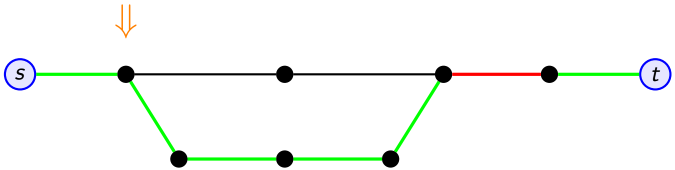

- As soon as RM is aware of the failed edge, it follows the shortest path in the residual graph.

Some Notations

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- For each $v \in V$,
  1. $\mathcal{P}_{v,t}$ is the set of v-t paths;
  2. $\pi(v) :=$ shortest v-t path length.
- For $e \in E$ and $u \in V$:
  1. let Q_u^{-e} be some fixed shortest u-t path in $G \setminus e$;
  2. $\pi^{-e}(u) = \ell(Q_u^{-e})$.

The Online Replacement Path Problem

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Definition (Detour)

Given a path $P \in \mathcal{P}_{v,t}$ and an edge $e = uu' \in E$, the detour P^{-e} is:

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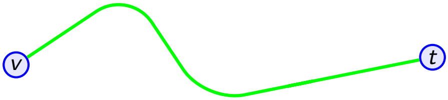

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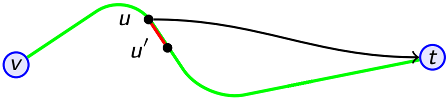

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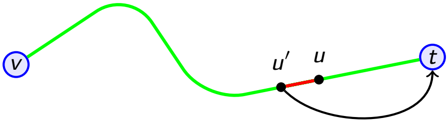

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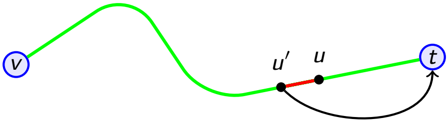

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The robust length of the v-t path P is

$$\mathrm{Val}(P) = \max_{e \in E} \ell(P^{-e}).$$

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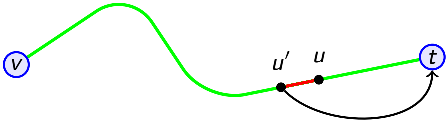

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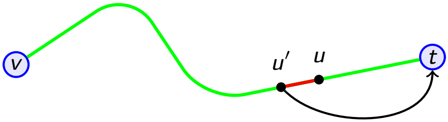

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- **Find:** *an s-t path with minimum robust length.*

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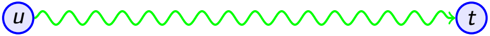

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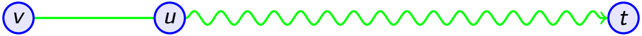

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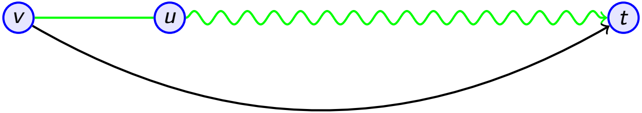

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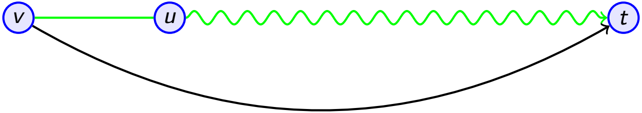

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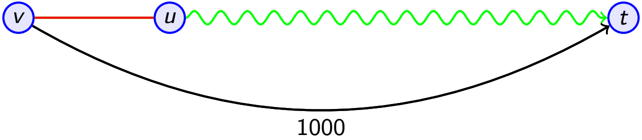

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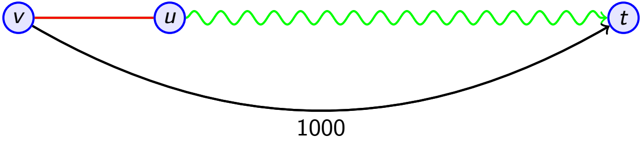

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Lemma (Weak optimality principle)

Let $P_v \in \mathcal{P}_{v,t}$ be an optimal path from v, $u \in V(P_v)$ and $P_u \in \mathcal{P}_{u,t}$ be an optimal path from u. Then the path $P'_v = P_v[v,u] \oplus P_u$ satisfies $\mathrm{Val}^k(P'_v) = \mathrm{Val}^k(P_v)$, namely it is also optimal from v.

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- We devise a label setting algorithm.

