The Split Delivery Vehicle Routing Problem

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Split Delivery VRP

- Capacitated Vehicle Routing Problem (CVRP)
  - given a set of customers and a depot
  - capacitated vehicles
  - tours for vehicles that start and end at the depot
  - visit each customer exactly once
  - respect the vehicle capacities
  - minimize the total transportation cost

- many variants: time windows, heterogeneous fleet, periodic VRP, pickup and delivery ...

- Split Delivery Vehicle Routing Problem (SDVRP)
  - relax "visit each customer exactly once"

History - exact approaches

- Dror and Trudeau (1989): split deliveries can lead to considerable cost savings
- Dror and Trudeau (1990): SDVRP is NP-hard, k-split cycles
- Dror et al. (1994): valid inequalities for a vehicle index formulation
- Belenguer et al. (2000): polyhedral study, cutting plane algorithm
- Lee et al. (2006): dynamic programming
- Jin et al. (2007): iterative two stage method, cluster and route with lower bounds on route lengths in clustering
- Jin et al. (2008): column generation
- Moreno et al. (2010): extended formulation with load and quantity delivered, generate columns and cuts
- Archetti et al. (2011): branch-and-price-and-cut
- Archetti et al. (2013): branch-and-cut

Notation

- $N = \{0, 1, \dots, n\}$, 0 depot, $1, \dots, n$ customers
- $G = (N, A)$ directed graph, complete
- c_a: cost of travelling on arc a
- m identical vehicles, each with capacity Q
- d_i: demand of customer i
- $\delta^-(S)$, $\delta^+(S)$: set of incoming and outgoing arcs of $S \subset N$, resp.
- Assumption: costs are symmetric and satisfy triangle inequality

Model

- SDVRP can be modeled using variables with vehicle index

$$
\begin{aligned}
\min \sum_{a \in A} \sum_{k=1}^{m} c_a x_a^k & \\
\text{s.t. } x^k(\delta^+(0)) = 1 & \quad k = 1, \dots, m, \\
x^k(\delta^-(i)) - x^k(\delta^+(i)) = 0 & \quad i \in N \setminus \{0\}, k = 1, \dots, m, \\
\sum_{k=1}^{m} w_{ik} = 1 & \quad i \in N \setminus \{0\}, \\
\sum_{i=1}^{n} d_i w_{ik} \leq Q & \quad k = 1, \dots, m, \\
w_{ik} \leq x^k(\delta^-(i)) & \quad i \in N \setminus \{0\}, k = 1, \dots, m, \\
\textit{subtour elimination constraints}, & \\
x_a^k \in \{0, 1\} & \quad a \in A, k = 1, \dots, m, \\
w_{ik} \geq 0 & \quad i \in N \setminus \{0\}, k = 1, \dots, m.
\end{aligned}
$$

- If $w_{ik} \in \{0, 1\}$ then CVRP.

Without vehicle index

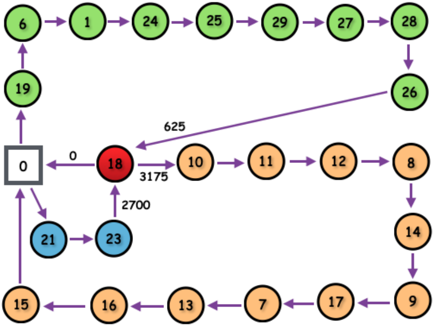
- aim : to come with a model without vehicle index
- **R-SDVRP**

$$
\begin{aligned}
\min \sum_{a \in A} & c_a x_a \\
\text{s.t. } & f(\delta^-(i)) - f(\delta^+(i)) = d_i && i \in N \setminus \{0\}, \\
& x(\delta^+(0)) = m, \\
& x(\delta^-(i)) - x(\delta^+(i)) = 0 && i \in N \setminus \{0\}, \\
& 0 \leq f_a \leq Q x_a && a \in A, \\
& x_a \in \mathbb{Z}_+ && a \in A.
\end{aligned}
$$

- remove second and third constraints: network loading

An example: *eil30*

- 30 nodes and 3 vehicles



- at node 18, incoming 2700 + 625, outgoing 3175 + 0
- virtual depot: customer node where unloading and loading take place

Belenguer, Martinez and Mota (2000)

- undirected graph $G = (N, E)$

$$
\begin{aligned}
\min \sum_{e \in E} c_e x_e & \\
\text{s.t. } x(\delta(0)) \geq 2m \text{ and even}, & \\
x(\delta(i)) \geq 2 \text{ and even} & \quad i \in N \setminus \{0\}, \\
x(\delta(S)) \geq 2 \left\lceil \frac{d(S)}{Q} \right\rceil & \quad S \subset N \setminus \{0\} : 2 \leq |S| \leq n-1, \\
x_e \in \mathbb{Z}_+ & \quad e \in E.
\end{aligned}
$$

