Lower Bounds on the Sizes of Integer Programs Without Additional Variables

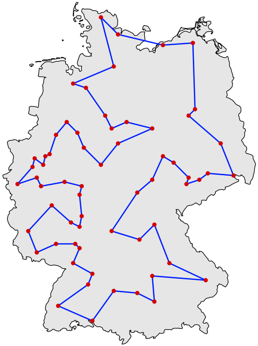
Volker Kaibel & Stefan Weltge

Aussois, 2014

Motivation

Traveling Salesman Problem

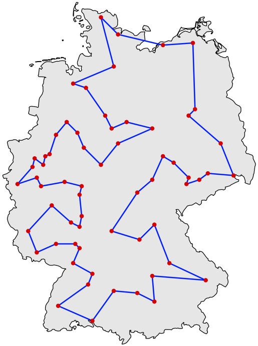
Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.



Motivation

Traveling Salesman Problem

Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.



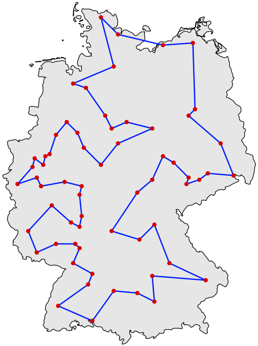
Natural approach:

- (binary) variable x_{ij} for each edge $\{i, j\}$

Motivation

Traveling Salesman Problem

Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.



Natural approach:

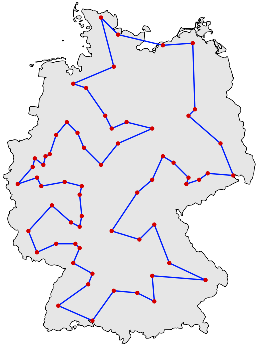
- (binary) variable x_{ij} for each edge $\{i, j\}$
- identify each tour T with its characteristic vector $\chi(T) \in \{0, 1\}^E$ $(\chi(T)_e = 1 \iff e \in T)$

Traveling Salesman Problem

Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.

Natural approach:

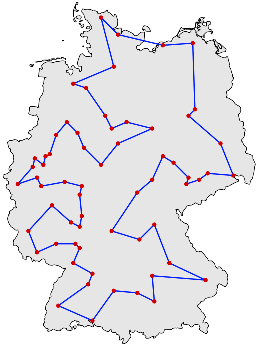
- (binary) variable x_{ij} for each edge $\{i, j\}$
- identify each tour T with its characteristic vector $\chi(T) \in \{0, 1\}^E$ $(\chi(T)_e = 1 \iff e \in T)$
- $\mathrm{TSP}_n := \{\chi(T) : T \text{ hamiltonian tour}\}$



Motivation

Traveling Salesman Problem

Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.



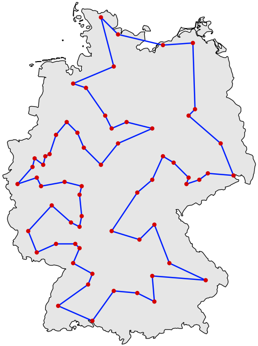
Natural approach:

- (binary) variable x_{ij} for each edge $\{i, j\}$
- identify each tour T with its characteristic vector $\chi(T) \in \{0, 1\}^E$ $(\chi(T)_e = 1 \iff e \in T)$
- $\mathrm{TSP}_n := \{\chi(T) : T \text{ hamiltonian tour}\}$
- given weights c_{ij}, we would like to solve:

$$\min \{\langle c, x\rangle : x \in \mathrm{TSP}_n\}$$

Traveling Salesman Problem

Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.



Natural approach:

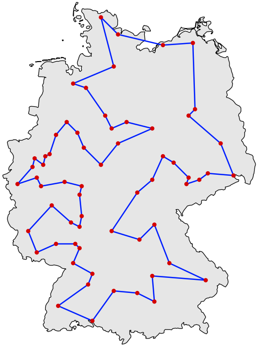
- (binary) variable x_{ij} for each edge $\{i, j\}$
- identify each tour T with its characteristic vector $\chi(T) \in \{0,1\}^E$ $(\chi(T)_e = 1 \iff e \in T)$
- $\mathrm{TSP}_n := \{\chi(T) : T \text{ hamiltonian tour}\}$
- given weights c_{ij}, we would like to solve:

$$\min\{\langle c, x\rangle : x \in \mathrm{TSP}_n\} = \min\{\langle c, x\rangle : x \in \mathrm{conv}(\mathrm{TSP}_n)\}$$

Motivation

Traveling Salesman Problem

Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.