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Lemma

Let $U \subset V$, with $t \in U$, be a set of nodes for which π_{rob} is known, and let $vu \in \delta(U)$:

$$vu = \underset{zw \in E : w \in U, z \notin U}{\arg\min} \max\{\ell(zw) + \pi_{rob}(w), \pi^{-zw}(z)\}.$$

Then $\pi_{rob}(v) = \mathrm{Val}(vu \oplus P_u)$, where P_u is any optimal (robust) u-t path .

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Theorem

Given an instance of ORP, the values π_{rob} and the corresponding paths can be computed in time $O(m + n \log n)$ in undirected graphs, and $O(mn + n^2 \log n)$ in directed graphs.

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- The same complexity bound is attained in Bar-Noy and Schieber '91, "The Canadian Traveller Problem".

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- At the same time we could have a budget B on the robust length.
- We aim at solving:

$$P^* := \operatorname*{arg\,min}_{P \in \mathcal{P}_{s,t}\colon \mathrm{Val}(P) \leq B} \ell(P).$$

Robust Length vs. Nominal Length

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Algorithm

$S = \emptyset; \quad \bar{S} = V \quad d(s) = 0; \quad d(u) = \infty \ \forall u \in V \setminus s$
while $t \notin S$ **do**
  Find $u = \arg\min_{z \in \bar{S}} d(z)$
  $S = S + u \quad \bar{S} = \bar{S} - u$
  for $q \in N(u) \setminus S$ **do**
    if $d(u) + \ell(uq) \leq d(q)$ *and* $d(u) + \pi^{-uq}(u) \leq B$ **then**
      $d(q) = d(u) + \ell(uq)$

Robust Length vs. Nominal Length

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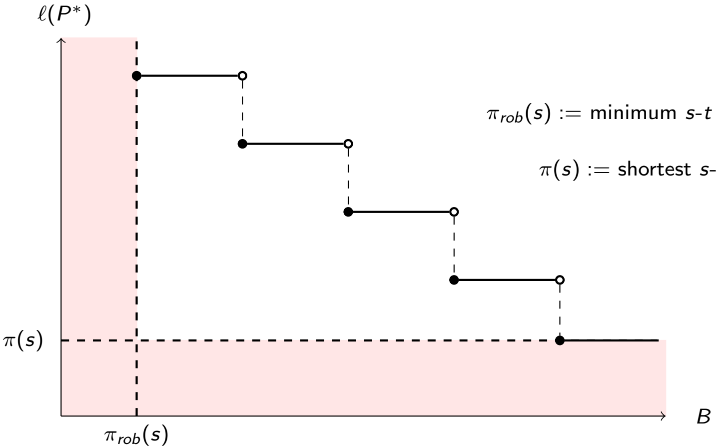

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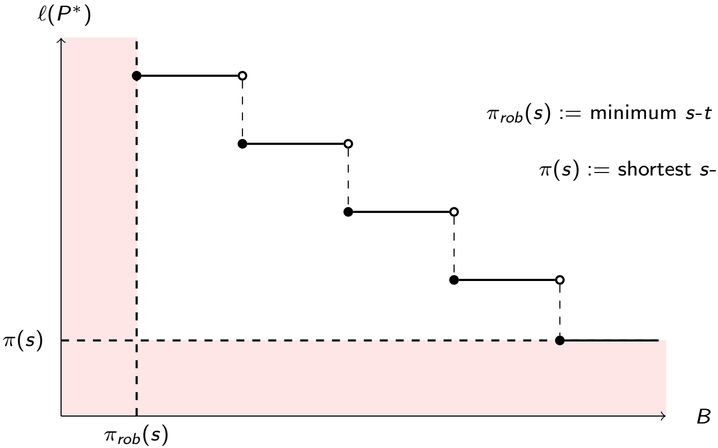

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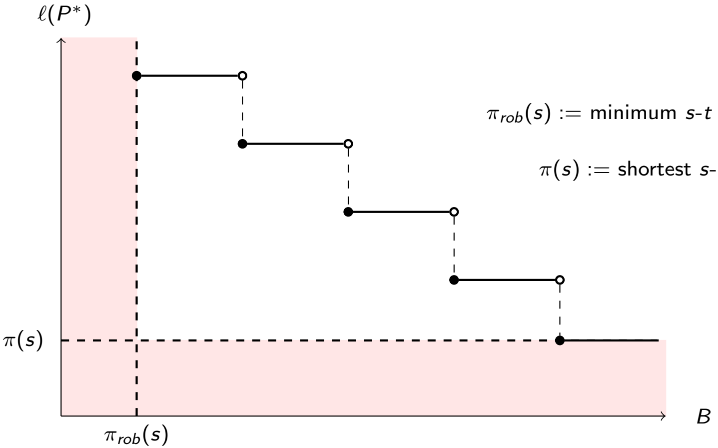

