Conclusions from classical parametric integer programming for stochastic optimization

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From parametric optimization to two-stage SP

Take a parametric mixed-integer program

$$(P_z) \quad \min_{x,y} \{c(x) + q(y) \mid x \in X, \ y \in C(x, z), \ y \in \mathbb{R}^{m_1} \times \mathbb{Z}^{m_2}\},$$
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add an information constraint

decide \(x \rightarrow\) observe \(z \rightarrow\) decide \(y\)

Task: Pick an “optimal” random variable taking into account risk aversion.

→ Mean risk models:

\[
\min_{x} \{\rho(f(x, z(\omega))) \mid x \in X\},
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decide \( x \) \( \rightarrow \) observe \( z \) \( \rightarrow \) decide \( y \)

and assume purely exogenous stochastic uncertainty \( z = z(\omega) \).
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→ Two-stage-formulation:

\[
\min \{ c(x) + \min \{ q(y) \mid y \in C(x,z(\omega)), \ y \in \mathbb{R}^{m_1} \times \mathbb{Z}^{m_2} \} \mid x \in X \}
= : f(x,z(\omega))
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From parametric optimization to two-stage SP

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\[ \min \{ \rho(f(x, z(\omega))) \mid x \in X \} \]
Stability in two-stage SP

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is a parametric problem w.r.t. the distribution of \( z \).

→ Stability of optimal values, solution sets?

For \( \epsilon \neq 0 \), every integer is an optimal solution with value 0.

Conclusion: No stability in two-stage SP if the underlying deterministic mixed-integer problem is not “well behaved.”
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**Example:**

\[ \min_{x \in \mathbb{Z}} \mathbb{E}[\chi_{\{0\}}(z(\omega))(x^2 + \lambda)], \]

where \( \lambda \in \mathbb{R} \) is fixed and \( \mathbb{P}(z(\omega) = 0) = 1 \).
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→ Unique optimal solution \( x = 0 \) yields the value \( \lambda \).
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\[ \min_{x \in \mathbb{Z}} \mathbb{E}[\chi_0(z(\omega))(x^2 + \lambda)], \]

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→ Unique optimal solution \( x = 0 \) yields the value \( \lambda \).

Consider the random variables \( z_\epsilon(\cdot) \) defined by \( \mathbb{P}(z_\epsilon(\omega) = \epsilon) = 1 \) and solve

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Stability in two-stage SP

**Question:** What has to be assumed of

\[ f(x, z) = c(x) + \min \{ q(y) \mid y \in C(x, z), \ y \in \mathbb{R}^{m_1} \times \mathbb{Z}^{m_2} \} \]
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**Sufficient:** \( f \) is defined by a MILP
\[
 f(x, z) = c^\top x + \min \{ q^\top y \mid Wy = z - Tx, \ y \in \mathbb{R}_{\geq 0}^{m_1} \times \mathbb{Z}_{\geq 0}^{m_2} \},
\]
the matrix \( W \) is rational and
\[
 (A1) \ W(\mathbb{R}_{\geq 0}^{m_1} \times \mathbb{Z}_{\geq 0}^{m_2}) = \mathbb{R}^s,
\]
\[
 (A2) \ \{ u \in \mathbb{R}^s \mid W^\top u \leq q \} \neq \emptyset.
\]

→ Schultz, Tiedemann (2006), Römisch, Vigerske (2008), ...

Improvement by Claus, Krätzschmer, Schultz (2015): Assume that \( f \) is continuous almost everywhere and fulfills a growth condition:
\[
 (G) \text{ There is a locally bounded mapping } \eta : \mathbb{R}^n \to (0, \infty) \text{ and a constant } \gamma > 0 \text{ such that } \]
\[
 |f(x, z)| \leq \eta(x)(\|z\|^{\gamma} + 1) \text{ for all } (x, z) \in \mathbb{R}^n \times \mathbb{R}^s.
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(A1) \( W(\mathbb{R}_{\geq 0}^{m_1} \times \mathbb{Z}_{\geq 0}^{m_2}) = \mathbb{R}^s \),

