

Lower Bounds on the Sizes of Integer Programs Without Additional Variables

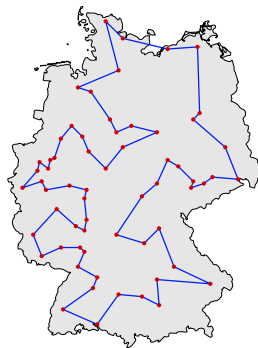
Volker Kaibel & Stefan Weltge



Aussois, 2014

Traveling Salesman Problem

Find a shortest cycle in a complete graph (V, E) that hits each node exactly once.

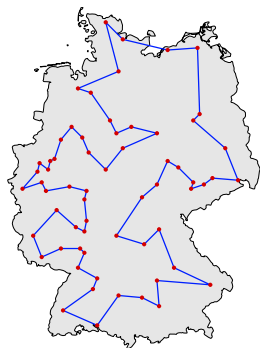


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- ▶ (binary) variable x_{ij} for each edge $\{i, j\}$

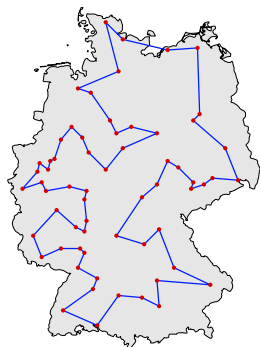


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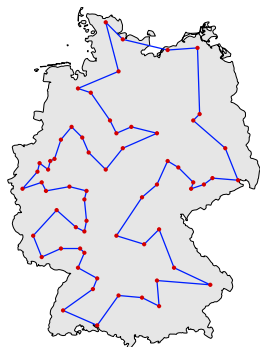


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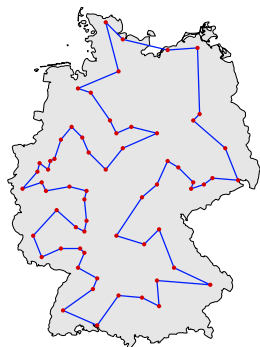
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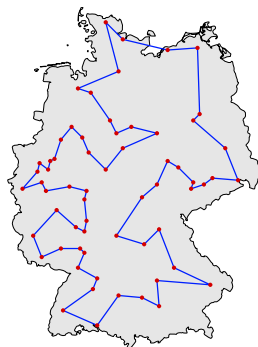
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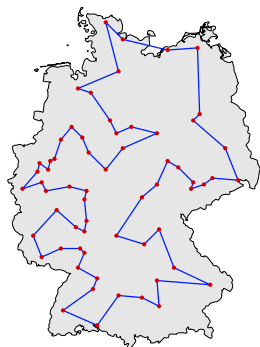
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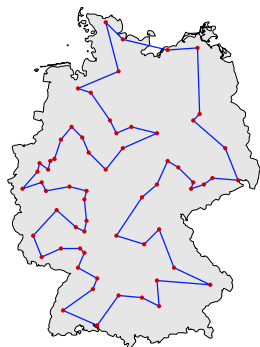
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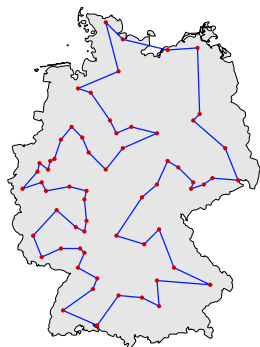
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Why not use additional variables?



Motivation (2)

Motivation

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Basics

○○

Lower bounds

○○○○○○○○

Conclusion

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$$\min \{ \langle c, x \rangle : x \in \text{TSP}_n \} = \min \{ \langle c, x \rangle : Ax + By \leq b \}$$

Theorem (Fiorini, Pokutta et al. 2012)

There is no polynomial size system $Ax + By \leq b$ such that

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But there are several polynomial size **integer programming** formulations of the form

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for solving the TSP (Miller-Tucker-Zemlin, flow based, ...)

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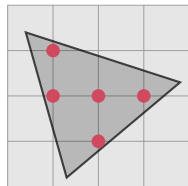
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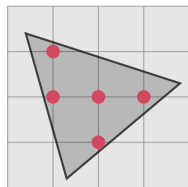
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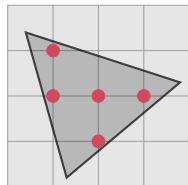
Is $rc(\text{TSP}_n)$ polynomial in n ?

