Convex hulls of graphs of bilinear functions on a box

Thomas Kalinowski

joint with Natasha Boland (Georgia Tech), Akshay Gupte (Clemson University), Fabian Rigterink (Newcastle) and Hamish Waterer (Newcastle)

CARMA and C-OPT
The University of Newcastle (Australia)

Aussois 2016
Bilinear functions on the unit cube

Let $f : [0, 1]^n \to \mathbb{R}$ be a bilinear function:

$$f(\mathbf{x}) = \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j.$$ 

$$f(\mathbf{x}) = x_1 x_2 - 2x_2 x_3 + 2x_3 x_4 + x_4 x_5 - x_1 x_5$$

There is an associated (weighted) graph $G = (V, E)$ with $V = \{1, \ldots, n\}$.

We are interested in the convex hull of the graph of $f$:

$$B = \operatorname{conv} \{ (\mathbf{x}, z) \in [0, 1]^n \times \mathbb{R} : z = f(\mathbf{x}) \}$$

$$= \operatorname{conv} \{ (\mathbf{x}, z) \in \{0, 1\}^n \times \mathbb{R} : z = f(\mathbf{x}) \}$$
An outer approximation (McCormick 1976)

Introduce a variable $y_{ij}$ for every edge $ij$, and look at

$$P = \left\{ \left( x, \sum_{ij \in E} a_{ij} y_{ij} \right) : y_{ij} \leq x_i, y_{ij} \leq x_j, y_{ij} \geq x_i + x_j - 1 \text{ for all } ij \right\} \supseteq B.$$

1. When is $P = B$?

2. How good is $P$ as an approximation for $B$?

3. Which inequalities do we need to add such that

$$B = \left\{ \left( x, \sum_{ij \in E} a_{ij} y_{ij} \right) : \ldots, \ldots \right\}?$$
Some answers

- If $G$ is bipartite and all coefficients are positive then $P = B$. (Coppersmith, Günlük, Lee, Leung 1999)

- If all coefficients are positive then $P$ approximates $B$ up to a factor of $\left(2 - \frac{1}{\lceil \chi(G)/2 \rceil}\right)$. (Luedtke, Namazifard, Linderoth 2012)

- In general $P$ approximates $B$ up to a factor of $n$. (Luedtke, Namazifard, Linderoth 2012)
Convex and concave envelopes

$\Lambda(x)$ – set of ways of writing $x$ as a convex combination of the cube vertices

\[
\Lambda(x) = \left\{ \lambda : \{0, 1\}^n \to [0, 1] : \sum_{\xi \in \{0, 1\}^n} \lambda(\xi) = 1, \sum_{\xi \in \{0, 1\}^n} \lambda(\xi)\xi = x \right\}.
\]

We have

\[
B = \{(x, z) : \text{vex}[f](x) \leq z \leq \text{cav}[f](x)\}
\]

where

\[
\text{vex}[f](x) = \min_{\lambda \in \Lambda(x)} \sum_{\xi \in \{0, 1\}^n} \lambda(\xi)f(\xi), \quad \text{cav}[f](x) = \max_{\lambda \in \Lambda(x)} \sum_{\xi \in \{0, 1\}^n} \lambda(\xi)f(\xi).
\]

The convex hull gap $\text{chgap}[f] : [0, 1] \to \mathbb{R}$ is defined by

\[
\text{chgap}[f](x) = \text{cav}[f](x) - \text{vex}[f](x).
\]
The McCormick envelopes

Similarly, we can write the McCormick hull as

\[ P = \{(x, z) : \text{mcl}[f](x) \leq z \leq \text{mcu}[f](x)\} \]

where

\[
\text{mcl}[f](x) = \min \left\{ \sum_{ij \in E} a_{ij} y_{ij} : \max\{0, x_i + x_j - 1\} \leq y_{ij} \leq \min\{x_i, x_j\} \right\}
\]

\[
\text{mcu}[f](x) = \max \left\{ \sum_{ij \in E} a_{ij} y_{ij} : \max\{0, x_i + x_j - 1\} \leq y_{ij} \leq \min\{x_i, x_j\} \right\}
\]

The **McCormick gap** \( \text{mcgap}[f] : [0, 1] \rightarrow \mathbb{R} \) is defined by

\[
\text{mcgap}[f](x) = \text{mcu}[f](x) - \text{mcl}[f](x).
\]
Our results

Theorem

If $G$ is the complete graph and the coefficients are chosen at random from $\{1, -1\}$ then for $x = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ asymptotically almost surely

$$\text{mcgap}[f](x) \geq \frac{\sqrt{n}}{4} \text{chgap}[f](x).$$

Theorem

We have $P = B$ if and only if every cycle in $G$ has an even number of positive edges and an even number of negative edges.
Proof strategy

- Let $T = T(x) = \{ i \in V : 0 < x_i < 1 \}$ be the set of fractional vertices.
- Luedtke, Namazifar and Linderoth prove $\text{mcgap}[f](x) \leq n \text{chgap}[f](x)$ in three steps.

