Problem Definition

Given $p$ points in $\mathbb{R}^n$.

Euclidean Steiner Tree Problem

Find a tree with minimal Euclidean length that spans these points using or not extra points, which are called Steiner points.
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**Euclidean Steiner Tree Problem**

Find a tree with minimal Euclidean length that spans these points using or not extra points, which are called Steiner points.
Problem Definition

Determine:

- The number of Steiner points to be used on the minimal tree.
- The arcs of the tree.
- Geometrical position of the Steiner points.
Properties of Steiner Minimal Trees (SMTs) on $p$ terminals

- No two edges meet at a point with angle less than $120^\circ$.
- Each terminal point has degree between 1 and 3.
- Each Steiner point has degree equal to 3.
- The number of Steiner points is $\leq p - 2$. 
A full Steiner topology (FST) for $p$ terminals is a topology with $p - 2$ Steiner points, where all the terminals have degree one and Steiner points have degree three.
Degenerate Steiner Topologies

A topology is called a degeneracy of another if the former can be obtained from the latter by shrinking edges.
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**Fact:** Any SMT either has a full Steiner topology or a degenerated full Steiner topology.
Solution algorithms

- $n = 2$

- $n > 2$
Maculan, Michelon, Xavier (2000)

Let $G(V, E)$ be a graph and

- $P := \{1, 2, \ldots, p - 1, p\}$ be the index set corresponding to the given terminals $a^1, \ldots, a^p$;
- $S := \{p + 1, p + 2, \ldots, 2p - 3, 2p - 2\}$ be the index set corresponding to Steiner points $x^{p+1}, \ldots, x^{2p-2}$;
- $V := P \cup S$;
- $E := E_1 \cup E_2$, where $E_1 := \{[i, j]|i \in S, j \in S, i < j\}$ and $E_2 := \{[i, j]|i \in P, j \in S\}$. 

MINLP Formulations for the Euclidean Steiner Problem

Maculan, Michelon, Xavier (2000)

MMX: \[ \text{min} \quad \sum_{[i,j] \in E_1} \|x^i - x^j\| y_{ij} + \sum_{[i,j] \in E_2} \|a^i - x^j\| y_{ij}, \]

s.t.:
\[ \sum_{j \in S} y_{ij} = 1, \quad i \in P, \]
\[ \sum_{i \in P} y_{ij} + \sum_{k < j, k \in S} y_{kj} + \sum_{k > j, k \in S} y_{jk} = 3, \quad j \in S, \]
\[ \sum_{i < j, i \in S} y_{ij} = 1, \quad j \in S - \{p + 1\}, \]
\[ y_{ij} \in \{0, 1\}, \quad [i,j] \in E_1 \cup E_2, \]
\[ x^i \in \mathbb{R}^n, \quad i \in S. \]
MINLP Formulations for the Euclidean Steiner Problem

Maculan, Michelon, Xavier (2000)

MMX: \[
\min \sum_{[i,j] \in E_1} \|x^i - x^j\| y_{ij} + \sum_{[i,j] \in E_2} \|a^i - x^j\| y_{ij},
\]
\[
s.t.: \sum_{j \in S} y_{ij} = 1, \quad i \in P,
\]
\[
\sum_{i \in P} y_{ij} + \sum_{k < j, k \in S} y_{kj} + \sum_{k > j, k \in S} y_{jk} = 3, \quad j \in S,
\]
\[
\sum_{i < j, i \in S} y_{ij} = 1, \quad j \in S - \{p + 1\},
\]
\[
y_{ij} \in \{0, 1\}, \quad [i, j] \in E_1 \cup E_2,
\]
\[
x^i \in \mathbb{R}^n, \quad i \in S.
\]

In “On a nonconvex MINLP formulation of the Euclidean Steiner tree problem in n-space,” with Jon Lee (U. Michigan), and Claudia D’Ambrosio (E. Polytechnique), we propose nonlinear cuts to MMX, and its solution by global solvers.
MINLP Formulations for the Euclidean Steiner Problem

Fampa and Maculan (2004)

FM: \( \text{min} \quad \sum_{[i,j] \in E} d_{ij} \),

s.t.:
\[
\begin{align*}
d_{ij} & \geq \|x^i - x^j\| - M(1 - y_{ij}), & [i,j] \in E_1, \\
d_{ij} & \geq \|a^i - x^j\| - M(1 - y_{ij}), & [i,j] \in E_2, \\
\sum_{j \in S} y_{ij} & = 1, & i \in P, \\
\sum_{i \in P} y_{ij} + \sum_{k < j, k \in S} y_{kj} + \sum_{k > j, k \in S} y_{jk} & = 3, & j \in S, \\
\sum_{i < j, i \in S} y_{ij} & = 1, & j \in S - \{p + 1\}, \\
y_{ij} & \in \{0, 1\}, & [i,j] \in E_1 \cup E_2, \\
x^i & \in \mathbb{R}^n, & i \in S.
\end{align*}
\]
Applying MINLP solvers