Natural approach:

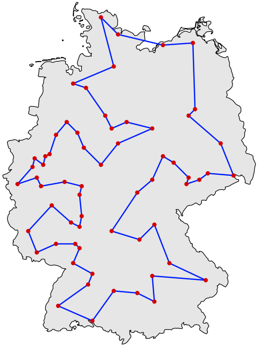
- (binary) variable x_{ij} for each edge $\{i, j\}$
- identify each tour T with its characteristic vector $\chi(T) \in \{0,1\}^E$ $(\chi(T)_e = 1 \iff e \in T)$
- $\mathrm{TSP}_n := \{\chi(T) : T \text{ hamiltonian tour}\}$
- given weights c_{ij}, we would like to solve:

$$\begin{aligned} \min\{\langle c, x\rangle : x \in \mathrm{TSP}_n\} &= \min\{\langle c, x\rangle : x \in \mathrm{conv}(\mathrm{TSP}_n)\} \\ &= \min\{\langle c, x\rangle : Ax \leq b\} \end{aligned}$$

Motivation

Traveling Salesman Problem

Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.



Natural approach:

- (binary) variable x_{ij} for each edge $\{i, j\}$
- identify each tour T with its characteristic vector $\chi(T) \in \{0,1\}^E$ $(\chi(T)_e = 1 \iff e \in T)$
- $\mathrm{TSP}_n := \{\chi(T) : T \text{ hamiltonian tour}\}$
- given weights c_{ij}, we would like to solve:

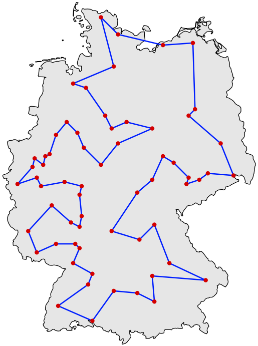
$$\begin{aligned} \min \{\langle c, x\rangle : x \in \mathrm{TSP}_n\} &= \min \{\langle c, x\rangle : x \in \mathrm{conv}(\mathrm{TSP}_n)\} \\ &= \min \{\langle c, x\rangle : Ax \le b\} \end{aligned}$$

- $Ax \le b$ is very large and complicated

Motivation

Traveling Salesman Problem

Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.



Natural approach:

- (binary) variable x_{ij} for each edge $\{i, j\}$
- identify each tour T with its characteristic vector $\chi(T) \in \{0,1\}^E$ $(\chi(T)_e = 1 \iff e \in T)$
- $\mathrm{TSP}_n := \{\chi(T) : T \text{ hamiltonian tour}\}$
- given weights c_{ij}, we would like to solve:

$$\begin{aligned} \min\{\langle c, x\rangle : x \in \mathrm{TSP}_n\} &= \min\{\langle c, x\rangle : x \in \mathrm{conv}(\mathrm{TSP}_n)\} \\ &= \min\{\langle c, x\rangle : Ax \le b\} \end{aligned}$$

- $Ax \le b$ is very large and complicated

Why not use additional variables?

Motivation (2)

$$\min\{\langle c, x\rangle : x \in \mathrm{TSP}_n\} = \min\{\langle c, x\rangle : Ax + By \leq b\}$$

Motivation (2)

Theorem (Fiorini, Pokutta et al. 2012)

There is no polynomial size system $Ax + By \leq b$ such that

$$\min\{\langle c, x\rangle : x \in \mathrm{TSP}_n\} = \min\{\langle c, x\rangle : Ax + By \leq b\}$$

holds for all $c \in \mathbb{R}^E$.

Motivation (2)

Theorem (Fiorini, Pokutta et al. 2012)

There is no polynomial size system $Ax + By \leq b$ such that

$$\min\{\langle c, x\rangle : x \in \mathrm{TSP}_n\} = \min\{\langle c, x\rangle : Ax + By \leq b\}$$

holds for all $c \in \mathbb{R}^E$.

But there are several polynomial size integer programming formulations of the form

$$\min\left\{\langle c, x\rangle : Ax + By \leq b,\ x \in \{0,1\}^E,\ y \in \mathbb{Z}^k\right\}$$

for solving the TSP (Miller-Tucker-Zemlin, flow based, ...)