1. $x \in \{0, 1/2, 1\}^n \implies \text{mcgap}[f](x) = \frac{1}{2} \sum_{ij \in E, i,j \in T} |a_{ij}|.$

2. $x \in \{0, 1/2, 1\}^n \implies \text{chgap}[f](x) \geq \frac{1}{2|T|} \sum_{ij \in E, i,j \in T} |a_{ij}|.$

3. $c \text{chgap}[f](x) - \text{mcgap}[f](x)$ is minimized at some $x \in \{0, 1/2, 1\}^n$.

- For $x = (1/2, \ldots, 1/2)$ and all $|a_{ij}| = 1$, $\text{mcgap}[f](x) = n(n - 1)/4$.
- We want to show that $\text{mcgap}[f](x) \geq c \sqrt{n} \text{chgap}[f](x)$, so we need $\text{chgap}[f](x) = O(n^{3/2})$. 
Some notation

- For $X \subseteq V$, $\gamma(X)$ is the set of edges with both vertices in $X$.
- $\delta(X, Y)$ is the set of edges with one vertex in $X$ and one vertex in $Y$.
- For $Z \subseteq E$, we put $a(Z) = \sum_{ij \in Z} a_{ij}$.
- Maximum and minimum weight of a cut in induced subgraphs

$$
\mu^+(X) = \max \left\{ \sum_{ij \in \delta(U_1, U_2)} a_{ij} : U_1 \cup U_2 = X,\ U_1 \cap U_2 = \emptyset \right\},
$$

$$
\mu^-(X) = \min \left\{ \sum_{ij \in \delta(U_1, U_2)} a_{ij} : U_1 \cup U_2 = X,\ U_1 \cap U_2 = \emptyset \right\}.
$$

- Points in $\xi \in \{0, 1\}^n$ are identified with vertex sets $\xi \subseteq V$.
- A point $x \in \{0, \frac{1}{2}, 1\}^n$ induces a partition $V = V_0 \cup V_{1/2} \cup V_1$. 
The convex hull gap for half integer points

If \( \mathbf{x} \in \{0, \frac{1}{2}, 1\}^n \) then \( \text{chgap}[f](\mathbf{x}) = \frac{1}{2} (\mu^+(V_{1/2}) - \mu^-(V_{1/2})) \).

Proof.

\[
\text{vex}[f](\mathbf{x}) = \min_{\lambda} \sum_{\xi \subseteq V_{1/2}} \lambda(\xi) a(\gamma(\xi \cup V_1))
\]

\[
= \min_{\lambda} \sum_{\xi \subseteq V_{1/2}} \lambda(\xi) [a(\gamma(V_1)) + a(\delta(V_1, \xi)) + a(\gamma(\xi))] 
\]

\[
= (\gamma(V_1)) + \frac{1}{2} a(\delta(V_1, V_{1/2})) + \min_{\lambda} \sum_{\xi \subseteq V_{1/2}} \lambda(\xi) a(\gamma(\xi))
\]
The convex hull gap for half integer points

If \( \mathbf{x} \in \{0, \frac{1}{2}, 1\}^n \) then \( \text{chgap}[f](\mathbf{x}) = \frac{1}{2} (\mu^+(V_{1/2}) - \mu^-(V_{1/2})) \).

Proof.

We only need \( \xi \in \{0, 1\}^n \) with \( \xi_i = 0 \) for \( i \in V_0 \) and \( \xi_i = 1 \) for \( i \in V_1 \).

\[ \sum_{\xi \subseteq V_{1/2}} \lambda(\xi) = 1 \]

\[ \sum_{\xi \subseteq V_{1/2}} \lambda(\xi)\xi_i = 1/2 \text{ for all } i \in V_{1/2} \]

\( \lambda(U_1) = \lambda(U_2) = 1/2 \) for a maximum cut \( (U_1, U_2) \) gives

\[ \min_{\lambda} \sum_{\xi \subseteq V_{1/2}} \lambda(\xi)a(\gamma(\xi)) \leq \frac{1}{2} \left[ a(\gamma(U_1)) + a(\gamma(U_2)) \right] = \frac{1}{2} \left[ a(\gamma(V_{1/2})) - \mu^+(V_{1/2}) \right] \]

and using LP duality we get equality.
The convex hull gap for half integer points

If \( x \in \{0, \frac{1}{2}, 1\}^n \) then \( \text{chgap}[f](x) = \frac{1}{2} \left( \mu^+(V_{1/2}) - \mu^-(V_{1/2}) \right) \).

