The quadratic assignment problem is easy for Robinsonian matrices

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Introduction

Seriation

Sir W.M. Flinders Petrie (1899)
Robinsonian similarities and Seriation

Definition (Robinson similarity)

$A \in S^n$ is a **Robinson similarity** if the values of its entries **decrease** monotonically along rows and columns when moving away from the diagonal.

Example:

$$A = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 4 & 5 & 3 & 2 \\ 2 & 3 & 5 & 4 \\ 1 & 2 & 4 & 5 \end{pmatrix} \quad \begin{pmatrix} * & \text{←} \\ \text{↔} & * \\ \text{→} & * \end{pmatrix}$$
Robinsonian similarities and Seriation

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$$A_{ik} \leq \min\{A_{ij}, A_{jk}\}, \quad 1 \leq i < j < k \leq n$$
Robinsonian similarities and Seriation

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\( A \in S^n \) is a **Robinson similarity** if the values of its entries decrease monotonically along rows and columns when moving away from the diagonal.

Example:

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\end{pmatrix}
\]

**Definition (Robinsonian similarity)**

\( A \in S^n \) is a **Robinsonian similarity** if there exists a permutation \( \pi \) such that \( A_\pi := \left( A_{\pi(i), \pi(j)} \right)_{i,j=1}^n \) is a **Robinson similarity**.
Robinsonian similarities and Seriation

Definition (Robinson similarity)

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\end{pmatrix} = \begin{pmatrix}
* & \leftarrow \\
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\uparrow & * \\
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Definition (Robinsonian similarity)

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$\Rightarrow$ the **Seriation** problem consists in finding such a $\pi$!
Seriation and QAP

...what if $A$ is not Robinsonian?
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**2-SUM:**

$$\min_{\pi \in \mathcal{P}} \sum_{i,j=1}^{n} A_{ij}(\pi(i) - \pi(j))^2$$

This is a special case of a well known problem ([Koopmans-Beckmann](#)): **QAP**

$$\text{QAP}(A,B): \min_{\pi \in \mathcal{P}} \sum_{i,j=1}^{n} A_{ij}B_{\pi(i)\pi(j)}$$

cost inferred by assigning object $i$ to position $\pi(i)$ and object $j$ to position $\pi(j)$

Sahni and Gonzalez [1976]

QAP is NP-hard and cannot be approximated within a constant factor in polynomial time.

IDEA:

find "easy cases" solvable by a fixed permutation (approximation algorithms, heuristics)
Seriation and QAP

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\textbf{IDEA:} find “easy cases” solvable by a \textit{fixed permutation} (approximation algorithms, heuristics)
Robinsonian dissimilarities

**Definition (Robinson dissimilarity)**

\[ A \in S^n \] is a **Robinson dissimilarity** if the values of its entries *increase* monotonically along rows and columns when moving away from the diagonal.

**Example:**

\[
A = \begin{pmatrix}
0 & 1 & 3 & 4 \\
1 & 0 & 2 & 3 \\
3 & 2 & 0 & 1 \\
4 & 3 & 1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
* & \rightarrow \\
* & \uparrow \\
\downarrow & * \\
\leftarrow & *
\end{pmatrix}
\]
Definition (Robinson dissimilarity)

\( A \in S^n \) is a **Robinson dissimilarity** if the values of its entries **increase** monotonically along rows and columns when moving away from the diagonal.

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Definition (Robinsonian dissimilarity)

\( A \in S^n \) is a **Robinsonian dissimilarity** if it exists a permutation \( \pi \) such that \( A_\pi \) is a Robinson dissimilarity.
Robinsonian dissimilarities

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Definition (Robinsonian dissimilarity)

$A \in S^n$ is a **Robinsonian dissimilarity** if it exists a permutation $\pi$ such that $A_\pi$ is a **Robinson dissimilarity**.

