Integer Quadratic Programming is in NP

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Slides by Santanu
1 Introduction and Main Result
Integer Quadratic Program: Definition

Definition (IQP)

\[
\min \quad x^T Q x + c^T x \\
\text{s.t.} \quad Ax \leq b \\
x \in \mathbb{Z}^n,
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We do not assume that \( x^\top Qx \) is convex
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\end{align*}
\]

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**Decision Version of IQP**

Does there exist \(x\) satisfying:

\[
\begin{align*}
x^\top Q x + c^\top x + d & \leq 0 \\
Ax & \leq b \\
x & \in \mathbb{Z}^n,
\end{align*}
\]

\[\mathcal{F}(Q, c, d, A, b)\]

where we assume all the data is rational.
Main Result

**Theorem**

Let $n, m \in \mathbb{Z}_{++}$. Let $Q \in \mathbb{Q}^{n \times n}$, $c \in \mathbb{Q}^n$, $d \in \mathbb{Q}$, $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$. If $F(Q, c, d, A, b)$ is non-empty, then there exists $x \in F(Q, c, d, A, b)$ such that the binary encoding size of $x$ is bounded from above by a polynomial function of the size of binary encoding of $Q$, $c$, $d$, $A$, $b$. 

**Consequences**

1. Integer Quadratic Programming is in NP. In particular, the decision version of IQP is NP-complete.
2. Broadly speaking, this implies that there exists an algorithm to solve IQP, i.e. not undecidable.
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2. Broadly speaking, this implies that there exists an algorithm to solve IQP, i.e. not undecidable.
Comparison 1: More quadratic inequalities?

1. Number of quadratic inequalities: $2(58^2) + 58 + 1 = 3424$.

2. Number of linear inequalities: $58$.

3. Number of integer variables: $(58^2 + 2 \times 58) = 1769$.

is undecidable.

Jones 82, discussion and additional references in Köppe 12.
Comparison 1: More quadratic inequalities?

Undecidable!

Determining the feasibility of a system with

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Comparison 2: Two quadratic inequalities?

Consider the system for \( d = 2n^2 + 1 \):

\[
\begin{align*}
x^2 - dy^2 + 1 & \leq 0, \\
x^2 - dy^2 - 1 & \leq 0,
\end{align*}
\]

\( x, y \in \mathbb{Z} \).

1. The binary encoding length of the smallest integer solution with minimal binary encoding length has an encoding length of: \( \Omega(5^n) \).

2. The binary encoding length of instance: \( \Theta(n) \).

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Comparison 2: Two quadratic inequalities?

Exponential size solution!
Consider the system for \( d = 5^{2n+1} \):

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\begin{align*}
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Comparison 3: More convex quadratic inequalities?

Consider the system:

\[ x_1 \geq 2x_j \geq x_{2j} - 1 \quad \forall j \in \{2, \ldots, n\} \]

\[ x_j \in \mathbb{Z} \quad \forall j \in \{1, \ldots, n\}. \]

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Comparison 3: More convex quadratic inequalities?

Exponential size solution!

Consider the system:

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\begin{align*}
  x_1 & \geq 2 \\
  x_j & \geq x_{j-1}^2 \quad \forall j \in \{2, \ldots, n\} \\
  x_j & \in \mathbb{Z} \quad \forall j \in \{1, \ldots, n\}.
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In Conclusion...