Motivation (2)

Theorem (Fiorini, Pokutta et al. 2012)

There is no polynomial size system $Ax + By \leq b$ such that

$$\min \{\langle c, x\rangle : x \in \mathrm{TSP}_n\} = \min \{\langle c, x\rangle : Ax + By \leq b\}$$

holds for all $c \in \mathbb{R}^E$.

But there are several polynomial size integer programming formulations of the form

$$\min \left\{\langle c, x\rangle : Ax + By \leq b,\ x \in \{0,1\}^E,\ y \in \mathbb{Z}^k\right\}$$

for solving the TSP (Miller-Tucker-Zemlin, flow based, ...)

Theorem

For any family of sets $X_d \subseteq \{0,1\}^d$ with the property that the problem

"Given $x \in \mathbb{Z}^d$, is $x \in X_d$?"

is in NP,

Motivation (2)

Theorem (Fiorini, Pokutta et al. 2012)

There is no polynomial size system $Ax + By \leq b$ such that

$$\min\{\langle c, x\rangle : x \in \mathrm{TSP}_n\} = \min\{\langle c, x\rangle : Ax + By \leq b\}$$

holds for all $c \in \mathbb{R}^E$.

But there are several polynomial size integer programming formulations of the form

$$\min\left\{\langle c, x\rangle : Ax + By \leq b,\ x \in \{0,1\}^E,\ y \in \mathbb{Z}^k\right\}$$

for solving the TSP (Miller-Tucker-Zemlin, flow based, ...)

Theorem

For any family of sets $X_d \subseteq \{0,1\}^d$ with the property that the problem

"Given $x \in \mathbb{Z}^d$, is $x \in X_d$?"

is in NP, we have polynomial size systems $Ax + By \leq b$ such that

$$X_d = \left\{x : Ax + By \leq b,\ x \in \{0,1\}^d,\ y \in \mathbb{Z}^k\right\}.$$

Motivation (3)

Observation

All such polynomial size formulations for the TSP use auxiliary variables.

Observation

All such polynomial size formulations for the TSP use auxiliary variables.

Motivating question

Is there a polynomial size system $Ax \leq b$ such that

$$\min \{\langle c, x\rangle : x \in \mathrm{TSP}_n\} = \min \left\{\langle c, x\rangle : Ax \leq b,\ x \in \mathbb{Z}^E\right\}$$

holds for all $c \subseteq \mathbb{R}^E$?

Motivation (3)

Observation

All such polynomial size formulations for the TSP use auxiliary variables.

Motivating question

Is there a polynomial size system $Ax \leq b$ such that

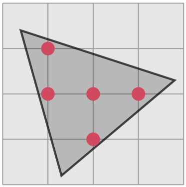
$$\min \{\langle c, x\rangle : x \in \mathrm{TSP}_n\} = \min \left\{\langle c, x\rangle : Ax \leq b,\ x \in \mathbb{Z}^E\right\}$$

holds for all $c \subseteq \mathbb{R}^E$?

Definition

For a set $X \subseteq \mathbb{Z}^d$, a polyhedron $P \subseteq \mathbb{R}^d$ is called a relaxation for X if

$$P \cap \mathbb{Z}^d = X.$$



Motivation (3)

Observation

All such polynomial size formulations for the TSP use auxiliary variables.

Motivating question

Is there a polynomial size system $Ax \leq b$ such that

$$\min \{\langle c, x\rangle : x \in \mathrm{TSP}_n\} = \min \left\{\langle c, x\rangle : Ax \leq b,\ x \in \mathbb{Z}^E\right\}$$

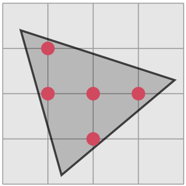
holds for all $c \subseteq \mathbb{R}^E$?

Definition

For a set $X \subseteq \mathbb{Z}^d$, a polyhedron $P \subseteq \mathbb{R}^d$ is called a relaxation for X if

$$P \cap \mathbb{Z}^d = X.$$

The relaxation complexity of X is the smallest number of facets of any relaxation for X. (short: $\mathrm{rc}(X)$)



Motivation (3)

Observation

All such polynomial size formulations for the TSP use auxiliary variables.

Motivating question

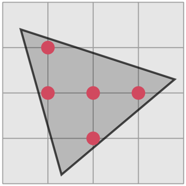
Is $\mathrm{rc}(\mathrm{TSP}_n)$ polynomial in n?

