Polytopes Associated with Symmetry Handling

Christopher Hojny

joint work with Marc Pfetsch

TECHNISCHE UNIVERSITÄT DARMSTADT

Technische Universität Darmstadt
Department of Mathematics

Discrete Optimization

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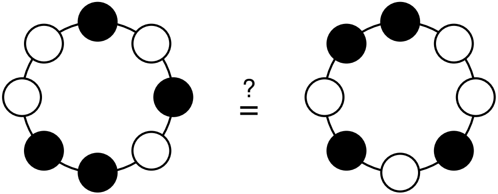
Motivation

- Consider a necklace with n black (0) and white (1) beads.
- Associate with a necklace a vector in $\{0, 1\}^n$.
- necklaces $x, x' \in \{0, 1\}^n$ are "equal" $\Leftrightarrow \exists$ cyclic shift γ such that $\gamma(x) = x'$.



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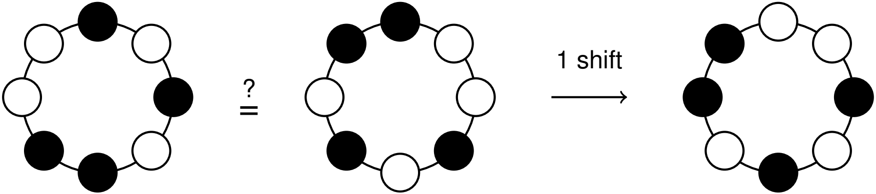

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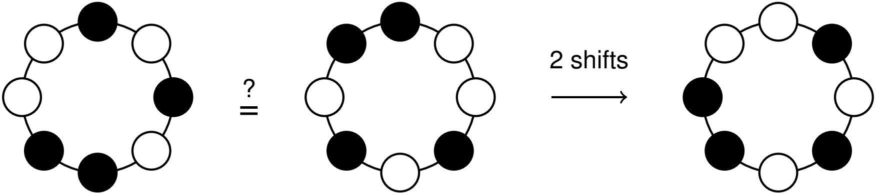

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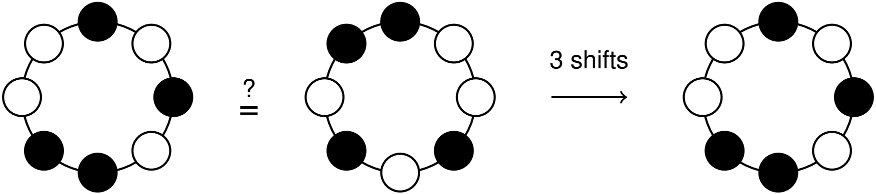

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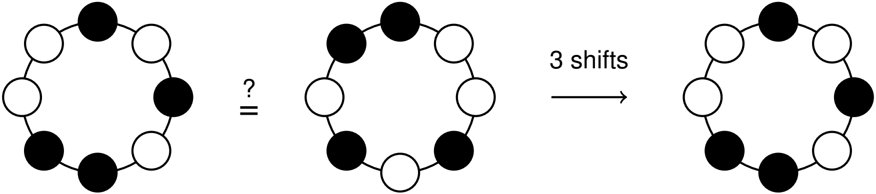

Question: How can we characterize (the convex hull of) a representative system of "distinct" necklaces?

Representative Systems

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Generalized Problem

What can be said about

$$\operatorname{conv}\left(\{x \in \{0,1\}^n \;:\; x \text{ is orbit representative}\}\right)$$

if $\Gamma \leq \mathcal{S}_n$ acts on $\{0,1\}^n$?

Lexicographic Representative Systems

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Friedman's approach [Friedman, 2007]:

- define $\bar{c} = (2^{n-1}, 2^{n-2}, \dots, 2, 1)^\top \in \mathbb{R}^n$
- Friedman inequality for $\gamma \in \Gamma$:

$$\begin{aligned} & \bar{c}^\top x \geq \bar{c}^\top \gamma(x) \\ \Leftrightarrow \quad & \sum_{i \in [n]} (2^{n-\gamma(i)} - 2^{n-i}) x_i \leq 0 \end{aligned}$$

- $x \in \{0,1\}^n$ with $\bar{c}^\top x \geq \bar{c}^\top \gamma(x) \; \forall \, \gamma \in \Gamma$ are lexicographically maximal in their Γ-orbit.

