Congestion Games Viewed from M-convexity

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A congestion game is given by a tuple $\Gamma = (N, A, (\mathcal{P}^{(i)} | i \in N), (c_a | a \in A))$, where

(a) $N$: a finite nonempty set of players,
(b) $A$: a set of resources,
(c) for each $i \in N$: $\mathcal{P}^{(i)}$ is a strategy set of subsets of $A$,
(d) for each $a \in A$: nondecreasing cost $c_a : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ with $c_a(0) = 0$.

Each player $i \in N$ selects a set $P_i \in \mathcal{P}^{(i)}$.
$\mathcal{P} = (P_i | i \in N)$: a strategy configuration
For $\forall a \in A$ define: $\nu_\mathcal{P}(a) = |\{i \in N | a \in P_i\}|$.
The incurred individual cost of player $i$ is given by

$$\pi_i(\mathcal{P}) = \sum_{a \in P_i} c_a(\nu_\mathcal{P}(a)).$$

$\forall i \in N, Q \in \mathcal{P}^{(i)}$: $(\mathcal{P}_{-i}, Q) = (\mathcal{P}$ with $P_i$ replaced by $Q)$
Potential Game (Rosenthal (1973))

A potential function $\Phi(\mathcal{P})$ for $\mathcal{P} = (P_i \mid i \in N)$:

$$\Phi(\mathcal{P}) = \sum_{a \in A} \hat{c}_a(\nu_P(a)),$$

$$\hat{c}_a(k) = \sum_{\ell=0}^k c_a(\ell) \quad (\forall a \in A, \forall k \in \mathbb{Z}_{\geq 0}).$$

We then have the following relation

$$\Phi(\mathcal{P}_{-i}, Q) - \Phi(\mathcal{P}) = \pi_i(\mathcal{P}_{-i}, Q) - \pi_i(\mathcal{P})$$

for any strategy configuration $\mathcal{P}$, $i \in N$, and $Q \in \mathcal{P}^{(i)}$.

Hence every (local) minimizer of $\Phi$ is a pure Nash equilibrium.
A congestion game $\Gamma = (N, A, (P^i \mid i \in N), (c_a \mid a \in A))$: $P^i$: a set of paths from source $s$ to sink $t$ ($st$-paths) in graph $G = (V, A)$ with vertex set $V$ and arc set $A$ (Arcs in $A$ are regarded as resources.)

Congestion games on extension-parallel networks (Holzman and Law-yone (2003))

An extension-parallel network with a source and a sink: constructed by finitely many repeated operations of source/sink extension and parallel join, starting from finitely many networks, each consisting of a single arc.
Consider a symmetric game $\Gamma$ such that $\forall i : \mathcal{P}^{(i)} = \mathcal{P}^{\text{all}}$ (the set of all $st$-paths in an extension-parallel network $G$).

**Theorem** (Fotakis (2010)): For any symmetric congestion game $\Gamma = (N, A, \mathcal{P}^{\text{all}}, (c_a \mid a \in A))$ on an extension-parallel network any best-response sequence reaches a pure Nash equilibrium in $n(= |N|)$ steps.

**Procedure** (*Best* _Response_)

1. Start from any strategy configuration $\mathcal{P} = (P_i \mid i \in N)$. Let $(i_1, i_2, \cdots, i_n)$ be any permutation of $N$.

2. For each $i = i_1, i_2, \cdots, i_n$ do the following.
   - Let $\hat{P} \in \mathcal{P}^{\text{all}}$ be a minimizer of $\Phi((\mathcal{P}_{-i}, P))$ in $P \in \mathcal{P}^{\text{all}}$.
   - Put $\mathcal{P} \leftarrow (\mathcal{P}_{-i}, \hat{P})$.

3. The obtained strategy configuration $\mathcal{P} = (P_i \mid i \in N)$ is a minimizer of $\Phi(\cdot)$ (a pure Nash equilibrium).
$Q_a$ (∀$a \in A$): the set of $st$-paths containing arc $a$

**Lemma:** The family $\mathcal{F}$ of path sets $Q_a$ ($a \in A$) is laminar, i.e., for any $a, a' \in A$ we have $Q_a \cap Q_{a'} = \emptyset$, $Q_a \subseteq Q_{a'}$, or $Q_a \supseteq Q_{a'}$.