Observation: $A$ is a **Robinson (Robinsonian) similarity** if and only if $-A$ is a **Robinson (Robinsonian) dissimilarity**.
Definition (Toeplitz matrix)

$B \in S^n$ is a Toeplitz matrix if has constant entries on its diagonals, i.e:

$$B_{ij} = B_{(i+1)(j+1)} \quad \forall 1 \leq i, j \leq n - 1$$

Example:

$$B = \begin{pmatrix}
0 & 1 & 4 & 9 \\
1 & 0 & 1 & 4 \\
4 & 1 & 0 & 1 \\
9 & 4 & 1 & 0
\end{pmatrix}$$
**Definition (Toeplitz matrix)**

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Example:

$$B = \begin{pmatrix}
0 & 1 & 4 & 9 \\
1 & 0 & 1 & 4 \\
4 & 1 & 0 & 1 \\
9 & 4 & 1 & 0
\end{pmatrix} = (|i - j|^2)^n_{i,j=1}$$
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$$B = \begin{pmatrix} 0 & 1 & 4 & 9 \\ 1 & 0 & 1 & 4 \\ 4 & 1 & 0 & 1 \\ 9 & 4 & 1 & 0 \end{pmatrix} = (|i - j|^2)_{i,j=1}^n$$

Observation

$B = (|i - j|^p)_{i,j=1}^n$ is a Toeplitz Robinson dissimilarity matrix, for $p \geq 1$. 
**Toeplitz Robinson dissimilarities**

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$$B = \begin{pmatrix} 0 & 1 & 4 & 9 \\ 1 & 0 & 1 & 4 \\ 4 & 1 & 0 & 1 \\ 9 & 4 & 1 & 0 \end{pmatrix} = (|i - j|^{2})_{i,j=1}^{n}$$

**Observation**

$B = (|i - j|^{p})_{i,j=1}^{n}$ is a **Toeplitz Robinson dissimilarity** matrix, for $p \geq 1$.

$\Rightarrow$ we can generalize the Seriation problem
The main result

\[ \text{QAP}(A,B): \min_{\pi \in \mathcal{P}} \sum_{i,j=1}^{n} A_{\pi(i)\pi(j)} B_{ij} \]
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QAP(A,B): \min_{\pi \in \mathcal{P}} \sum_{i,j=1}^{n} A_{\pi(i)\pi(j)} B_{ij}
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**Theorem [Laurent & Seminaroti, 2014]**

Given \( A, B \in S^n \), suppose \( A \) is a **Robinson similarity** and \( B \) is a **Robinson dissimilarity**, and \( A \) or \( B \) is a **Toeplitz** matrix. Then, the identity permutation is optimal for \( QAP(A, B) \).
The main result

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This result is a generalization of two previous results:

1. (Fogel et al., 2013)
   - \( A \) is a conic combination of interval CUT matrices;
   - \( B = (|i-j|^2)^n \) \( i,j=1 \).

2. (Christopher et al., 1996)
   - \( A \) is the adjacency matrix of the path \((1, \ldots, n)\);
   - \( B \) is a Robinson dissimilarity **metric** and **strongly monotonic**.
Intuition behind the main result

\[
\sum_{i,j=1}^{n} A_{\pi(i)\pi(j)} B_{ij} \geq \sum_{i,j=1}^{n} A_{ij} B_{ij}
\]

where:

\[
A = \begin{pmatrix}
* & \leftarrow & \downarrow \\
\uparrow & * & \downarrow \\
\rightarrow & * & \\
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
* & \rightarrow & \uparrow \\
\downarrow & * & \uparrow \\
& \leftarrow & *
\end{pmatrix}
\]
Intuition behind the main result

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where:

\[ A = \begin{pmatrix} * & \leftarrow & \downarrow \\ \uparrow & * & \downarrow \\ \rightarrow & \downarrow & * \end{pmatrix} \quad \quad B = \begin{pmatrix} * & \rightarrow & \uparrow \\ \downarrow & * & \uparrow \\ \leftarrow & \uparrow & * \end{pmatrix} \]

This is the analogous for matrices of the rearrangement inequality:

\[ \sum_{i=1}^{n} x_{\pi(i)} y_i \geq \sum_{i=1}^{n} x_i y_i \]

where:

\[ x_1 \geq \cdots \geq x_n \]
\[ y_1 \leq \cdots \leq y_n \]
QAP is easy over Robinsonian matrices

\[
\text{QAP}(A,B): \quad \min_{\pi \in \mathcal{P}} \sum_{i,j=1}^{n} A_{\pi(i)\pi(j)} B_{ij}
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**Theorem [Laurent & Seminaroti, 2014]**

Given \( A, B \in S^n \), suppose \( A \) is a **Robinson similarity** and \( B \) is a **Robinson dissimilarity**, and \( A \) or \( B \) is a **Toeplitz** matrix. Then, the identity permutation is optimal for \( QAP(A,B) \).