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$$R_n^{\text{sub}} := \{x \in [0, 1]^E : x(\delta(v)) = 2 \text{ for all } v \in V \\ x(\delta(S)) \geq 2 \text{ for all } \emptyset \neq S \subset V\}$$

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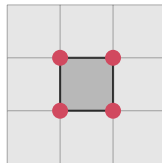
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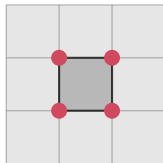
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Clearly: $rc(\{0, 1\}^d) \leq 2d$

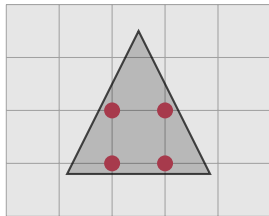


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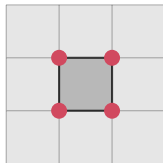


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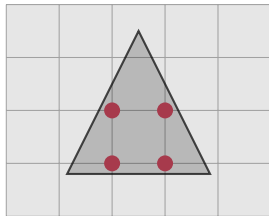


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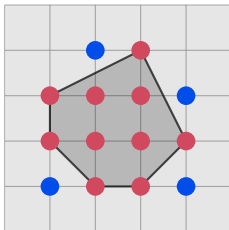
$$\{0, 1\}^d = \left\{ x \in \mathbb{Z}^d : 0 \leq x_1 + \sum_{i=2}^d \frac{1}{2^i} x_i \right.$$

$$\left. x_k \leq 1 + \sum_{i=k+1}^d \frac{1}{2^i} x_i \text{ for } k = 1, \dots, d \right\}$$

Definition

Let $X \subseteq \mathbb{Z}^d$. A set $H \subseteq \mathbb{Z}^d \setminus X$ is called a **hiding set** for X if:

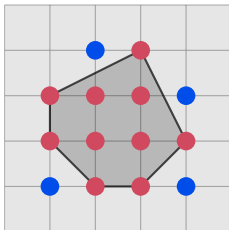
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Proposition

$$H \text{ hiding set for } X \Rightarrow \text{rc}(X) \geq |H|$$

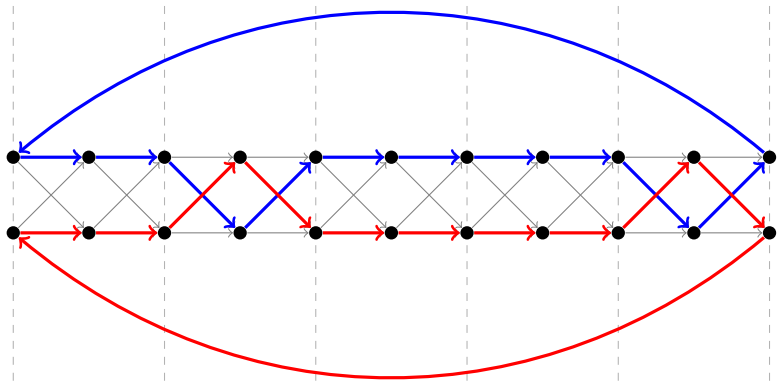
Hiding set for TSP

Motivation
○○

Basics
○○

Lower bounds
●○○○○○

Conclusion
○○



Proof strategy

Motivation
○○

Basics
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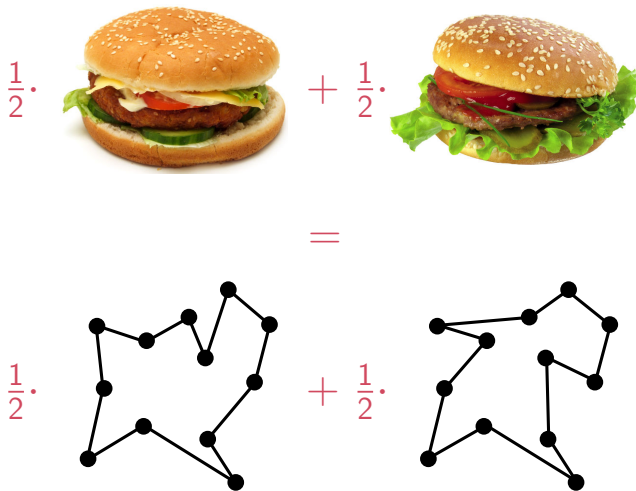
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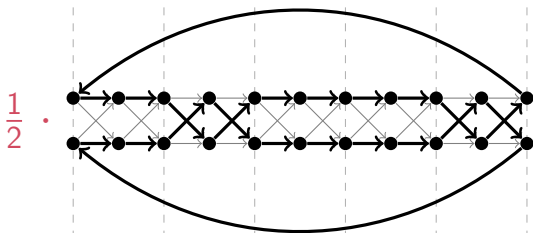




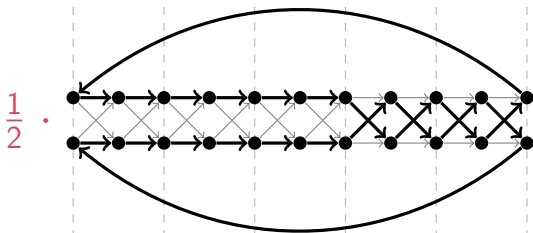
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Hiding set for TSP (2)