- Weak lower bounds.
- Long time to decrease the upper bound.
- Symmetry.
Dealing with isomorphism

Create a set of representative FSTs, with one topology saved for each group of isomorphic FSTs. Later, during the B&B execution, we only solve subproblems corresponding to these representative topologies, pruning all the others.
Let $T(V(T), E(T))$ be a spanning tree in $G$, of $p - 2$ Steiner points.

$V(T) := \{p + 1, \ldots, 2p - 2\}$

$E(T) := \{[i_1, j_1], \ldots, [i_{p-3}, j_{p-3}]\} \in E_1$

Consider the bijection

$$f : E_1 \rightarrow L := \left\{1, \ldots, \frac{(p - 2)(p - 3)}{2}\right\}$$

$$f([p + i, p + j]) := i + \frac{(j - 1)(j - 2)}{2}$$

We represent $T$ by $f(E(T))$ defined as

$$f(E(T)) := \{f([i_1, j_1]), \ldots, f([i_{p-3}, j_{p-3}])\}$$

We map into vectors all spanning trees on the Steiner points, and save for each class of isomorphic spanning trees, a representative given by the lexicographically smaller one.
Isomorphic trees, 4 Steiner points
Isomorphic trees in the feasible set of FM, 4 Steiner points

\begin{align*}
&\{1,2,4\} \\
&\{1,3,5\}
\end{align*}
Isomorphic trees in the feasible set of FM, 4 Steiner points

\{1,2,4\}
Constructing Representative FSTs - Phase 1

\[ p + 3 \quad p + 1 \quad p + 2 \]

\[ p + 4 \]

\[ p + 3 \quad p + 1 \quad p + 2 \quad p + 4 \]
Constructing Representative FSTs - Phase 2

Marcia Fampa, Jon Lee, Wendel Melo

Solving the Euclidean Steiner Tree Problem

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### Number of Representative FSTs vs Number of FSTs in the feasible set of (FM)

<table>
<thead>
<tr>
<th>$p$</th>
<th>Rep. FSTs</th>
<th>FSTs in (FM)</th>
<th>Reduction (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
<td>6</td>
<td>50</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>30</td>
<td>50</td>
</tr>
<tr>
<td>6</td>
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</tr>
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<td>8</td>
<td>10395</td>
<td>264600</td>
<td>96</td>
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<tr>
<td>9</td>
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<td>9525600</td>
<td>98</td>
</tr>
<tr>
<td>10</td>
<td>2027025</td>
<td>428652000</td>
<td>99</td>
</tr>
</tbody>
</table>
Branch-and-Bound - Preprocessing Phase

Discarding representative FSTs based on geometric conditions satisfied by SMTs.

Let

- $\eta_i$ be the minimum Euclidean distance between $a^i$ and all other terminals.
- $T$ be a minimum-length spanning tree on the terminals.
- $\beta_{ij}$ be the length on the longest edge on the path between $a^i$ and $a^j$ in $T$.

Then:

1. Two terminals $a^i$ and $a^j$ may be connected to a common Steiner point only if
   \[ \|a^i - a^j\| \leq \eta_i + \eta_j. \]

2. Two terminals $a^i$ and $a^j$ may be connected by two or fewer Steiner points only if
   \[ \|a^i - a^j\| \leq \eta_i + \eta_j + \beta_{ij}. \]
Fixing variables

Let
- $E^+$ be the set of edges of $G$ that belong to every representative FST.
- $E^-$ be the set of edges of $G$ that do not belong to any representative FST.

Then:
- If $[i, j] \in E^+$, set $y_{ij} = 1$.
- If $[i, j] \in E^-$, set $y_{ij} = 0$. 
B&B enumeration scheme starts, at its first level, with one node corresponding to each representative of spanning trees of Steiner points only.
Each node on the enumeration tree has at most \( p - 2 \) children. A terminal is connected to a different Steiner point at each child of the node, dividing the feasible set according to the constraint \( \sum_{j \in S} y_{ij} = 1, \quad i \in P \).
Branch-and-Bound

Each node on the enumeration tree has at most $p - 2$ children. A terminal is connected to a different Steiner point at each child of the node, dividing the feasible set according to the constraint $\sum_{j \in S} y_{ij} = 1, \ i \in P$.