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(A1) \quad & W(\mathbb{R}^{m_1}_{\geq 0} \times \mathbb{Z}^{m_2}_{\geq 0}) = \mathbb{R}^s, \\
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\[(G) \quad \text{There is a locally bounded mapping } \eta : \mathbb{R}^n \to (0, \infty) \text{ and a constant } \gamma > 0 \text{ such that}
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\[ |f(x, z)| \leq \eta(x) (\|z\|^{\gamma} + 1) \quad \text{for all } (x, z) \in \mathbb{R}^n \times \mathbb{R}^s. \]
Back to parametric mixed integer programming

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**Theorem (Blair, Jeroslow 1977)**

Assume (A1), (A2) and the rationality of \( W \). Then

(i) \( f \) is real valued and lower semicontinuous on \( \mathbb{R}^n \times \mathbb{R}^s \).

(ii) \( f \) is continuous on \( (\mathbb{R}^n \times \mathbb{R}^s) \setminus A \), where the \((n + s)\)-dim. Lebesgue measure of

\[ A = (-T, I)^{-1}(\text{bd } W(\mathbb{R}^{m_1}_{\geq 0} \times \mathbb{Z}^{m_2}_{\geq 0})) \]

is equal to zero.

(iii) There exist constants \( C, D \geq 0 \) such that

\[ |f(x, z) - f(x', z')| \leq C \|(x, z) - (x', z')\| + D \]

for all \( (x, z), (x', z') \in \mathbb{R}^n \times \mathbb{R}^s \).
(G) There is a locally bounded mapping $\eta : \mathbb{R}^n \rightarrow (0, \infty)$ and a constant $\gamma > 0$ such that

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**MILP case:**

\[
|f(x, z)| \leq |f(x, z) - f(0, 0)| + |f(0, 0)| \leq C\|(x, z)\| + D + |f(0, 0)|
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$\Rightarrow$ (G) holds with $\eta(x) := C\|x\| + D + |f(0, 0)| + 1.$
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**Observation:** $(x, z)$ can be replaced with $h(x, z)$ if the growth condition is fulfilled for the mapping $(x, z) \mapsto |h(x, z)|$. In this case, $\gamma_f = \gamma_h$.

$\rightarrow$ This is especially the case if $h$ is Hölder continuous.
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**Conclusion:** The proposed growth condition is more general than the standard MILP setting.
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**Conclusion:** The proposed growth condition is more general than the standard MILP setting.

$\rightarrow$ Which other (mixed-)integer parametric problems are covered?
Theorem (Cook, Gerards, Schrijver, Tardos 1986)

Assume that

(i) $A$ is an integral matrix,

(ii) $Ay \leq b$ has an integral solution and

(iii) $\min\{q^\top y \mid Ay \leq b\}$ exists.

Then

$$d_\infty(\arg\min\{q^\top y \mid Ay \leq b\}, \arg\min\{q^\top y \mid Ay \leq b, y \in \mathbb{Z}^m\}) \leq m\Delta(A).$$

Here, $\Delta(A)$ denotes the maximum of the absolute values of the determinants of square submatrices of $A$. 
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Here, \( \Delta(A) \) denotes the maximum of the absolute values of the determinants of square submatrices of \( A \).

→ This proximity result can be generalized to quadratic integer problems.
Back to parametric integer programming

\[ f(x, z) = c(x) + \min \{ y^\top Q y + q^\top y \mid Ay \leq h(x, z), \ y \in \mathbb{Z}^m \} \]
Back to parametric integer programming

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**Theorem (Garnot, Skorin-Kapov 1990)**

Assume that

(i) \( A \) is integral, rank \( A = m \) and \( Q \) is a positive definite diagonal matrix,

(ii) \( \{ y \in \mathbb{Z}^m \mid Ay \leq h(x, z) \} \neq 0 \) for all \( (x, z) \in \mathbb{R}^n \times \mathbb{R}^s \) and

(iii) \( \min \{ y^\top Q y + q^\top y \mid Ay \leq b \} \) exists.