Definition

For a set $X \subseteq \mathbb{Z}^d$, a polyhedron $P \subseteq \mathbb{R}^d$ is called a relaxation for X if

$$P \cap \mathbb{Z}^d = X.$$

The relaxation complexity of X is the smallest number of facets of any relaxation for X. (short: $\mathrm{rc}(X)$)



Separation

Subtour elimination polytope

$$
\begin{aligned}
R_n^{\text{sub}} := \{x \in [0,1]^E : \; & x(\delta(v)) = 2 \text{ for all } v \in V \\
& x(\delta(S)) \geq 2 \text{ for all } \emptyset \neq S \subset V\}
\end{aligned}
$$

Separation

Subtour elimination polytope

$$R_n^{\text{sub}} := \{x \in [0,1]^E : x(\delta(v)) = 2 \text{ for all } v \in V \\ x(\delta(S)) \geq 2 \text{ for all } \emptyset \neq S \subset V\}$$

- R_n^{sub} is a relaxation for TSP_n

Separation

Subtour elimination polytope

$$R_n^{\text{sub}} := \{x \in [0,1]^E : x(\delta(v)) = 2 \text{ for all } v \in V \\ x(\delta(S)) \geq 2 \text{ for all } \emptyset \neq S \subset V\}$$

- R_n^{sub} is a relaxation for TSP_n
- R_n^{sub} has exponentially many facets

Separation

Subtour elimination polytope

$$R_n^{\text{sub}} := \{x \in [0,1]^E : x(\delta(v)) = 2 \text{ for all } v \in V \\ x(\delta(S)) \geq 2 \text{ for all } \emptyset \neq S \subset V\}$$

- R_n^{sub} is a relaxation for TSP_n
- R_n^{sub} has exponentially many facets
- But: We can optimize over R_n^{sub} in polynomial time!

Separation

Subtour elimination polytope

$$R_n^{\text{sub}} := \{x \in [0,1]^E : x(\delta(v)) = 2 \text{ for all } v \in V \\ x(\delta(S)) \geq 2 \text{ for all } \emptyset \neq S \subset V\}$$

- R_n^{sub} is a relaxation for TSP_n
- R_n^{sub} has exponentially many facets
- But: We can optimize over R_n^{sub} in polynomial time!

Theorem

For any family of sets $X_d \subseteq \{0,1\}^d$ with the property that the problem

$$\text{“Given } x \in \mathbb{Z}^d \text{, is } x \in X_d\text{?”}$$

is in P,

Separation

Subtour elimination polytope

$$R_n^{\text{sub}} := \{x \in [0,1]^E : x(\delta(v)) = 2 \text{ for all } v \in V$$
$$x(\delta(S)) \geq 2 \text{ for all } \emptyset \neq S \subset V\}$$

- R_n^{sub} is a relaxation for TSP_n
- R_n^{sub} has exponentially many facets
- But: We can optimize over R_n^{sub} in polynomial time!

Theorem

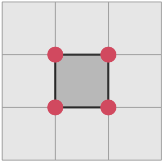
For any family of sets $X_d \subseteq \{0,1\}^d$ with the property that the problem

"Given $x \in \mathbb{Z}^d$, is $x \in X_d$?"

is in P, there are relaxations R_d for X_d and we can optimize over R_d in polynomial time.

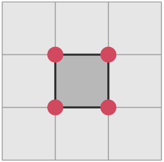
Cube

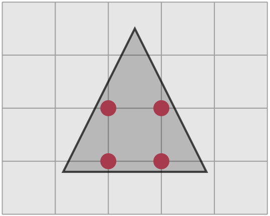
Clearly: $\mathrm{rc}(\{0,1\}^d) \leq 2d$



Cube

Clearly: $\mathrm{rc}(\{0,1\}^d) \leq 2d$

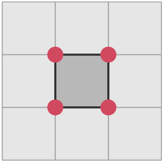




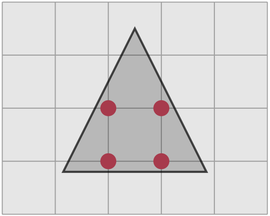
Theorem

$$\mathrm{rc}(\{0,1\}^d) = d + 1$$

Cube



Clearly: $\mathrm{rc}(\{0,1\}^d) \leq 2d$



Theorem

$$\mathrm{rc}(\{0,1\}^d) = d + 1$$

$$\{0,1\}^d = \Big\{ x \in \mathbb{Z}^d : 0 \leq x_1 + \sum_{i=2}^{d} \frac{1}{2^i} x_i$$