(Proof) $a \in A(G_1), a' \in A(G_2), G_1$ and $G_2$ joined $\implies Q_a \cap Q_{a'} = \emptyset$, $a \in A(G_1), G_1$ is extended by $a'$ $\implies Q_a \subseteq Q_{a'}$.
Consider a mapping of $\mathcal{P} = (P_i \mid i \in N)$ to $x_\mathcal{P} \in \mathbb{Z}\mathcal{P}^{\text{all}}$ given by

$$x_\mathcal{P} = \sum_{i \in N} \chi_{P_i},$$

where $\chi_{P_i}$ is a unit vector in $\mathbb{Z}\mathcal{P}^{\text{all}}$ such that $\chi_{P_i}(P) = 1$ if $P = P_i$ and $= 0$ otherwise.

For a given strategy configuration $\mathcal{P} = (P_i \mid i \in N)$,

$$\Phi(\mathcal{P}) = \sum_{a \in A} \hat{c}_a(\nu_\mathcal{P}(a)) = \sum_{a \in A} \hat{c}_a(x_\mathcal{P}(Q_a)) \equiv \tilde{\Phi}(x_\mathcal{P}),$$

where $x_\mathcal{P}(Q_a) = \sum_{P \in Q_a} x_\mathcal{P}(P)$ for each $a \in A$.

The function $\tilde{\Phi}(x)$ in $x \in \mathbb{Z}\mathcal{P}^{\text{all}}$ is a laminar convex function with its effective domain

$$\Delta_n = \{x \in \mathbb{Z}_{\geq 0}^{\mathcal{P}^{\text{all}}} \mid x(\mathcal{P}^{\text{all}}) = n\},$$

where $n = |N|$.
Laminar convex functions are known to be M-convex functions (Danilov-Koshevoy-Murota (2001)).

Lemma: The function $\tilde{\Phi}(x)$ is an M-convex function.

The M-convexity of $\tilde{\Phi}$ validates the $n$ step convergence of any best-response sequence, due to Fotakis (2010).
M-convex Function (Murota (1996))

$W$: a finite nonempty set

$f : \mathbb{Z}^W \rightarrow \mathbb{R} \cup \{+\infty\}$

$\text{dom}(f) = \{x \in \mathbb{Z}^W \mid f(x) < +\infty\}$

$f : \mathbb{Z}^W \rightarrow \mathbb{R} \cup \{+\infty\}$ is called an M-convex function if $\text{dom}(f) \neq \emptyset$ and it satisfies:

(M-EXC) $\forall x, y \in \mathbb{Z}^W, \forall u \in W$ with $x(u) > y(u)$, $\exists v \in W$ with $x(v) < y(v)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where $\chi_u$ is the characteristic vector of $\{u\}$ and we allow $+\infty \geq +\infty$. 

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**Proposition** (A characterization of M-convex functions): 

\( f : \mathbb{Z}^W \rightarrow \mathbb{R} \cup \{+\infty\} \): a convex-extensible function with bounded \( \text{dom}(f) \neq \emptyset \).

\( \tilde{f} \): the convex extension of \( f \).

Then \( f \) is an M-convex function if and only if for every non-vertical edge \( L \) of the epigraph \( \text{epi}(\tilde{f}) \)

- a direction vector of the line segment obtained by the projection \( ((x, \beta) \mapsto x \text{ onto } \mathbb{R}^W) \) of \( L \) belongs to

\[ \{ \chi_u - \chi_v \mid u, v \in W, u \neq v \}. \]
**Proposition (A characterization of M-convex functions):**

$f: \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$: a convex-extensible function with bounded $\text{dom}(f) \neq \emptyset$.

Then $f$ is an M-convex function if and only if for every non-vertical edge $L$ of the epigraph $\text{epi}(\tilde{f})$ a direction vector of the line segment obtained by the projection $((x, \beta) \mapsto x$ onto $\mathbb{R}^W$) of $L$ belongs to $\{\chi_u - \chi_v \mid u, v \in W, u \neq v\}$.
$f: \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$: an M-convex function with
\[ \text{dom}(f) \subseteq \mathbb{Z}^W_{\geq 0}, \quad \forall x \in \text{dom}(f): x(W) = n \geq 1 \]
\[ N = \{1, 2, \ldots, n\} \quad \text{(We call the integer } n \text{ the \textbf{rank} of } f.) \]

\section*{Greedy Procedure}

1. Start from any $x = x_0 \in \text{dom}(f)$.
   Choose any mapping $\sigma: N \to W$ such that $x = \sum_{i \in N} \chi_{\sigma(i)}$.