\( \Rightarrow \) Can we recognize Robinsonian matrices in polynomial time? **YES!**
QAP is easy over Robinsonian matrices

\[ \text{QAP}(A,B) : \min_{\pi \in \mathcal{P}} \sum_{i,j=1}^{n} A_{\pi(i)\pi(j)} |i - j|^2 \]

**Theorem [Laurent & Seminaroti, 2014]**

Given \( A, B \in S^n \), suppose \( A \) is a **Robinson similarity** and \( B \) is a **Robinson dissimilarity**, and \( A \) or \( B \) is a **Toeplitz** matrix. Then, the identity permutation is optimal for \( \text{QAP}(A,B) \).

**Corollary**

Given \( A \in S^n \) and \( B = (|i - j|^2) \), assume that \( A \) is a **Robinsonian similarity** and \( \pi \) is a permutation which reorders \( A \) as a Robinson similarity. Then \( \pi \) is optimal for \( \text{QAP}(A,B) \).
QAP is easy over Robinsonian matrices

QAP(A,B): \( \min_{\pi \in \mathcal{P}} \sum_{i,j=1}^{n} A_{\pi(i)\pi(j)} |i - j|^2 \)

**Theorem [Laurent & Seminaroti, 2014]**
Given \( A, B \in S^n \), suppose \( A \) is a **Robinson similarity** and \( B \) is a **Robinson dissimilarity**, and \( A \) or \( B \) is a **Toeplitz** matrix. Then, the identity permutation is optimal for \( QAP(A, B) \).

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⇒ Can we recognize Robinsonian matrices in polynomial time? **YES!**
Given a graph $G(V, E)$:
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1. **Closed neighborhood:**
   
   $N[x] = N(x) \cup \{x\}, \forall x \in V$

2. **Block:**
   
   $x, y \in B \iff N[x] = N[y]$

3. **Adjacent blocks:**
   
   $B_i, B_j$ adjacent if $x \in B_i, y \in B_j$ and $xy \in E$

4. **Straight enumeration:**
   
   A linear order of the blocks of $G$ such that for every block, the block and its neighboring blocks are consecutive.
Given a graph $G(V, E)$:

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Straight enumeration of a graph

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![Diagram](image)

- **Example:**
  - $B_1 \rightarrow B_2 \rightarrow B_3$ is a straight enumeration
  - $B_2 \rightarrow B_1 \rightarrow B_3$ is NOT a straight enumeration
A graph $G(V, E)$ is an **unit interval graph** (uig) if there exists unit intervals $I_1, \ldots, I_n$ of the real line such that, for each $x \neq y \in V$, then:

$$
\{x, y\} \in E \iff I_x \cap I_y \neq \emptyset
$$

**Theorem (Deng et al., 1996)**

$G$ is a uig if and only if it has a straight enumeration. Furthermore, if $G$ is connected, then such a straight enumeration is unique (up to reversal).
Unit interval graphs (uigs)

A graph $G(V, E)$ is an **unit interval graph** (uig) if there exists unit intervals $I_1, \ldots, I_n$ of the real line such that, for each $x \neq y \in V$, then:

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**Theorem (Deng et al., 1996)**

$G$ is a uig if and only if it has a straight enumeration. Furthermore, if $G$ is connected, then such a straight enumeration is unique (up to reversal).
A straight enumeration \((B_1, \ldots, B_p)\) of \(G(V, E)\) is an ordered partition of \(V\) and (thus) induces a **weak linear order** \(\psi\):

- \(x =_{\psi} y\) if \(x, y \in B_i\)
- \(x <_{\psi} y\) if \(x \in B_i, y \in B_j\) and \(i < j\)

A linear order \(\pi\) of \(V\) is compatible with \(\psi\) if:

- \(x <_{\pi} y\) implies \(x \leq_{\psi} y\) for all \(x, y \in V\) with \(x \neq y\)
Binary Robinsonian matrices

A straight enumeration \((B_1, \ldots, B_p)\) of \(G(V, E)\) is an ordered partition of \(V\) and (thus) induces a \textbf{weak linear order} \(\psi\):

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A linear order \(\pi\) of \(V\) is \textbf{compatible} with \(\psi\) if:

\[ x <_\pi y \implies x \leq_\psi y \quad \forall x \neq y \in V \]

### Lemma

Let \(G(V, E)\) be a graph and \(A_G\) its extended adjacency matrix. Then:

- \(A_G\) is Robinsonian if and only if \(G\) is uig;
- A linear order \(\pi\) of \(V\) reorders \(A_G\) as a Robinson matrix if and only if there exists a straight enumeration \(\psi\) of \(G\) compatible with \(\pi\).
Binary Robinsonian matrices

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A linear order \(\pi\) of \(V\) is **compatible** with \(\psi\) if:

\[
x <_{\pi} y \implies x \leq_{\psi} y \quad \forall x \neq y \in V
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**Lemma**

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- \(x <_\psi y\) if \(x \in B_i\), \(y \in B_j\) and \(i < j\)

A linear order \(\pi\) of \(V\) is compatible with \(\psi\) if:

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- A linear order \(\pi\) of \(V\) reorders \(A_G\) as a Robinson matrix if and only if there exists a straight enumeration \(\psi\) of \(G\) compatible with \(\pi\).

⇒ What about general Robinsonian matrices?
Definition (Level graphs)

Given a non-negative matrix $A \in S^n$, we let $0 < \alpha_1 < \cdots < \alpha_L$ denote the distinct values taken by its entries. For $\ell \in [L]$, the $\ell$-th level graph $G^{(\ell)} = (V, E_\ell)$ of $A$ is defined as:

- $V = [n]$
- $\{x, y\} \in E_\ell$ if $A_{xy} \geq \alpha_\ell$
Robinson matrix decomposition in uigs

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Lemma (Roberts, 1978)
Let $A \in S^n$ a non-negative matrix with level graphs $G^{(1)}, \ldots, G^{(L)}$. Then:

i. $A = \sum_{\ell=1}^L (\alpha_\ell - \alpha_{\ell-1}) A_{G^{(\ell)}}$
Robinson matrix decomposition in uigs

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Given a non-negative matrix $A \in S^n$, we let $0 < \alpha_1 < \cdots < \alpha_L$ denote the distinct values taken by its entries. For $\ell \in [L]$, the $\ell$-th level graph $G^{(\ell)} = (V, E_\ell)$ of $A$ is defined as:

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Lemma (Roberts, 1978)

Let $A \in S^n$ a non-negative matrix with level graphs $G^{(1)}, \ldots, G^{(L)}$. Then:

i. $A = \sum_{\ell=1}^{L} (\alpha_\ell - \alpha_{\ell-1}) A_{G^{(\ell)}}$

ii. $A$ is Robinsonian if there exists a permutation $\pi$ such that $(A_{G^{(\ell)}})_\pi$ is a Robinson matrix for all $\ell \in [L]$
### Definition (Level graphs)

Given a non-negative matrix $A \in S^n$, we let $0 < \alpha_1 < \cdots < \alpha_L$ denote the distinct values taken by its entries. For $\ell \in [L]$, the $\ell$-th level graph $G^{(\ell)} = (V, E_\ell)$ of $A$ is defined as:

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### Lemma (Roberts, 1978)

Let $A \in S^n$ a non-negative matrix with level graphs $G^{(1)}, \ldots, G^{(L)}$. Then:

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ii. $A$ is Robinsonian if there exists a permutation $\pi$ such that $(A_{G^{(\ell)}})_\pi$ is a Robinson matrix for all $\ell \in [L]$

How can I find $\pi$?
Example

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 0 & 2 & 1 & 1 & 1 \\
2 & 0 & 2 & 1 & 0 & 0 & 0 \\
3 & 2 & 1 & 2 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 2 & 2 & 1 \\
5 & 1 & 0 & 0 & 2 & 2 & 2 \\
6 & 1 & 0 & 0 & 1 & 2 & 2 \\
\end{pmatrix}
\]

\[\alpha_1 = 1, \quad \alpha_2 = 2\]

\[
A_G^{(1)} = \begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}
\]

\[
A_G^{(2)} = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]
Common refinement of two compatible weak linear orders

\[ \psi = (B_1, \ldots, B_p) \]
Common refinement of two compatible weak linear orders

\[ \psi = (B_1, \ldots, B_p) \]

\[ \phi = (C_1, \ldots, C_q) \]
Common refinement of two compatible weak linear orders

\[ \psi = (B_1, \ldots, B_p) \]

\[ \phi = (C_1, \ldots, C_q) \]

common refinement

\[ \Phi = (B'_1, \ldots, B'_r) \]
The common refinement does not always exist

\[ \psi = (B_1, \ldots, B_p) \]

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The common refinement does not always exist

\[ \psi = (B_1, \ldots, B_p) \]

\[ \phi = (C_1, \ldots, C_q) \]