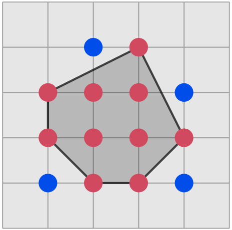
$$x_k \leq 1 + \sum_{i=k+1}^{d} \frac{1}{2^i} x_i \text{ for } k = 1, \ldots, d \Big\}$$

Lower bounds

Definition

Let $X \subseteq \mathbb{Z}^d$. A set $H \subseteq \mathbb{Z}^d \setminus X$ is called a hiding set for X if:

- $H \subseteq \text{aff}(X)$
- For any two distinct points $a, b \in H$: $\text{conv}(\{a, b\}) \cap \text{conv}(X) \neq \emptyset$

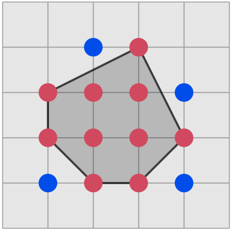


Lower bounds

Definition

Let $X \subseteq \mathbb{Z}^d$. A set $H \subseteq \mathbb{Z}^d \setminus X$ is called a hiding set for X if:

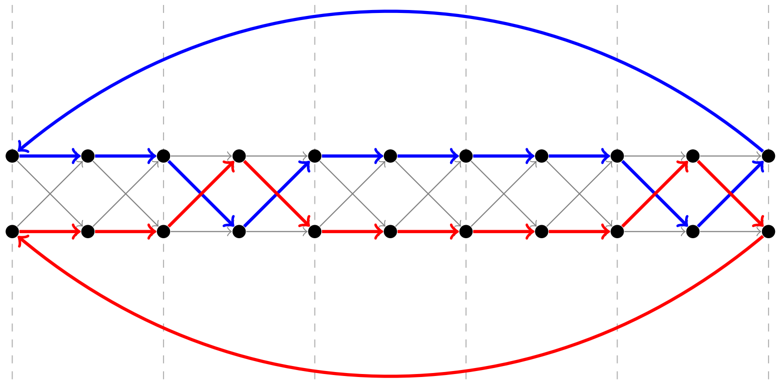
- $H \subseteq \operatorname{aff}(X)$
- For any two distinct points $a, b \in H$: $\operatorname{conv}(\{a, b\}) \cap \operatorname{conv}(X) \neq \emptyset$



Proposition

$$H \textit{ hiding set for } X \quad \Rightarrow \quad \operatorname{rc}(X) \geq |H|$$

Hiding set for TSP



Proof strategy

Proof strategy

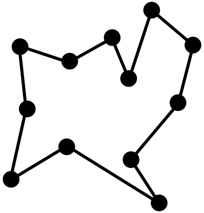
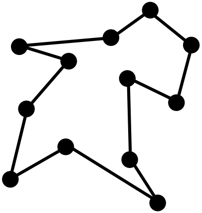
$\frac{1}{2}\cdot$  $+\ \frac{1}{2}\cdot$ 

$=$

Proof strategy

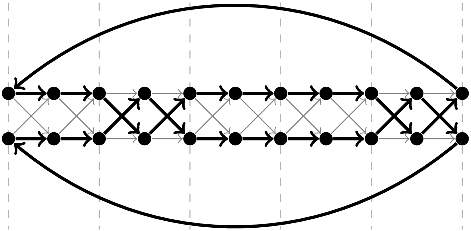
$\frac{1}{2}\cdot$  $+\ \frac{1}{2}\cdot$ 

$=$

$\frac{1}{2}\cdot$  $+\ \frac{1}{2}\cdot$ 

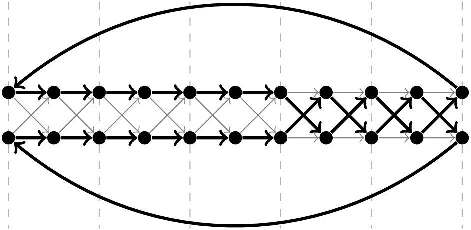
Hiding set for TSP (2)