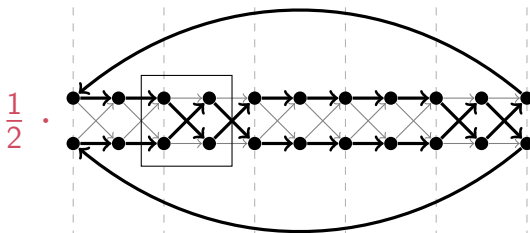


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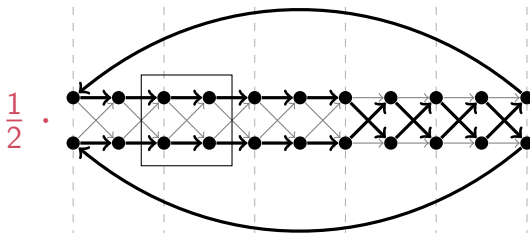


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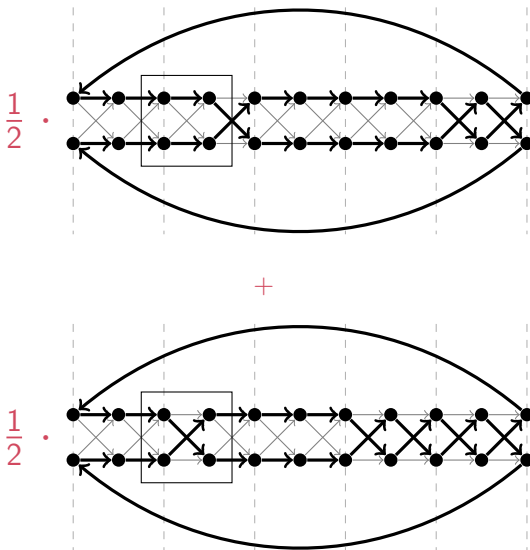


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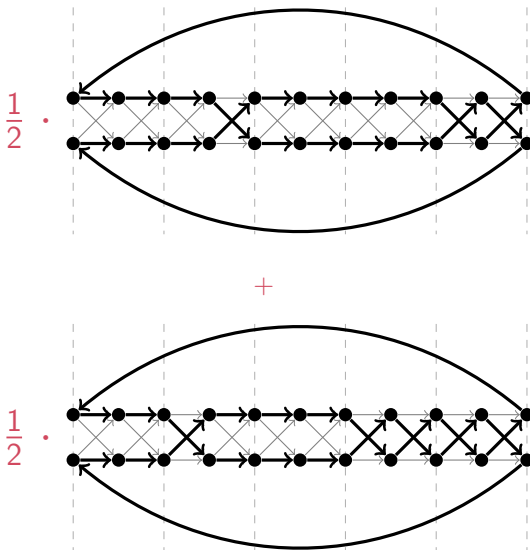


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$\in \text{conv}(\text{TSP}_n)$

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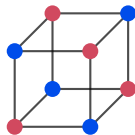
Theorem

The asymptotical growth of $rc(\star)$ is $2^{\Theta(n)}$, where \star is the set of characteristic vectors of ...

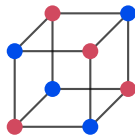
- ▶ *spanning trees*
- ▶ *arborescences*
- ▶ *forests*
- ▶ *branchings*
- ▶ *connected edge sets*

for the complete (undirected/directed) graph on n nodes.

$$\text{EVEN}_n := \left\{ x \in \{0, 1\}^n : \sum_{i=1}^n x_i \text{ even} \right\}$$
$$\text{ODD}_n := \left\{ x \in \{0, 1\}^n : \sum_{i=1}^n x_i \text{ odd} \right\}$$



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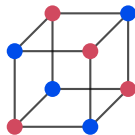


Theorem (Jeroslow 1973)

ODD_n is a hiding set for EVEN_n .

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Theorem (Jeroslow 1973)

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Corollary

$$\text{rc}(\text{EVEN}_n) \geq 2^{n-1}$$

Further Results (2)

For $T \subseteq V$ with $n, |T|$ even:

T -JOINS $_n :=$ set of characteristic vectors of T -joins in (V, E)

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$$\text{rc}(\text{DIFF}_{2,n}) \geq 2^n$$

Hard to model without additional variables:

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Everything becomes easy with additional variables!

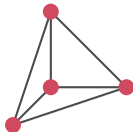
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- Distinctness

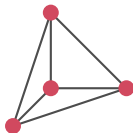
Everything becomes easy with additional variables!

⇒ Projection is a powerful tool!

What is the relaxation complexity of the standard simplex' vertices?

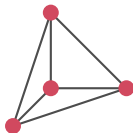


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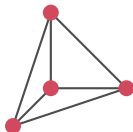
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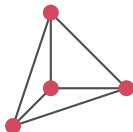
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(If yes, P must be unbounded and hence irrational!)