- relaxation
- eliminate solutions that are not feasible with cuts
- they obtain the same solution as ours for *eil*30

Comparing the two relaxations

- Projecting out the flow variables in the LP relaxation gives the fractional capacity inequalities

$$x(\delta^-(S)) \geq \frac{d(S)}{Q} \quad S \subseteq N \setminus \{0\}$$

- In the integer problem, the rounded capacity inequalities

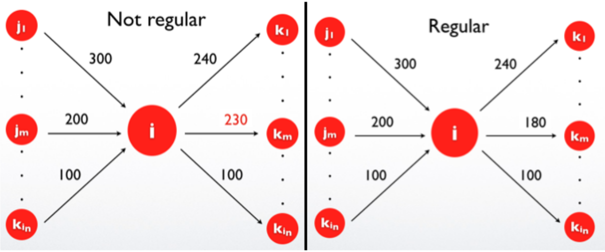
$$x(\delta^-(S)) \geq \left\lceil \frac{d(S)}{Q} \right\rceil \quad S \subseteq N \setminus \{0\}$$

  are satisfied.

- NP-hard to check whether there exists a feasible routing when x is fixed ($x_{0i} = 2$ for all $i \in N \setminus \{0\}$, binpacking).

Checking feasibility

- A solution (f, x) is called regular if at each customer node, one can match each incoming arc with an outgoing arc so that the flow of the incoming arc is greater than or equal to the one on its outgoing arc.



- A solution (f, x) of R-SDVRP is regular iff it is feasible for the SDVRP.

- Regularity can be checked in polynomial time.

Archetti, Bianchessi, Speranza (2013)

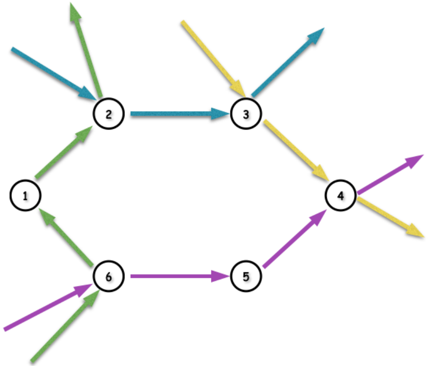
- Similar formulations to the one of Belenguer et al. (2000) and to R-SDVRP.
- Without flows: build the routes for given x and choose the best by solving a model.
- With flows: same idea as regularity check.
- Branch and cut: eliminate vectors x for which there are no feasible routes by branching
- The formulation without flows is better.

Aim

- To solve the R-SDVRP more quickly
- Eliminate virtual depots

k-splits by Dror and Trudeau

- There exists an optimal solution to SDVRP with no k-splits.



$\Rightarrow$ There exists an optimal solution to SDVRP where $x_a \in \{0, 1\}$ for all $a \in A \setminus (\delta^-(0) \cup \delta^+(0))$.

Improvements for R-SDVRP

- There exists an optimal solution to SDVRP where $x_a \in \{0, 1\}$ for all outgoing arcs of the depot.
- For $i \in V \setminus \{0\}$, $x(\delta^-(i)) \geq 1$ is often used to strengthen.
- We use the cutset inequalities for singletons at the root node.

Cutset inequalities

- Let $S \subseteq N \setminus \{0\}$.

$$
\begin{aligned}
& f(\delta^-(S)) - f(\delta^+(S)) = d(S), \\
& 0 \le f_a \le Q x_a, \;\; x_a \in Z_+ \qquad\qquad a \in \delta^-(S) \cup \delta^+(S).
\end{aligned}
$$

- Atamturk (2002): convex hull = trivial + cutset.

$$
A^- \subseteq \delta^-(S), \;\; A^+ \subseteq \delta^+(S)
$$

$$
\eta = \left\lceil \frac{d(S)}{Q} \right\rceil, \;\; r = d(S) - \left\lfloor \frac{d(S)}{Q} \right\rfloor Q
$$

The cutset inequality

$$
f(\delta^-(S) \setminus A^-) + r x(A^-) + (Q - r) x(A^+) - f(A^+) \ge r\eta
$$

is valid.

- $A^- = \delta^-(S)$ and $A^+ = \emptyset$, rounded capacity inequality.