$\frac{1}{2}\cdot$



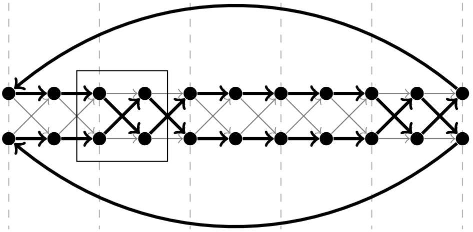
$+$

$\frac{1}{2}\cdot$

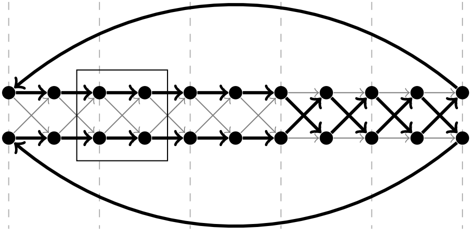


$=$

Hiding set for TSP (2)

$\frac{1}{2}\cdot$ 

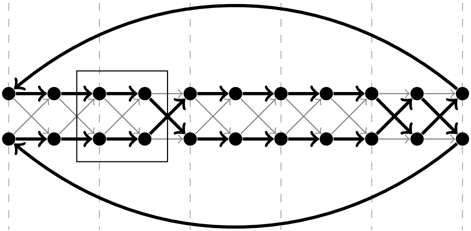
$+$

$\frac{1}{2}\cdot$ 

$=$

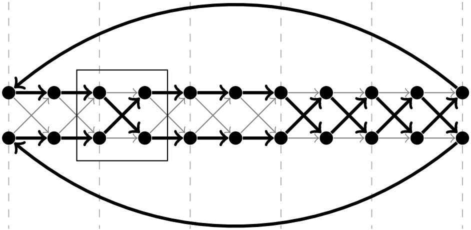
Hiding set for TSP (2)

$\frac{1}{2} \cdot$



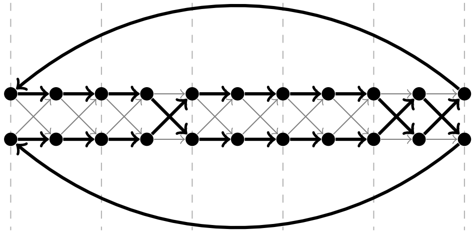
$+$

$\frac{1}{2} \cdot$



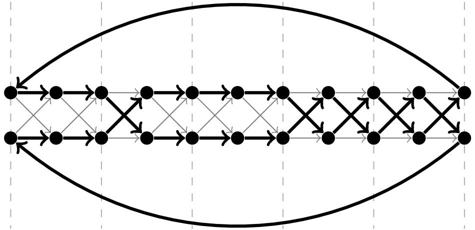
Hiding set for TSP (2)

$\frac{1}{2}$ ·



$+$

$\frac{1}{2}$ ·



$\in \text{conv}(\mathrm{TSP}_n)$

Results for TSP

Theorem

The asymptotical growth of $\mathrm{rc}(\mathrm{TSP}_n)$ *is* $2^{\Theta(n)}$.

Results for TSP

Theorem

The asymptotical growth of $\mathrm{rc}(\mathrm{TSP}_n)$ *is* $2^{\Theta(n)}$.

$\rightsquigarrow$ The subtour elimination relaxation is asymptotically smallest possible.

Results for TSP

Theorem

The asymptotical growth of $\mathrm{rc}(\mathrm{TSP}_n)$ *is* $2^{\Theta(n)}$.

$\rightsquigarrow$ The subtour elimination relaxation is asymptotically smallest possible.

Theorem

The asymptotical growth of $\mathrm{rc}(\star)$ *is* $2^{\Theta(n)}$, *where* $\star$ *is the set of characteristic vectors of ...*

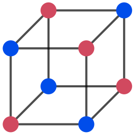
- *spanning trees*
- *arborescences*
- *forests*
- *branchings*
- *connected edge sets*

for the complete (undirected/directed) graph on n *nodes.*

Further Results

$$\mathrm{EVEN}_n := \left\{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \text{ even} \right\}$$

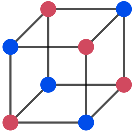
$$\mathrm{ODD}_n := \left\{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \text{ odd} \right\}$$



Further Results

$$\mathrm{EVEN}_n := \left\{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \text{ even} \right\}$$

$$\mathrm{ODD}_n := \left\{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \text{ odd} \right\}$$



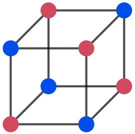
Theorem (Jeroslow 1973)

ODD_n *is a hiding set for* EVEN_n.