**Proof.**

1. We only need \( \xi \in \{0, 1\}^n \) with \( \xi_i = 0 \) for \( i \in V_0 \) and \( \xi_i = 1 \) for \( i \in V_1 \).
2. \( \sum_{\xi \subseteq V_{1/2}} \lambda(\xi) = 1 \)
3. \( \sum_{\xi \subseteq V_{1/2}} \lambda(\xi) \xi_i = 1/2 \) for all \( i \in V_{1/2} \)

Putting everything together we get

\[
\text{vex}[f](x) = a(\gamma(V_1)) + \frac{1}{2} a(\delta(V_1, V_{1/2})) + \frac{1}{2} a(\gamma(V_{1/2})) - \frac{1}{2} \mu^+(V_{1/2})
\]

\[
\text{cav}[f](x) = a(\gamma(V_1)) + \frac{1}{2} a(\delta(V_1, V_{1/2})) + \frac{1}{2} a(\gamma(V_{1/2})) - \frac{1}{2} \mu^-(V_{1/2})
\]

and the result follows.
A random function does the job

- Choose the coefficients $a_{ij} \in \{1, -1\}$ independent and uniformly at random.

- Chernoff bound: The probability for a cut $(U_1, U_2)$ to have weight of absolute value $> 0.6n^{3/2}$ is less than $2 \exp(-0.72n)$.

- Union bound over all $2^{n-1}$ cuts:

$$
P\left(\text{All cuts have weights in } [-0.6, 0.6]n^{3/2}\right) \geq 1 - 2^n \exp(-0.72n)
$$

- Therefore $\lim_{n \to \infty} P\left(\mu^+(V) - \mu^-(V) \leq 1.2n^{3/2}\right) = 1$ and our first theorem follows.
An explicit bad function

- \( k = \lceil \log_2 n \rceil \)

- With vertex \( i \in \{1, \ldots, n\} \) we associate the vector \( i \in \{0, 1\}^k \) of the binary digits of \( i - 1 \).

- Put \( a_{ij} = (-1)^{\langle i,j \rangle} \) where \( \langle i,j \rangle = i_1j_1 + \cdots + i_kj_k \) is the standard scalar product.

- Then every cut has weight in \([-1/2, 1/2]n^{3/2}\), hence \( \text{chgap}[f](1/2, \ldots, 1/2) \leq n^{3/2} \), and consequently

\[
\text{mcgap}[f](1/2, \ldots, 1/2) = \frac{n(n-1)}{4} \geq \left( \frac{\sqrt{n}}{4} - \frac{1}{4\sqrt{n}} \right) \text{chgap}[f](1/2, \ldots, 1/2)
\]
Characterizing hull equality (sufficiency)

- Assume every cycle has an even number of positive and an even number of negative edges.

- We need $\text{mcgap}[f](x) = \text{chgap}[f](x)$ for all $x \in [0, 1]^n$.

- W.l.o.g. $x$ is half-integer (Luedtke, Namazifar and Linderoth).

- Let $G^+$ and $G^-$ be the graphs obtained by contracting the positive and negative edges, respectively.

- By assumption, $G^+$ and $G^-$ are bipartite.

- Bipartitions of $G^+$ and $G^-$ induce cuts $(U_1, U_2)$ and $(U'_1, U'_2)$ with
  $\delta(U_1, U_2) = \{ij \in E : a_{ij} < 0\}$ and $\delta(U'_1, U'_2) = \{ij \in E : a_{ij} > 0\}$.

- So $\mu^+(X) - \mu^-(X) = \sum_{ij \in \gamma(X)} |a_{ij}|$ for all $X \subseteq V$, in particular for $X = V_{1/2}$.

- This implies $\text{mcgap}[f](x) = \text{chgap}[f](x)$. 
Assume there is a cycle with an odd number of negative edges. Every cut that contains all negative edges must contain at least one positive edge.

This implies \( \mu^-(V) > a(\{ij : a_{ij} < 0\}) \), and therefore

\[
\text{chgap}[f] \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) = \frac{1}{2} (\mu^+(V) - \mu^-(V)) < \frac{1}{2} \sum_{ij \in E} |a_{ij}| = \text{mcgap}[f] \left( \frac{1}{2}, \ldots, \frac{1}{2} \right)
\]
An open problem

Nail down the correct ratio
Is it true that $\text{mcgap}[f](x) = O(\sqrt{n}) \text{chgap}[f](x)$?

An equivalent question about discrepancy
Let $G = (V, E)$ be the complete graph on $n$ vertices, and let $a = (a_{ij})$ be an arbitrary weight vector. Is it true that

$$\max_{U \subseteq V} \left| \sum_{ij \in \delta(U, V \setminus U)} a_{ij} \right| = \Omega \left( \frac{1}{\sqrt{n}} \right) \sum_{ij \in E} |a_{ij}|?$$

For weights $a_{ij} \in \{\pm 1\}$ the answer is YES.
Another open problem

What can be done if the McCormick hull is not equal to the convex hull?

We can add

- odd-cycle inequalities: $x(C) - y(C) \leq k$ for any $(2k + 1)$-cycle $C$,
- clique inequalities: $sx(Q) - y(Q) \leq \binom{s+1}{2}$ for any $k$-clique $Q$ and $s = 1, \ldots, k - 2$.

Sometimes these are sufficient, but in general they are not.

**Problem:** Characterize the graphs $G = (V, E)$ for which the these inequalities are sufficient to describe the convex hull of the graph of the function $f(x) = \sum_{ij \in E} x_i x_j$.

Thank you!