2. For each $i = 1, 2, \ldots, n$ do the following.
   - Find an element $w^*$ of $W$ such that
     \[ (*) \quad f(x - \chi_{\sigma(i)} + \chi_{w^*}) = \min \{ f(x - \chi_{\sigma(i)} + \chi_w) \mid w \in W \}. \]
   - Put $x \leftarrow x - \chi_{\sigma(i)} + \chi_{w^*}$.

3. The obtained $x$ is a minimizer of $f$. 

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**Theorem:** The greedy procedure described above computes a minimizer of any M-convex function $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ with $\text{dom}(f) \subseteq \mathbb{Z}^W_{\geq 0}$ in $n$ steps, where $n$ is the rank of $f$.

Fotakis’ theorem (the $n$-step convergence of best-response sequences) is a special case of this theorem.
**Theorem:** The greedy procedure described above computes a minimizer of any M-convex function $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ with $\text{dom}(f) \subseteq \mathbb{Z}^W_{\geq 0}$ in $n$ steps, where $n$ is the rank of $f$.

(Proof) Let $x_i$ be the $x$ obtained after the $i$th execution of Step 2 for $i = 1, 2, \cdots, n$. Also denote by $w_i^*$ the element $w^* \in W$ found at the $i$th execution of Step 2. It suffices to prove the following local optimality (Murota 1996):

$$\forall u, v \in W : f(x_n - \chi_u + \chi_v) \geq f(x_n). \quad (1)$$

We show that for any M-convex function $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ of rank $n \geq 1$ the greedy procedure obtains $x = x_n$ satisfying (1), by induction on the rank $n$ of $f$, where the effective domain of $f$ lies on the hyperplane $x(W) = n$. Note that we fix $W$ in the following arguments.

For any M-convex function of rank $n = 1$, (1) holds. Hence, let $k$ be an integer with $k \geq 1$ and suppose that for any M-convex function of rank $n = k$ the greedy procedure obtains $x = x_n$ satisfying (1), i.e., the greedy procedure finds a minimizer of any M-convex function $f$ when $x(W) = k$ for all $x \in B \equiv \text{dom}(f)$. 

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Now suppose $n = k + 1$. Since $f$ remains to be M-convex by the restriction of its effective domain $B$ to

$$B_1 = B \cap \{ x \in \mathbb{Z}^W \mid x(\sigma(n)) \geq 1 \},$$

it follows from the induction hypothesis that

(*) $x_{n-1}$ is a minimizer of $f$ restricted on $B_1$.

For any fixed distinct $u, v \in W$ consider $y = x_n - \chi_u + \chi_v$. We show $f(y) \geq f(x_n)$. Hence we assume $y \in \text{dom}(f)$, i.e., $f(y) < +\infty$. If $v = \sigma(n)$, then since $y \in B_1$ and by (*) we have

$$f(y) \geq f(x_{n-1}) \geq f(x_n).$$

(2)

Hence suppose $v \neq \sigma(n)$.

If $x_n = x_{n-1}$ and $u \neq \sigma(n)$, then $y$ belongs to $B_1$, so that we have $f(y) \geq f(x_{n-1}) = f(x_n)$. Also, if $x_n = x_{n-1}$ and $u = \sigma(n)$, then $f(y) = f(x_{n-1} - \chi_{\sigma(n)} + \chi_v) \geq f(x_n)$ by the definition of $x_n$. Hence we further suppose $x_n \neq x_{n-1}$, i.e., $x_n(\sigma(n)) < x_{n-1}(\sigma(n))$ and $w^*_n \neq \sigma(n)$.
Now, since \( y(\sigma(n)) = x_n(\sigma(n)) < x_{n-1}(\sigma(n)) \), there exists \( p \in \{v, w^*_n\} \) such that

\[
f(x_{n-1}) + f(y) \geq f(x_{n-1} - \chi_{\sigma(n)} + \chi_p) + f(y + \chi_{\sigma(n)} - \chi_p).
\]