Further Results

$$\mathrm{EVEN}_n := \left\{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \text{ even} \right\}$$

$$\mathrm{ODD}_n := \left\{ x \in \{0,1\}^n : \sum_{i=1}^{n} x_i \text{ odd} \right\}$$



Theorem (Jeroslow 1973)

ODD_n *is a hiding set for* EVEN_n.

Corollary

$$\mathrm{rc}(\mathrm{EVEN}_n) \geq 2^{n-1}$$

Further Results (2)

For $T \subseteq V$ with $n, |T|$ even:

$T\text{-JOINS}_n :=$ set of characteristic vectors of T-joins in (V, E)

Further Results (2)

For $T \subseteq V$ with $n, |T|$ even:
$T\text{-JOINS}_n :=$ set of characteristic vectors of T-joins in (V, E)

Theorem

$$\mathrm{rc}(T\text{-JOINS}_n) \geq 2^{\frac{n}{4}-1}$$

Further Results (2)

For $T \subseteq V$ with $n, |T|$ even:
$T\text{-JOINS}_n :=$ set of characteristic vectors of T-joins in (V, E)

Theorem

$$\mathrm{rc}(T\text{-JOINS}_n) \geq 2^{\frac{n}{4}-1}$$

$$\mathrm{DIFF}_{m,n} := \left\{x \in \{0,1\}^{m \times n} : x \text{ has pairwise distinct rows}\right\}$$

Further Results (2)

For $T \subseteq V$ with $n, |T|$ even:
$T\text{-JOINS}_n :=$ set of characteristic vectors of T-joins in (V, E)

Theorem

$$\mathrm{rc}(T\text{-JOINS}_n) \geq 2^{\frac{n}{4}-1}$$

$$\mathrm{DIFF}_{m,n} := \left\{x \in \{0,1\}^{m \times n} : x \text{ has pairwise distinct rows}\right\}$$

Theorem

$$\mathrm{rc}(\mathrm{DIFF}_{2,n}) \geq 2^n$$

Hard to model without additional variables:

- Acyclicity

Hard to model without additional variables:

- Acyclicity
- Connectivity

Conclusion

Hard to model without additional variables:

- Acyclicity
- Connectivity
- Parity

Conclusion

Hard to model without additional variables:

- Acyclicity
- Connectivity
- Parity
- Distinctness

Conclusion

Hard to model without additional variables:

- Acyclicity
- Connectivity
- Parity
- Distinctness

Everything becomes easy with additional variables!

Conclusion

Hard to model without additional variables:

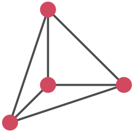
- Acyclicity
- Connectivity
- Parity
- Distinctness

Everything becomes easy with additional variables!

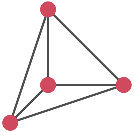
$\rightsquigarrow$ Projection is a powerful tool!

My Favorite Open Question

What is the relaxation complexity of the standard simplex' vertices?



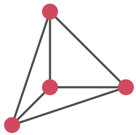
What is the relaxation complexity of the standard simplex' vertices?



- Let $\Delta_d := \{\mathbb{0}, \mathbb{e}_1, \ldots, \mathbb{e}_d\}$

My Favorite Open Question

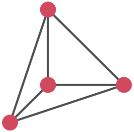
What is the relaxation complexity of the standard simplex' vertices?



- Let $\Delta_d := \{\mathbb{O}, \mathbb{e}_1, \dots, \mathbb{e}_d\}$
- Clearly: $\mathrm{rc}(\Delta_d) \leq d + 1$

My Favorite Open Question

What is the relaxation complexity of the standard simplex' vertices?

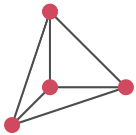


- Let $\Delta_d := \{\mathbb{0}, \mathbb{e}_1, \dots, \mathbb{e}_d\}$
- Clearly: $\mathrm{rc}(\Delta_d) \leq d + 1$

Is there a polyhedron P with less than $d + 1$ facets such that $P \cap \mathbb{Z}^d = \Delta_d$?

My Favorite Open Question

What is the relaxation complexity of the standard simplex' vertices?



- Let $\Delta_d := \{\mathbb{0}, \mathbb{e}_1, \ldots, \mathbb{e}_d\}$
- Clearly: $\mathrm{rc}(\Delta_d) \leq d + 1$

Is there a polyhedron P with less than $d + 1$ facets such that $P \cap \mathbb{Z}^d = \Delta_d$?

(If yes, P must be unbounded and hence irrational!)