Eliminating virtual depots: Patching

- solve R-SDVRP
- add vehicle indexed variables at a node where regularity is not satisfied, iterate

$$
\begin{aligned}
& f^k(\delta^-(i)) - f^k(\delta^+(i)) \geq 0 && k = 1, \ldots, m, \\
& x^k(\delta^-(i)) - x^k(\delta^+(i)) = 0 && k = 1, \ldots, m, \\
& x^k(\delta^-(i)) \leq 1 && k = 1, \ldots, m, \\
& 0 \leq f_a^k \leq Q x_a^k && a \in \delta^-(i) \cup \delta^+(i), k = 1, \ldots, m, \\
& x_a^k \in \{0, 1\} && a \in \delta^-(i) \cup \delta^+(i), k = 1, \ldots, m, \\
& x_a = \sum_{k=1}^{m} x_a^k, \; f_a = \sum_{k=1}^{m} f_a^k && a \in \delta^-(i) \cup \delta^+(i).
\end{aligned}
$$

Eliminating virtual depots: Node splitting

- Idea: split nodes as necessary to force regularity at every customer node
  - solve R-SDVRP
  - create a duplicate of the node violating regularity, enlarge the node & arc sets accordingly

- N_i: the set of nodes containing the original customer i and its duplicates.

- Order N_i so that a node $j \in N_i$ is represented with (i, l) where l is the order of j in N_i.

- Let $N' = \cup_{i \in N \setminus \{0\}} N_i$. Define A' to be the set of all arcs except arcs between duplicates of the same node.

- $v_{i,l}$ is 1 if node (i, l) is visited; 0 otherwise.

Eliminating virtual depots: Node splitting

$$
\begin{aligned}
\min \sum_{a \in A'} c_a x_a & \\
\text{s.t.} \sum_{j \in N_i} \left(f(\delta^-(j)) - f(\delta^+(j)) \right) = d_i & \qquad i \in N \setminus \{0\}, \\
f(\delta^-(j)) - f(\delta^+(j)) \geq 0 & \qquad j \in N', \\
x(\delta^+(0)) = m, & \\
x(\delta^-(j)) - x(\delta^+(j)) = 0 & \qquad j \in N', \\
x(\delta^-(i,l)) = v_{i,l} & \qquad (i,l) \in N' : |N_i| \geq 2, l \neq |N_i|, \\
x(\delta^-(i,|N_i|)) \leq (m - |N_i| + 1) v_{i|N_i|} & \qquad i \in N \setminus \{0\} : |N_i| \geq 2, \\
v_{i,l} \geq v_{i,l+1} & \qquad (i,l) \in N' : |N_i| \geq 2, l \neq |N_i|, \\
0 \leq f_a \leq Q x_a & \qquad a \in A', \\
v_{i,l} \in \{0,1\} & \qquad (i,l) \in N' : |N_i| \geq 2, \\
x_a \in \{0,1\} & \qquad a \in A' \setminus \delta^-(0), \\
x_a \in \mathbb{Z}_+ & \qquad a \in \delta^-(0).
\end{aligned}
$$

- If $|N_i| = m$, same as patching but we break symmetry with $v_{i,l} \geq v_{i,l+1}$.

Eliminating virtual depots: Cutting

- $H \subseteq V \setminus \{0\}$ and $S_1, \ldots, S_t$ disjoint nonempty subsets of H with $d(S_u) \leq Q$ for $u = 1, \ldots, t$.
- $b(S_1, \ldots, S_t)$: optimal value of binpacking problem with items $1, \ldots, t$ of size $d(S_1), \ldots, d(S_t)$.
- The framed capacity inequality (Pochet (1998), Augerat (1995), Naddef and Rinaldi (2002))

$$x(\delta^-(H)) + \sum_{u=1}^{t} x(\delta^-(S_u)) \geq t + b(S_1, \ldots, S_t) \tag{1}$$

  is valid.
- The inequality used by Belenguer et al. is a special case where $H = V \setminus \{0\}$ (known as the generalized capacity inequality).

Results: Patching

instance			with $x(\delta^-(i)) \geq 1$			with cutset inequalities		
name	nodes	vehicles	gap	iters	cpu	gap	iters	cpu
eil22	22	4	0	1	4.09	0	1	2.60
eil23	23	3	0	1	0.56	0	1	0.57
eil30	30	3	0	3	1804.22	0	4	1818.54
eil33	33	4	0.22	3	13540.53	0	1	**1037.77**
eil51	51	5	0	1	2951.48	0	1	**1383.03**
S51D1	51	3	0	1	18.71	0	1	23.51
S51D2	51	9	2.86	N/A	5210.24	3.08	N/A	5226.98
p01-110	51	3	0	1	19.43	0	1	15.15
p01-1030	51	11	2.41	N/A	6100.37	2.41	N/A	5946.72
p01-1050	51	16	1.37	N/A	4817.89	1.38	N/A	4621.67