Application in Binary Programs

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- Let $\Gamma \leq \mathcal{S}_n$ be a symmetry group of a binary program, i.e.,
  - $\gamma \in \Gamma$ transforms feasible solutions x to feasible solutions $\gamma(x)$,
  - $\gamma(x)$ and x have the same objective value for each feasible x.
- Add Friedman's inequalities for each $\gamma \in \Gamma$.
- Restricted binary program contains only solutions which are lexicographically maximal w.r.t. Γ.

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- Restricted binary program contains only solutions which are lexicographically maximal w.r.t. Γ.

But:

- Coefficients of Friedman inequalities are exponentially large.
- If Γ is large, many inequalities have to be added.

Further Symmetry Handling Techniques

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Examples:

- isomorphism pruning [Margot, 2002]
- orbital branching [Ostrowski et al., 2011]
- symmetry breaking inequalities [Liberti, 2012]

Outline

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Symmetry Breaking Polytopes

Optimization Complexity over Symresacks

Separation of Minimal Cover Inequalities for Symresacks

Relaxed Symretopes

Symmetry Breaking Polytopes

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Definition

The symmetry breaking polytope (symretope) for $\Gamma \leq \mathcal{S}_n$ is

$$\mathrm{S}(\Gamma) \coloneqq \operatorname{conv}\left(\{x \in \{0,1\}^n \;:\; \bar{c}^\top x \geq \bar{c}^\top \gamma(x) \;\forall\, \gamma \in \Gamma\}\right).$$

A relaxed symretope for Γ is any polytope S with

$$\operatorname{conv}\left(S \cap \{0,1\}^n\right) = \mathrm{S}(\Gamma).$$

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Proposition (cf. [Babai and Luks, 1983])

The linear optimization problem over $\mathrm{S}(\Gamma)$ is NP-hard.

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A relaxed symretope for Γ is any polytope S with

$$\operatorname{conv}\left(S \cap \{0,1\}^n\right) = \mathrm{S}(\Gamma).$$

Questions:

- Can we find a complete linear description of $\mathrm{S}(\Gamma)$?
- Can we find tractable IP-formulations for $\mathrm{S}(\Gamma)$ via relaxed symretopes?
- Are all Friedman inequalities necessary for a relaxed symretope?

Symresacks

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Special case: only one permutation, knapsack polytope

Definition

The symresack for a permutation $\gamma \in \mathcal{S}_n$ is

$$P_\gamma \coloneqq \operatorname{conv}\left(\{x \in \{0,1\}^n \;:\; \bar{c}^\top x \geq \bar{c}^\top \gamma(x)\}\right) = \operatorname{conv}\left(\{x \in \{0,1\}^n \;:\; x \succeq \gamma(x)\}\right).$$

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Examples:

- $\gamma = (1,2) \in \mathcal{S}_n$:

$$P_\gamma = \operatorname{conv}\left(\{x \in \{0,1\}^n \ : \ x_1 \geq x_2\}\right)$$

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- $\gamma = (1,2)(3,4)\dots(2n-1,2n) \in \mathcal{S}_{2n}$:

$$P_\gamma = \operatorname{conv}\left(\{x \in \{0,1\}^{2n} \;:\; (x_1, x_3, \dots, x_{2n-1}) \succeq (x_2, x_4, \dots, x_{2n})\}\right)$$

  This is the orbisack O_n, see [Kaibel and Loos, 2011].

Orbisacks

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Orbisack

$$\mathrm{O}_n \coloneqq \operatorname{conv}\left(\{X = (X^1, X^2) \in \{0,1\}^{n \times 2} \;:\; X^1 \succeq X^2\}\right)$$

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$$O_n \coloneqq \operatorname{conv}\left(\{X = (X^1, X^2) \in \{0,1\}^{n \times 2} \; : \; X^1 \succeq X^2\}\right)$$

Properties of vertices of O_n:

- either all rows are constant

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$$

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Properties of vertices of O_n:

- either all rows are constant
- or the first non-constant row c is $(1, 0)$, c is called critical row

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Discrete Optimization

Orbisacks and Symresacks

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Orbisackoidal Symresack

$$Q_\gamma \coloneqq \text{conv}\left(\{X = (X^1, X^2) \in \{0,1\}^{n\times 2} \;:\; X^1 \succeq X^2,\; X^2 = \gamma(X^1)\}\right)$$

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Lemma

P_γ and Q_γ are linearly equivalent.