Exactly one rational quadratic inequality is the threshold where we can guarantee existence of poly-size feasible solutions.
2
Proof Outline
Getting Started

- small size = poly-size = numerators/denominators have size polynomial wrt to input size
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\[ x^TQx + c^Tx + d \leq 0 \]
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- Show that there is feasible solution of small size
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  - Only the recession cone matters for bounding the size of solutions
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- Focus \( x^T Qx + c^T x + d \leq 0 \)
- **Strategy:** Focus on higher-order term \( x^T Qx \).
Getting Started

- "Slice" the cone $\mathcal{P}$ with a "carefully selected" hyperplane $\mathcal{H}$
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Let $x^*$ be a poly-size rational optimal solution to the problem

$$x^* \top Q x^* := \min_x x \top Q x$$

s.t. $x \in \mathcal{P} \cap \mathcal{H}$
Getting Started

- "Slice" the cone \( \mathcal{P} \) with a "carefully selected" hyperplane \( \mathcal{H} \)
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- The quadratic problem \( \min \{ x \top V x \mid x \in \text{rational polytope} \} \) (where \( V \) is a rational matrix) has a rational globally optimal solution of poly-size with respect to the size of the instance. [Vavasis 1990]
Case 1: $x^* \top Q x^* < 0$
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- There is $\lambda$ such that $\lambda x^* \in P \cap \mathbb{Z}^n$ and $(\lambda x^*)^\top Q(\lambda x^*) + c^\top (\lambda x^*) + d \leq 0$
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- Poly-size $\lambda$ suffices
Case 2: $x^* \top Q x^* > 0$
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Bound size of all potential solutions
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- Again poly-size, bounded away from 0
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Higher-order term $x \top Q x$ is strictly positive on $\mathcal{P} \cap \mathcal{H}$

Bound size of all potential solutions

- Again poly-size, bounded away from 0
- $\Rightarrow$ any solution more than poly-size far away from $\mathcal{P} \cap \mathcal{H}$ has $x \top Q x + c \top x + d > 0$, infeasible
Case 3: $x^* \top Q x^* = 0$

$x^*$ such that $x^* \top Q x^* = 0$
Case 3: $x^* \top Qx^* = 0$

Not easy to find feasible solution of small size
Not easy to bound size of feasible solution
Lemma

There exists a family of cones $C^i$, $i \in I$ such that

(a) $\bigcup_{i \in I} C^i = \mathcal{P}$,

(b) each $C^i$ has poly-size description

(c) for every $C^i$, if a face $F$ of $C^i$ has point $x$ with $x^\top Qx = 0$, then there exists an extreme ray $v$ of $F$ with $v^\top Qv = 0$, 

Decomposing cone $\mathcal{P}$

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Working with one of these cones $\mathcal{C}$

1. $\mathcal{C}$ has integral extreme rays $r^1, \ldots, r^k$. 

2. $x \in \mathcal{C} \cap \mathbb{Z}^n$ can be written $x = x_0 + \sum_{j=1}^{k} r_j y_j$, $y_j \in \mathbb{Z}^+$ for $x_0 \in X_0$.

Converse: this gives only points in $\mathcal{C} \cap \mathbb{Z}^n$.

3. For each ray $r_j$ with $r_j^\top Q r_j = 0$ function decomposes nicely:

$$x^\top Q x + c^\top x + d = y_j \cdot \text{affine}(x_0, y_j) + f(x_0, y_j)$$

4. If there is ray $r_j$ with $r_j^\top Q r_j = 0$ and $\text{affine}(x_0, y_j) < 0$ for some $x_0, y_j$ ⇒ find feasible poly-sized solution

5. Else all rays with $r_j^\top Q r_j = 0$ have $\text{affine}(x_0, y_j) \geq 0$ ⇒ ignore them

6. Working on face induced by rays with $r_j^\top Q r_j > 0$: by decomposition have $x^\top Q x > 0$ in the whole face ⇒ bound size of solutions
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$$x = \underbrace{x_0}_{\text{poly-size integer point}} + \sum_{j=1}^{k} r^j y_j, \quad y_j \in \mathbb{Z}_+ \quad \forall j \in \{1, \ldots, k\}. \quad \text{for } x_0 \in \mathcal{X}_0. \text{ Converse: this gives only points in } C \cap \mathbb{Z}^n.$$

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Open Problem

Is Integer Quadratic Programming in P for \textit{fixed dimension}?
Thank You!