Results: Node splitting

instance			with $x(\delta^{-}(i)) \geq 1$			with cutset inequalities		
name	nodes	vehicles	gap	iters	cpu	gap	iters	cpu
eil22	22	4	0	1	4.06	0	1	2.58
eil23	23	3	0	1	0.64	0	1	0.58
eil30	30	3	0	4	**248.4**	0	5	2403.13
eil33	33	4	0	3	7129.49	0	1	**1033.82**
eil51	51	5	0	1	2953.74	0	1	**1382.24**
S51D1	51	3	0	1	18.77	0	1	23.58
S51D2	51	9	2.86	N/A	5296.85	2.82	N/A	5098.22
p01-110	51	3	0	1	19.70	0	1	15.04
p01-1030	51	11	2.41	N/A	6111.64	2.41	N/A	6050.16
p01-1050	51	16	1.37	N/A	4983.22	1.38	N/A	4767.24

Comparison

			patching with		node splitting with	
instance	nodes	vehicles	$x(\delta^-(i)) \geq 1$	cutsets	$x(\delta^-(i)) \geq 1$	cutsets
eil30	30	3	1804.22 (3)	1818.54 (4)	**248.4 (4)**	2403.13 (5)
eil33	33	4	13540.53 (3)	**1037.77 (1)**	7129.49 (3)	**1033.82 (1)**

*eil*30 with cutting planes

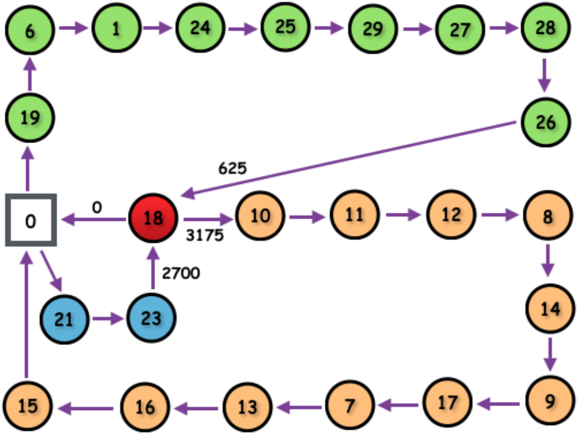


Figure : Initial solution with objective 508

*eil*30 with cutting planes

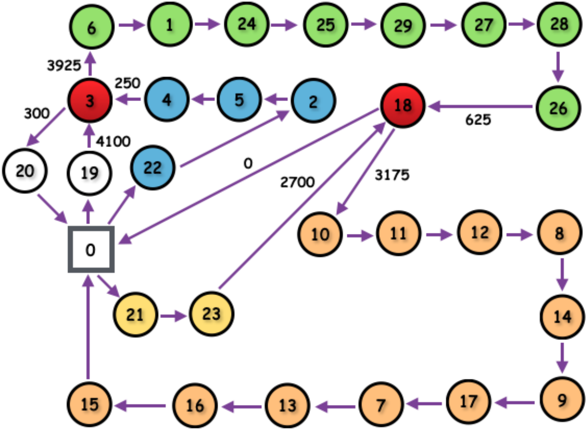


Figure : Seventh solution with objective 510

*eil*30 with cutting planes

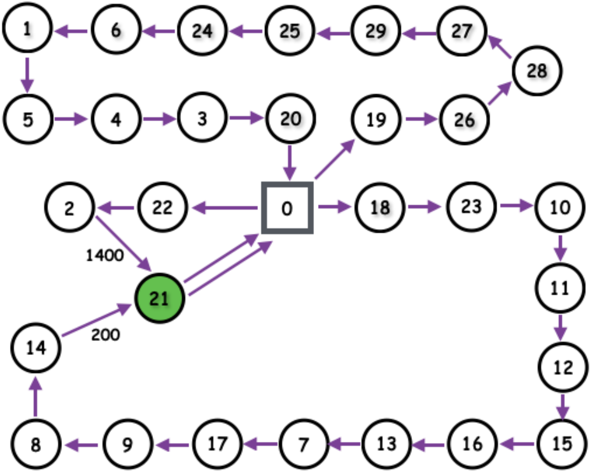


Figure : Ninth solution with objective 510: optimal!

To be done

- Strengthen R-SDVRP
- Properties of optimal solutions
- Separation of framed capacity constraints
- Complete model ?
- Combine different approaches
- Variants: at most 2 splits, splitting and/or stocking costs ...