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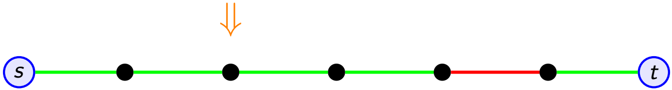

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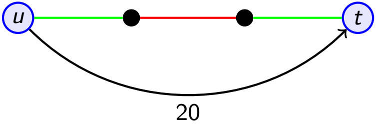

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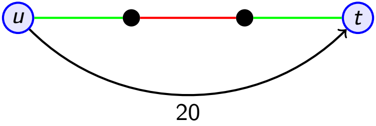

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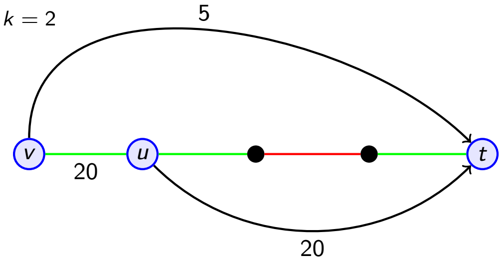

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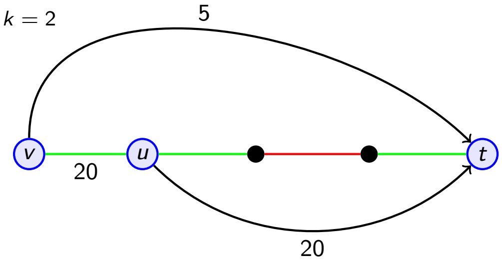

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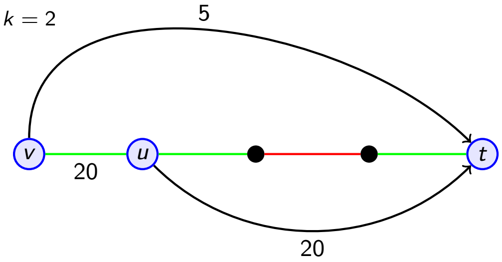

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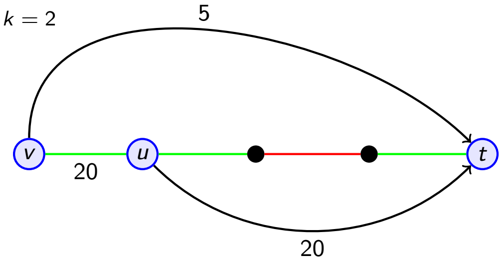

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- **Weak optimality principle** still holds for k-Hop ORP

k-Hop ORP and Radius ORP

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Theorem

Given an instance of k-Hop ORP we can compute the v-t path with minimum k-robust length for any $v \in V$ in time $O(m + n \log n)$ in undirected graphs, and $O(mn + n^2 \log n)$ in directed graphs.

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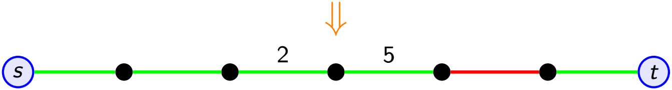
- Our algorithm for k-Hop ORP solves Radius ORP as well.

$R = 6$



Strong k-Hop ORP

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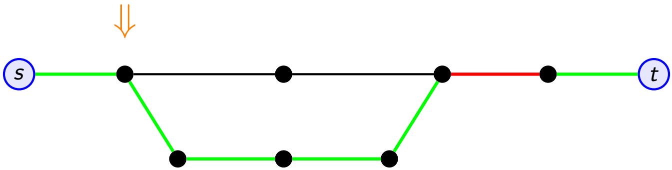

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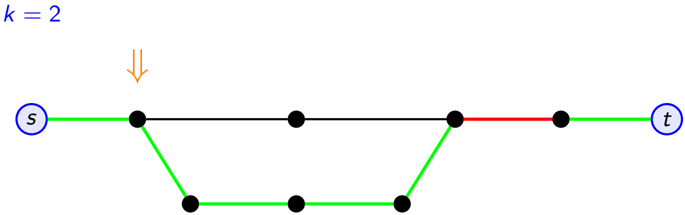

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For any $\epsilon > 0$ it is NP-hard to approximate Strong 1-Hop ORP within a factor of $3 - \epsilon$ in undirected graphs. In directed graphs it is strongly NP-hard to decide if there exists a path with finite robust length.