Then \( f \) is finite, the infimum is attained and there exists constants \( C, D \geq 0 \) such that

\[ |f(x, z)| \leq C \Delta(x, z) \|(x, z)\| + D \]

holds true for all \( (x, z) \in \mathbb{R}^n \times \mathbb{R}^s \). Here, \( \Delta(x, z) \) denotes the maximum of the absolute values of the determinants of square submatrices of

\[
\begin{pmatrix}
A & 0 & h(x, z) \\
-sQ & A^\top & q
\end{pmatrix}.
\]
\[ \Delta(x, z) = \max \{| \det B | \mid B \text{ is a square sub-matrix of } \begin{pmatrix} A & 0 \\ -sQ & A^\top \\ q & \end{pmatrix} \} \]
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→ Laplace expansion yields a constant \( E \) such that
\[ \Delta(x, z) \leq E(\|h(x, z)\| + 1) \]
for all \((x, z) \in \mathbb{R}^n \times \mathbb{R}^s\).
Back to parametric integer programming

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→ If the growth condition holds for \((x, z) \mapsto |h(x, z)|\) and \(c\), then it also holds for \(f\) with \(\gamma_f = \max\{\gamma_c, 2\gamma_h\}\).
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→ If \(c\) and \(h\) are continuous and \(h^{-1}(\mathbb{R}^k \setminus (\mathbb{R} \setminus \mathbb{Z})^k)\) has Lebesgue measure zero, then the Lebesgue measure of the set of discontinuities of \(f\) is equal to zero.
Back to parametric integer programming

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Conclusion: Our approach allows to derive stability for two-stage SPs with QIP recourse problems.
Consider the case where

\[ f(x, z) = c(x) + \min \{ q(y) \mid y \in C(x, z), \ y \in \mathbb{R}^{m_1} \times \mathbb{Z}^{m_2} \} \]

is defined by

\[ C(x, z) = \{ y \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \mid g(y) \leq h(x, z) \}, \]

$q$ is convex and $g = (g_1, \ldots, g_k)^\top$ is such that $g_i$ is convex and $\text{epi } g_i$ is closed for every $i = 1, \ldots, k$. 

Helpful fact: Convex functions are Lipschitz continuous on bounded subsets.

Assumptions:

(C1) $C(0, 0)$ is bounded and

(C2) $C(x, z) \cap (\mathbb{R}^{m_1} \times \mathbb{Z}^{m_2}) \neq \emptyset$ for all $(x, z) \in \mathbb{R}^n \times \mathbb{R}^s$. 

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Conclusions from classical parametric integer programming

January 4, 2016
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A stability result from convex analysis

**Theorem (Auslender, Crouzeix 1988)**

Assume (C1) and (C2), then for every $R > 0$ there exists a constant $K(R)$, such that

$$d_\infty(C(x, z), C(x', z')) \leq K(R)\|h(x, z) - h(x', z')\|$$

whenever $\|(x, z) - (x', z')\| \leq R$. Set

$$\Theta(x, z, y) := (\max\{g_1(y) - h_1(x, z), 0\}, \ldots, \max\{g_k(y) - h_k(x, z), 0\})^\top \in \mathbb{R}^k,$$

then

$$K(R) = \sup_{(x, z, y): \|h(x, z)\| \leq R, y \notin C(x, z)} \frac{d_{C(x, z)}(y)}{\|\Theta(x, z, y)\|_\infty} < \infty$$

holds.
A stability result from convex analysis

Theorem (Auslender, Crouzeix 1988)

Assume (C1) and (C2), then for every $R > 0$ there exists a constant $K(R)$, such that

$$d_\infty(C(x, z), C(x', z')) \leq K(R)\|h(x, z) - h(x', z')\|$$

whenever $\|(x, z) - (x', z')\| \leq R$. Set

$$\Theta(x, z, y) := (\max\{g_1(y) - h_1(x, z), 0\}, \ldots, \max\{g_k(y) - h_k(x, z), 0\})^\top \in \mathbb{R}^k,$$

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→ $C(x, z)$ is compact for all $(x, z) \in \mathbb{R}^n \times \mathbb{R}^s$. 
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\[\rightarrow C(x, z) \text{ is compact for all } (x, z) \in \mathbb{R}^n \times \mathbb{R}^s.\]

\[\rightarrow \text{The continuous relaxation is solvable.}\]
Consequences for the recourse problem