1 \( \prec_\psi \) 3 and 3 \( \prec_\psi \) 1 \( \Rightarrow \) \( \psi \) and \( \phi \) are NOT compatible!
Theorem (Laurent & Seminaroti, 2014)

Let $A \in S^n$ non-negative with level graphs $G^{(1)}, \ldots, G^{(L)}$. Then:

i. $A$ is a Robinsonian matrix if and only if there exist straight enumerations of $G^{(1)}, \ldots, G^{(L)}$ whose corresponding weak linear orders $\psi_1, \ldots, \psi_L$ are pairwise compatible;

ii. a linear order $\pi$ of $V$ reorders $A$ as a Robinson matrix if and only if there exist straight enumerations of $G^{(1)}, \ldots, G^{(L)}$, whose corresponding common refinement is compatible with $\pi$. 

Idea: Given a symmetric non-negative matrix $A$:

1. compute the level graphs $G^{(1)}, \ldots, G^{(L)}$ of $A$;
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Example 1/2

\[
A_{G(1)} = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 0 & 1 & 1 & 1 \\
2 & 0 & 1 & 1 & 0 & 0 \\
3 & 1 & 1 & 1 & 0 & 0 \\
4 & 1 & 0 & 0 & 1 & 1 \\
5 & 1 & 0 & 0 & 1 & 1 \\
6 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}
\]

\[\psi = (\{2\}, \{3\}, \{1\}, \{4, 5, 6\})\]

\[\phi = (\{2\}, \{1, 3\}, \{4\}, \{5\}, \{6\})\]

The common refinement is:

\[\Phi = (\{2\}, \{3\}, \{1\}, \{4\}, \{5\}, \{6\})\]
Example 1/2

\[ A_{G(1)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 1 & 1 \\ 5 & 1 & 0 & 0 & 1 & 1 \\ 6 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \]

\[ A_{G(2)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 1 \\ 5 & 0 & 0 & 0 & 1 & 1 \\ 6 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

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\[ A_G^{(2)} = \begin{pmatrix}
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$$\pi = (\{2\}, \{3\}, \{1\}, \{4\}, \{5\}, \{6\})$$

$$A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 0 & 2 & 1 & 1 & 1 \\
2 & 0 & 2 & 1 & 0 & 0 & 0 \\
3 & 2 & 1 & 2 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 2 & 2 & 1 \\
5 & 1 & 0 & 0 & 2 & 2 & 2 \\
6 & 1 & 0 & 0 & 1 & 2 & 2
\end{pmatrix} \Rightarrow A_\pi = \begin{pmatrix}
1 & 2 & 3 & 1 & 4 & 5 & 6 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 \\
3 & 1 & 2 & 2 & 0 & 0 & 0 \\
1 & 0 & 2 & 2 & 1 & 1 & 1 \\
4 & 0 & 0 & 1 & 2 & 2 & 1 \\
5 & 0 & 0 & 1 & 2 & 2 & 2 \\
6 & 0 & 0 & 1 & 1 & 2 & 2
\end{pmatrix}$$
### Recognition algorithms for Robinsonian matrices

<table>
<thead>
<tr>
<th>Name</th>
<th>Year</th>
<th>Complexity</th>
<th>Enum.</th>
<th>Sub.</th>
<th>Char.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chepoi &amp; Fichet</td>
<td>1997</td>
<td>$O(n^3)$</td>
<td>no</td>
<td>none</td>
<td>interval hypergraphs</td>
</tr>
<tr>
<td>Préa &amp; Fortin</td>
<td>2014</td>
<td>$O(n^2)$</td>
<td>yes</td>
<td>PQ-tree</td>
<td>interval hypergraphs</td>
</tr>
<tr>
<td>Laurent &amp; Seminaroti</td>
<td>2014</td>
<td>$O(d(m + n))$</td>
<td>yes</td>
<td>Lex-BFS</td>
<td>unit interval graphs</td>
</tr>
</tbody>
</table>

- $L$: number of distinct values of $A$
- $d$: depth of the recursion tree (bounded by $L$)
- $m$: number of nonzero entries of $A$
Open questions

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V. Chepoi, B. Fichet and M. Seston.
Seriation in the Presence of Errors: NP-Hardness of $l_\infty$ Fitting
Robinson Structures to Dissimilarity Matrices, 2009.
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F. Fogel, A. d’ Aspremont, and M. Vojnovic.
References


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