Optimization over Q_γ and P_γ

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Theorem

The linear optimization problem $\max\{W^\top X \ : \ X \in Q_\gamma\}$, $W \in \mathbb{R}^{n\times 2}$, can be solved in time $\mathcal{O}(n^2)$ for any $\gamma \in \mathcal{S}_n$.

Proof idea:

- each vertex of Q_γ is a vertex of O_n
- fix a critical row $c \in [n+1]$
- find a maximizer in Q_γ with critical row c in time $\mathcal{O}(n)$
- iterate over all possible critical rows

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Corollary

The linear optimization problem $\max\{w^\top x \; : \; x \in P_\gamma\}$, $w \in \mathbb{R}^n$, can be solved in time $\mathcal{O}(n^2)$ for any $\gamma \in \mathcal{S}_n$.

Consequences of the Optimization Algorithm

- There is an extended formulation of P_γ of size $\mathcal{O}(n^2)$.
- The algorithm can be adapted to optimize over $\mathrm{S}(\Gamma)$ in time $\mathcal{O}(|\Gamma| n^{|\Gamma|})$.
  - If $|\Gamma| \in \mathcal{O}(1)$, optimization over $\mathrm{S}(\Gamma)$ can be done in polynomial time.
  - Examples for such symretopes: geometric symmetries in fixed dimension, e.g., cyclic group, or full orbitopes with a fixed number of columns [Kaibel and Pfetsch, 2008]
- There is an extended formulation of $\mathrm{S}(\Gamma)$ of size $\mathcal{O}(n^{|\Gamma|})$.

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Minimal Cover Inequalities for Symresacks

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- How do covers for Friedman inequalities $\bar{c}^\top x \geq \bar{c}^\top \gamma(x)$ look like?
- $C \subseteq [n]$ is cover $\Leftrightarrow$ $\bar{c}^\top \chi_C \leq \bar{c}^\top \gamma(\chi_C) + 1$
- cover inequality for Friedman inequality:

$$\sum_{i \in C^+} x_i - \sum_{i \in C^-} x_i \leq |C^+| - 1,$$

where $C^+ = \{i \in [n] \ : \ i \geq \gamma(i)\}$ and $C^- = C \setminus C^+$

Separation of Minimal Cover Inequalities

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Classical approach: Formulate separation problem of minimal cover inequalities for $\bar{x} \in \mathbb{R}^n$ as IP, see [Crowder et al., 1983]:

$$\begin{aligned}
\max \sum_{i : i \geq \gamma(i)} \bar{x}_i y_i - \sum_{i : i < \gamma(i)} \bar{x}_i (1 - y_i) - (\sum_{i : i \geq \gamma(i)} y_i - 1) \\
\bar{c}^\top \gamma(y) - \bar{c}^\top y \geq 1, \\
y \in \{0, 1\}^n.
\end{aligned}$$

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$$\bar{c}^\top \gamma(y) - \bar{c}^\top y \geq 1 \quad \Leftrightarrow \quad y \prec \gamma(y),$$

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Discrete Optimization

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Theorem

The separation problem of minimal cover inequalities and $\bar{x} \in \mathbb{R}^n$ for P_γ can be solved in time $\mathcal{O}(n^2)$ for all $\gamma \in \mathcal{S}_n$.

Discrete Optimization

IP-formulations for Symretopes

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An IP-formulation for $\mathrm{S}(\Gamma)$ is given by

$$\mathrm{S}_C(\Gamma) \coloneqq \operatorname{conv}\Big(\bigcap_{\gamma\in\Gamma}\{x \in \{0,1\}^n \;:\; x \text{ fulfills all cover inequalities for } P_\gamma\}\Big)$$

since $P_\gamma = \operatorname{conv}\big(\{x \in \{0,1\}^n \;:\; x \text{ fulfills all cover inequalities for } P_\gamma\}\big)$.

Properties:

- $\mathrm{S}_C(\Gamma)$ is defined by inequalities with coefficients in $\{0, \pm 1\}$.
- The LP-relaxation of $\mathrm{S}_C(\Gamma)$ can be separated in time $\mathcal{O}(|\Gamma| n^2)$.

Outline

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Discrete Optimization

Friedman Inequalities and Facets

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Proposition

- $(i, i+1) \in \Gamma$, $i \in [n-1]$: Friedman inequality for $(i, i+1)$ defines a facet of $\mathrm{S}(\Gamma)$.
- $\gamma \in \Gamma$ not a transposition: Friedman inequality for γ does not define a facet of $\mathrm{S}(\Gamma)$.