Setting

\[ f(x, z) = c(x) + \min \{ q(y) \mid y \in C(x, z), \ y \in \mathbb{R}^{m_1} \times \mathbb{Z}^{m_2} \}, \]

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Consequence: If \( c \) is real-valued \( c \), then so is
\[ f(x, z) = c(x) + \min_{y_2 \in Z(x, z)} \min_{y_1 \in C_{y_2}(x, z)} q(y). \]
Consequences for the recourse problem

Additional assumptions:

(C3) There exist constants $\beta_1, L_1 > 0$ such that $L(r) \leq L_1 r^{\beta_1}$ for all $r > 0$, where $L(r)$ denotes the (minimal) Lipschitz constant for $q$ on $B_r(0)$.

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Theorem

Assume that the growth condition is fulfilled for $c$ and $(x, z) \mapsto |h(x, z)|$ and that (C1) - (C4) hold true. Then the growth condition is also fulfilled for $f$ with $\gamma_f = \max\{\gamma_c, (\beta_1 + \beta_2 + 1)\gamma_h\}$. 
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Conclusion: Under a compactness condition, our approach allows to derive stability for two-stage SPs with recourse problems from a fairly general class.
Conclusion for two-stage SP

\[ \min \{ \rho(f(x, z(\omega))) \mid x \in X \} \]
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Assumption on the recourse problem:
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**Assumption on the risk measure:**

The risk measure \( \rho \) is induced by a mapping that is convex, nondecreasing w.r.t. the \( \mathbb{P} \)-almost sure partial order and law invariant.
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- Every convex risk measure, especially every coherent one
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- Conditional Value-at-risk: Expectation of the $(1 - \alpha) \times 100\%$ worst outcomes
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- Every conic combination of covered risk functionals
Conclusion for two-stage SP

Reformulation of the objective function:

\[ \rho(f(x, z(\omega))) = Q(x, \nu) = R_\rho ((\delta_x \otimes \nu) \circ f^{-1}), \text{ where } \nu = \mathbb{P} \circ z^{-1} \]
Reformulation of the objective function:

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Theorem (Claus, Krätschmer, Schultz 2015)

Let \( M \subseteq M_\gamma^p \) be a locally uniformly \( \| \cdot \|_{s,2}^p \)-integrating subset, and let \( D_f \) denote the set of discontinuity points of \( f \). If \( x \in \mathbb{R}^n \) and \( \nu \in M \) satisfy \( \delta_x \otimes \nu(D_f) = 0 \), then under the growth condition \((G)\) the mapping \( Q|_{\mathbb{R}^n \times M} \) is continuous at \( (x, \nu) \) with respect to the product topology of the standard topology on \( \mathbb{R}^n \) and the relative topology of weak convergence on \( M \).
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Key idea: Considering \( \Psi \)-weak topologies on suitable subclasses of Borel probability measures.
Implications for stability

\[ \varphi(\nu) := \inf \{ Q(x, \nu) \mid x \in X \} \]
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**Corollary**

*In addition to the previous assumptions, let \( \delta_x \otimes \nu(\mathcal{D}_f) = 0 \) hold for all \( x \in X \). Then \( \varphi|_M \) is upper semicontinuous in \( \nu \) with respect to the relative topology of weak convergence.*
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\[ \Phi(\nu) := \{x \in X \mid Q(x, \nu) = \varphi(\nu)\} \]
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**Corollary**

In addition to the previous assumptions, let \( \delta_x \otimes \nu(D_f) = 0 \) hold for all \( x \in X \) and \( X \) be compact. Then \( \varphi|_M \) is continuous on \( M \) and \( \Phi|_M \) is upper semicontinuous on \( M \) with respect to the relative topology of weak convergence.
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Upper semicontinuity: For any \( \mu_0 \in M \) and any open set \( \mathcal{O} \subseteq \mathbb{R}^n \) such that \( \Phi|_M(\mu_0) \subseteq \mathcal{O} \) there exists a neighborhood \( \mathcal{N} \) of \( \mu_0 \) with respect to the topology of weak convergence such that \( \Phi|_M(\mu) \subseteq \mathcal{O} \) for all \( \mu \in \mathcal{N} \).
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→ **Interpretation:** The solution set does not ”explode” under small perturbations.
Thank you for your attention!