Terminals are added to the tree in a fixed order. The terminal added at each level is the farthest one to the terminals already in the tree. The first terminal added is the farthest one to the centroid of the set of terminals.
Branch-and-Bound

Pruning by isomorphism

At each node of the B&B tree, we use the set of representative FSTs to prune all descendant nodes where the variables fixed at 1 do not correspond to edges in any representative FST.

Fixing more variables

Let

- $\mathcal{F}$ be the set of representative FSTs that contain all the edges corresponding to $y_{ij} = 1$ at a given node of the B&B tree.
- $E^+$ be the set of additional edges of $G$ that also belong to every topology in $\mathcal{F}$.
- $E^-$ be the set of edges of $G$ that do not belong to any topology in $\mathcal{F}$.

Then:

- If $[i,j] \in E^+$, set $y_{ij} = 1$.
- If $[i,j] \in E^-$, set $y_{ij} = 0$. 
Branch-and-Bound

Dynamic Constraint Set (DCS)

The idea is to dynamically change the set of constraints of the subproblems, eliminating redundant nonlinear constraints.

Constraints

\[ d_{ij} \geq \|x^i - x^j\| - M(1 - y_{ij}), \quad [i, j] \in E_1, \]
\[ d_{ij} \geq \|a^i - x^j\| - M(1 - y_{ij}), \quad [i, j] \in E_2, \]

concerning to \( y_{ij} \) fixed at 0, i.e., the redundant constraints

\[ d_{ij} \geq \|x^i - x^j\| - M, \quad [i, j] \in E_1: y_{ij} = 0, \]
\[ d_{ij} \geq \|a^i - x^j\| - M, \quad [i, j] \in E_2: y_{ij} = 0, \]

are removed from the model.
Branch-and-Bound

Improving upper bounds and generating more valid inequalities

- Let \( \hat{y}_{ij} \) be the fractional value of the binary variable \( y_{ij} \) at the optimal solution of the relaxation solved at a node of the B&B tree.
- Assign to each edge \([i, j]\) of \( G \), the weight given by \( w_{ij} := 1 - \hat{y}_{ij} \).
- Apply a greedy heuristic to compute the minimum spanning tree of \( G \) with a FST.
- Let \( \hat{y} \) be the characteristic vector of the tree, i.e. \( \hat{y}_{ij} = 1 \) if edge \([i, j]\) belongs to the tree, and \( \hat{y}_{ij} = 0 \), otherwise.
- Compute the Steiner minimal tree corresponding to the FST given by \( \hat{y} \), and update the current upper bound if possible.
- Add to each unsolved subproblem, the cut

\[
\sum_{[i,j] \in E} \hat{y}_{ij} y_{ij} \leq 2p - 4, \quad (1)
\]
Numerical experiments

- We show the impact of the main specialized procedures on a standard b&b algorithm for convex MINLP (coded in a solver named Muriqui), and compare Muriqui to our specialized b&b algorithm named SAMBA (Steiner Adaptations on Muriqui B&b Algorithm).
- Algorithms are implemented in C++.
- Runs were conducted on a 3.60 GHz core i7-4790 CPU, 8 MB, 16 GB, running under Linux.
- Time limit of 4 hours.
- MOSEK was used to solve the subproblems relaxations in b&b, and also to solve the convex problem obtained by fixing all integer variables in (FM), in the heuristic procedure used to improve the upper bounds.
- Instances considered were the same used in used by Fampa & Anstreicher in 2008, and Laarhoven & Anstreicher in 2013: 30 instances with 10 terminals randomly distributed in the hypercube $[0,10]^n$, 10 in each dimension $n = 3, 4, 5$. 
Impact of the specialized procedures on B&B

![Bar chart showing the impact of specialized procedures on B&B. The chart compares cpu time (s) for different dimensions (R3, R4, R5) and different procedures: original, fix. var., isom. prune.]

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Impact of the specialized procedures on B&B

![Bar chart showing CPU time for different rules in dimensions R3, R4, and R5. The chart compares the effect of no rules, rule 1, rule 2, and rules 1 and 2.]
Impact of the specialized procedures on B&B

![Bar chart showing the impact of specialized procedures on B&B. The chart compares CPU time (s) for different dimensions (R3, R4, R5) with and without DCS.]
**Numerical results**

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Muriqui gap (%)</th>
<th>Branch-and-Bound cpu time (sec)</th>
<th>SAMBA gap (%)</th>
<th>SAMBA cpu time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 3$</td>
<td>84</td>
<td>14400.38</td>
<td>0</td>
<td>240.96</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>84</td>
<td>14401.12</td>
<td>0</td>
<td>740.87</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>82</td>
<td>14400.34</td>
<td>0</td>
<td>1301.54</td>
</tr>
</tbody>
</table>