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Example 1

- $\Gamma = \mathcal{S}_n$
- only next neighbor transpositions define facets
- Friedman inequality for next neighbor transposition $(i, i+1)$:
  $(2^{n-(i+1)} - 2^{n-i})x_i + (2^{n-i} - 2^{n-(i+1)})x_{i+1} \leq 0 \quad \Leftrightarrow \quad -x_i + x_{i+1} \leq 0$
- $S(\mathcal{S}_n) = \{x \in [0,1]^n \ : \ -x_i + x_{i+1} \leq 0 \ \forall i \in [n-1]\}$

Friedman Inequalities and Facets

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Example 2

- $\Gamma = \mathcal{C}_n$, $n \geq 3$
- no Friedman inequality is facet defining
- But not all Friedman inequalities from

$$\mathrm{conv}\left(\left\{x \in \{0,1\}^n \;:\; \bar{c}^\top x \geq \bar{c}^\top \gamma(x) \;\forall\, \gamma \in \mathcal{C}_n\right\}\right)$$

can be dropped to define a relaxed symretope.

Discrete Optimization

Sufficient Criterion

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Proposition

Let $\Gamma = \langle \gamma_1, \dots, \gamma_m \rangle \leq \mathcal{S}_n$. Then $S^\star \coloneqq \bigcap_{i=1}^m \mathrm{S}(\langle \gamma_i \rangle)$ is a relaxed symretope for Γ if for each $x \in S^\star \cap \mathbb{Z}^n$ and $\gamma \in \Gamma$ there is a sequence of powers of generators such that

$$\gamma = \gamma_{i_1}^{p_1} \dots \gamma_{i_k}^{p_k}, \quad i_1, \dots, i_k \in [m],\ p_1, \dots, p_k \in \mathbb{Z}_{\geq 0},$$

and such that

$$\left(\gamma_{i_j}^{p_j} \dots \gamma_{i_k}^{p_k}\right)(x) \in \mathrm{S}(\langle \gamma_{i_{j-1}} \rangle) \quad \forall j \leq k.$$

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Examples:

- $\Gamma = \mathcal{S}_n$: Friedman inequalities for $(i, i+1)$, $i \in [n-1]$, sufficient
- $\Gamma = \mathcal{A}_n$: Friedman inequalities for $(i, i+1, i+2)$, $i \in [n-2]$, sufficient

Conclusion and Outline

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- preliminary computational experiments indicate effectiveness of symretopes in symmetry handling
- Questions:
  - Can we find a complete linear description of $S(\Gamma)$? ($\checkmark$)
  - Can we find tractable IP-formulations for $S(\Gamma)$ via relaxed symretopes? ($\checkmark$)
  - Are all Friedman inequalities necessary for a relaxed symretope? ($\checkmark$)

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Thank you for your attention!

Literature

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Babai, L. and Luks, E. M. (1983).
Canonical labeling of graphs.
In Proceedings of the Fifteenth Annual ACM Symposium on Theory of Computing, STOC '83, pages 171–183, New York, NY, USA. ACM.

Crowder, H., Johnson, E. L., and Padberg, M. (1983).
Solving large-scale zero-one linear programming problems.
Operations Research, 31(5):803–834.

Friedman, E. J. (2007).
Fundamental domains for integer programs with symmetries.
In Dress, A., Xu, Y., and Zhu, B., editors, Combinatorial Optimization and Applications, volume 4616 of Lecture Notes in Computer Science, pages 146–153. Springer Berlin Heidelberg.

Kaibel, V. and Loos, A. (2011).
Finding descriptions of polytopes via extended formulations and liftings.
In Mahjoub, A. R., editor, Progress in Combinatorial Optimization. Wiley.

Kaibel, V. and Pfetsch, M. E. (2008).
Packing and partitioning orbitopes.
Mathematical Programming, 114(1):1–36.

Liberti, L. (2012).
Reformulations in mathematical programming: automatic symmetry detection and exploitation.
Mathematical Programming, 131(1-2):273–304.

Margot, F. (2002).
Pruning by isomorphism in branch-and-cut.
Mathematical Programming, 94(1):71–90.

Ostrowski, J., Linderoth, J., Rossi, F., and Smriglio, S. (2011).
Orbital branching.
Mathematical Programming, 126(1):147–178.