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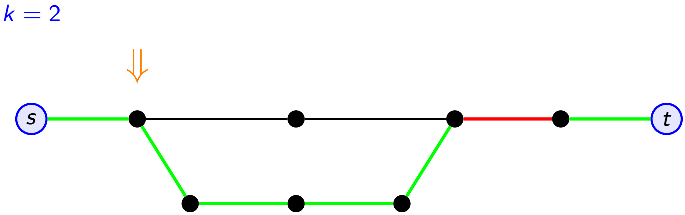

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- We call the ORP Game, that game in which both the path builder and the interdictor communicate their strategies at the **same time**.

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Let P and e be optimal solutions to the ORP and MVA instances on $G = (V, E)$. Then (P, e) is a pure NE of the ORP Game if and only if $\mathrm{Val}(P) = \pi^{-e}(s)$*. Moreover, in this case,* $\mathrm{Val}(P) = z^*(ORP) = z^*(MVA) = \pi^{-e}(s)$.

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- **Question:** Is it possible to find a mixed NE in polynomial time?

Stochastic ORP

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- In this variant of ORP:
  1. at most one edge can fail;
  2. for each $e \in E$, $p(e) \in [0,1]$ is the probability that e will be the unique edge to fail;
  3. $p(\emptyset) \in [0,1]$ is the probability that no edge fails;
  4. it holds $p(\emptyset) + \sum_{e \in E} p(e) = 1$.

Definition

Given a node $v \in V$, the *expected robust length* of the v-t path P is

$$\mathbb{E}\mathrm{Val}(P) = \Big(p(\emptyset) + \sum_{uu' \notin P} p(uu')\Big)\ell(P) + \sum_{uu' \in P} \Big[p(uu')(\ell(P[v,u]) + \pi^{-uu'}(u))\Big].$$

- Stochastic ORP is to find, for some $v \in V$, a path P minimizing $\mathbb{E}\mathrm{Val}(P)$ over all paths $P \in \mathcal{P}_{v,t}$.
- Stochastic ORP is NP-hard by reduction from the Hamiltonian Path problem.
- **Open question:** Is it possible to find a mixed NE in polynomial time?

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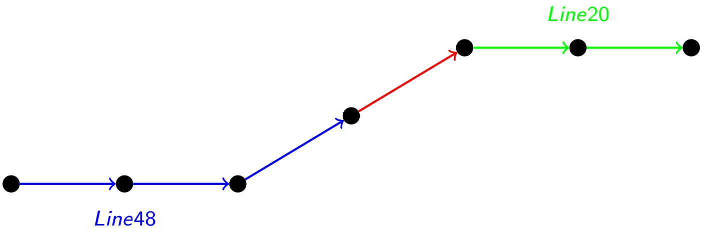

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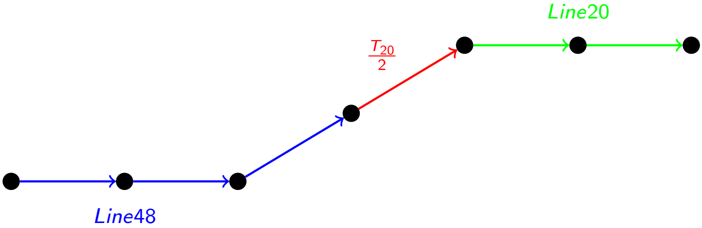

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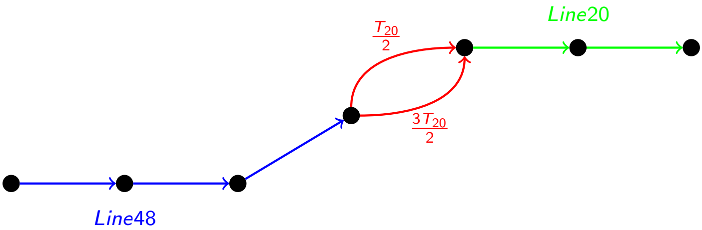

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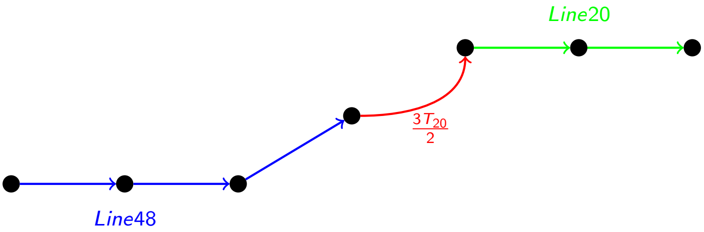

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Thank You...