(3)

Because of the optimality of \( x_{n-1} \) within \( B_1 \) we have

\[
f(y + \chi_{\sigma(n)} - \chi_p) \geq f(x_{n-1})
\]

(4)

since \( p \neq \sigma(n) \) and \( y + \chi_{\sigma(n)} - \chi_p \in B_1 \). Also, because of the definition of \( x_n \) we have

\[
f(x_{n-1} - \chi_{\sigma(n)} + \chi_p) \geq f(x_n).
\]

(5)

It follows from (3)–(5) that \( f(y) \geq f(x_n) \) since \( f(x_{n-1}) < +\infty \).

This completes the proof. \( \Box \)
Similarly as shown by Dress-Wenzel (1990) for valuated matroids, we have a converse of this theorem, which shows the equivalence between the greediness and M-convexity.

**Theorem:** Let \( f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\} \) be a function with \( \emptyset \neq \text{dom}(f) \subseteq \mathbb{Z}^W_{\geq 0} \). Suppose that \( f \) is convex-extensible on \( \mathbb{R}^W \).

Then, \( f \) is an M-convex function if and only if for every \( d \in \mathbb{R}^W \)

Greedy Procedure works for the function

\[
f^d(x) = f(x) + \langle d, x \rangle.
\]
Similarly as shown by Dress-Wenzel (1990) for valuated matroids, we have a converse of this theorem, which shows the equivalence between the greediness and M-convexity.

Theorem: Let $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ be a function with $\emptyset \neq \text{dom}(f) \subseteq \mathbb{Z}^W_{\geq 0}$. Suppose that $f$ is convex-extensible on $\mathbb{R}^W$.

Then, $f$ is an M-convex function if and only if for every $d \in \mathbb{R}^W$

Greedy Procedure works for the function

$$f^d(x) = f(x) + \langle d, x \rangle.$$  

(Proof) It suffices to show the if part.

Since $f$ is convex-extensible, denoting by $\tilde{f}$ the convex extension of $f$, it suffices to prove that every non-vertical edge vector of the epigraph of $\tilde{f}$ projected onto $\mathbb{R}^W$ belongs to

$$\{\chi_u - \chi_v \mid u, v \in W, u \neq v\},$$

due to Proposition on a characterization of M-convex functions.
Let $L$ be an arbitrary non-vertical edge of the epigraph of $\bar{f}$ and let $\hat{L}$ be the projection (onto $\mathbb{R}^W$) of $L$. Also let $x_1, x_2 \in B$ be the end points of $\hat{L}$. Let $z \in B$ be the point in $(\hat{L} \setminus \{x_1\}) \cap B$ nearest to $x_1$. Then there exists a vector $d \in \mathbb{R}^W$ such that $x_1$ is the unique minimizer of $f^d$ and

$$\{x \in B \mid f^d(x) \leq f^d(z)\} = \{z, x_1\}.$$  

Hence, starting from $z$, Greedy Procedure for $f^d$ must move from $z$ to $x_1$ by the first improving step. By the definition of Greedy Procedure the direction of the movement from $z$ to $x_1$, which is a direction vector of $\hat{L}$, belongs to $\{\chi_u - \chi_v \mid u, v \in W, u \neq v\}$.  

$\square$

$\rightarrow$
We call a transformation from $x \in \mathbb{Z}^W$ to $x - \chi_u + \chi_v$ for $u, v \in W$ a basic local transformation.

Theorem: Let $f : \mathbb{Z}^W \to \mathbb{R} \cup \{+\infty\}$ be a convex-extensible function with a nonempty bounded $\text{dom}(f)$. Suppose that there exists a procedure $P$ such that for every $d \in \mathbb{R}^W$ and every initial solution $x_0 \in \text{dom}(f)$ Procedure $P$ finds a finite sequence of solutions $(x_0, x_1, \cdots, x_k)$ for some integer $k \geq 0$ satisfying

(a) $f^d(x_0) \geq f^d(x_1) \geq \cdots \geq f^d(x_k)$,

(b) each $x_i$ for $i = 1, \cdots, k$ is obtained by a basic local transformation of $x_{i-1}$.

(c) $x_k$ is a minimizer of $f^d$.

Then, $f$ is an M-convex function.